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Combinatorial Reciprocity Theorems*

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CONTENTS

A *combinatorial reciprocity theorem* is a result which establishes a kind of duality between two related enumeration problems. This rather vague concept will become clearer as more and more examples of such theorems are given. We will begin with simple, known results and see to what extent they can be generalized. The culmination of our efforts will be the "Monster Reciprocity Theorem" of Section 10,

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which will include many (but not all) of our previous results. Our main new results are Proposition 8.3, Theorem 10.2, and Proposition 10.3. The statement and proof of Proposition 8.3 can be read independently of the rest of the paper, except for some terminology introduced in Section 7. Theorem 10.2 and Proposition 10.3 complement one another and can be understood by beginning this paper with Section 9. (The long computational proof of Lemma 9.2 can be omitted without significant loss of understanding.)

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The reader may find some of our results to be of independent interest. For instance, in Sections 3b and 3c we give a generalization of the Dehn–Sommerville equations [13, Section 9.2] valid for *all* convex polytopes (or even spherical polytopes), while in Proposition 11.5 we strengthen Stiemke's theorem [36] to an *explicit* criterion for a system of linear homogeneous equations with integer (or real) coefficients to possess a positive solution.

The following notation will be used throughout.

Symbol	Meaning
$\mathbf{N}$	Set of nonnegative integers
$\mathbf{P}$	Set of positive integers
$\mathbf{Z}$	Set of integers
$\mathbf{C}$	Set of complex numbers
$\mathbf{R}$	Set of real numbers
$\mathbf{R}^s$	s-dimensional Euclidean space
$\mathbf{Q}$	Set of rational numbers
$\mathbf{m}$	The set $\{1, 2, \ldots, m\}$, where $m \in \mathbf{N}$ (so $\mathbf{0} = \varnothing$)
$\{m_1, \ldots, m_s\}_<$	The set $\{m_1, \ldots, m_s\} \subset \mathbf{R}$, where $m_1 < m_2 < \cdots < m_s$
$\lvert S \rvert$	Cardinality of the finite set S

1. Binomial Coefficients

Let p and n be non-negative integers, and define $C_p(n)$ to be the number of combinations of n things taken p at a time allowing repetitions.

Also define $\bar{C}_p(n)$ to be the number of combinations of n things taken p at a time not allowing repetitions. We think of p as fixed and $C_p(n)$ and $\bar{C}_p(n)$ as being functions of n. Of course,

$$C_p(n) = \binom{n+p-1}{p}, \qquad \bar{C}_p(n) = \binom{n}{p}.$$

Hence for fixed p, $C_p(n)$ and $\bar{C}_p(n)$ are polynomials in n of degree p, and it makes sense (algebraically, though *a priori* not combinatorially) to evaluate them at negative integers. Since $C_p(n)$ and $\bar{C}_p(n)$ are known explicitly, it is a matter of trivial computation to verify our first example of a combinatorial reciprocity theorem.

PROPOSITION 1.1. $\bar{C}_p(n) = (-1)^p C_p(-n)$.

At first, one might dismiss this result as merely a "coincidence," but we shall soon see that this is not the case. (Proposition 1.1 was observed implicity by Riordan [26, pp. 4–7] without further comment. The remainder of this paper may be regarded as "further comment.")

Suppose $\{a_1, a_2, \ldots, a_p\}_<$ is a combination from $\mathbf{n+p-1}$ without repetitions, so $1 \leqslant a_1 < a_2 < \cdots < a_p \leqslant n + p - 1$. If we define $b_i = a_i - i + 1$, then $1 \leqslant b_1 \leqslant b_2 \leqslant \cdots \leqslant b_p \leqslant n$, so $\{b_1, b_2, \ldots, b_p\}$ is a combination from $\mathbf{n}$ with repetitions allowed. Conversely, given $1 \leqslant b_1 \leqslant b_2 \leqslant \cdots \leqslant b_p \leqslant n$, we can recover the a_i's by $a_i = b_i + i - 1$. Thus we have defined a bijection between the objects enumerated by $\bar{C}_p(n + p - 1)$ and by $C_p(n)$. Therefore, if we know Proposition 1.1 we can conclude that

$$C_p(n) = (-1)^p C_p(-p + 1 - n), \tag{1}$$

without knowing $C_p(n)$ or $\bar{C}_p(n)$ explicitly. Moreover, it is evident from the definition of $\bar{C}_p(n)$ (without bothering to make an explicit calculation) that if $p \geqslant 1$, then $\bar{C}_p(0) = \bar{C}_p(1) = \cdots = \bar{C}_p(p - 1) = 0$. Hence, it follows from Proposition 1.1 that

$$C_p(0) = C_p(-1) = \cdots = C_p(-p + 1) = 0. \tag{2}$$

By similar reasoning we will frequently be able to prove results analogous to (1) and (2) for other classes of polynomials, even though we cannot give an explicit formula for them.

2. The Order Polynomial

In the previous section we regarded a p-combination σ from $\mathbf{n}$ with repetitions as a sequence $1 \leqslant b_1 \leqslant b_2 \leqslant \cdots \leqslant b_p \leqslant n$, and a p-combination τ from $\mathbf{n}$ without repetitions as a sequence $1 \leqslant a_1 < a_2 < \cdots < a_p \leqslant n$. Thus σ corresponds to an order-preserving map $\sigma: \mathbf{p} \to \mathbf{n}$ and τ to a strict order-preserving map $\tau: \mathbf{p} \to \mathbf{n}$ by the rules $\sigma(i) = b_i$ and $\tau(j) = a_j$. This suggests looking at the generalization obtained by replacing the chain $\mathbf{p}$ by an arbitrary finite partially ordered set P of cardinality p.

Definition. Let P be a finite partially ordered set, and let n be a non-negative integer. Define $\Omega(P, n)$ to be the number of order-preserving maps $\sigma: P \to \mathbf{n}$, i.e., the number of maps $\sigma: P \to \mathbf{n}$ such that if $x \leqslant y$ in P, then $\sigma(x) \leqslant \sigma(y)$ as integers. Define $\bar{\Omega}(P, n)$ to be the number of *strict* order-preserving maps $\tau: P \to \mathbf{n}$, i.e., the number of maps $\tau: P \to \mathbf{n}$ such that if $x < y$ in P, then $\tau(x) < \tau(y)$ as integers. Regarded as a function of n, $\Omega(P, n)$ is called the *order polynomial* of P, and $\bar{\Omega}(P, n)$ is called the *strict order polynomial* of P.

Thus, when $P = \mathbf{p}$, $\Omega(P, n) = C_p(n)$ and $\bar{\Omega}(P, n) = \bar{C}_p(n)$ (as defined in Section 1).

If e_s (respectively, $\bar{e}_s$) denotes the number of *surjective* order-preserving maps (respectively, surjective strict order-preserving maps) $P \to \mathbf{s}$, then it is clear that for $p \geqslant 1$,

$$\Omega(P, n) = \sum_{s=1}^{p} e_s \binom{n}{s}, \qquad \bar{\Omega}(P, n) = \sum_{s=1}^{p} \bar{e}_s \binom{n}{s},$$

where $p = |P|$. Hence, $\Omega(P, n)$ and $\bar{\Omega}(P, n)$ are polynomials in n of degree p, so it makes sense to evaluate them at negative integers. It turns out that Proposition 1.1 generalizes directly to order polynomials.

Proposition 2.1. *If P is a finite partially ordered set of cardinality p, then*

$$\bar{\Omega}(P, n) = (-1)^p \, \Omega(P, -n).$$

This result was first stated in [30, Theorem 3], while the first published detailed proof appeared in [32, Proposition 13.2(i)]. Another proof appears in [33, Example 2.4]. We shall sketch two proofs, leading to two different types of generalizations. The first proof has not been published before, though it essentially has been given in [31, p. 149]. The second proof is the one sketched in [30].

First Proof. Let I be an *order ideal* of P, i.e., a subset of P such that if $y \in I$ and $x < y$, then $x \in I$. Any order-preserving map $\sigma: P \to \mathbf{n}$ is equivalent to a chain $\varnothing = I_0 \leqslant I_1 \leqslant \cdots \leqslant I_n = P$ of order ideals of P via the rule $\sigma(x) = i$ if $x \in I_i - I_{i-1}$. Similarly, a strict order-preserving map $\tau: P \to \mathbf{n}$ is equivalent in the same way to a chain $\varnothing = I_0 \leqslant I_1 \leqslant \cdots \leqslant I_n = P$ of order ideals such that each $I_i - I_{i-1}$, $1 \leqslant i \leqslant n$, is an *antichain* (totally unordered subset) of P.

Let $J(P)$ denote the partially ordered set of all order ideals of P, ordered by inclusion. Then $J(P)$ is a distributive lattice (and every finite distributive lattice arises in this way [2, III.3, Theorem 3]). If ζ denotes the zeta function [23] of $J(P)$, then the number of chains $\varnothing = I_0 \leqslant I_1 \leqslant \cdots \leqslant I_n = P$ of elements of $J(P)$ is just $\zeta^n(\varnothing, P)$. If μ denotes the Möbius function [23] of $J(P)$, then $\mu(I, I') = 0$ if $I' - I$ is not an antichain of P and $\mu(I, I') = (-1)^k$ if $I' - I$ is an antichain of cardinality k [23, Section 5, Example 1]. Hence the number of chains $\varnothing = I_0 \leqslant I_1 \leqslant \cdots \leqslant I_n = P$ of elements of $J(P)$ such that each $I_i - I_{i-1}$ is an antichain is just $(-1)^p \mu^n(\varnothing, P)$. It follows that

$$\Omega(P, n) = \zeta^n(\varnothing, P), \qquad (-1)^p \bar{\Omega}(P, n) = \mu^n(\varnothing, P).$$

Since $\mu = \zeta^{-1}$, we have formally $\Omega(P, n) = (-1)^p \bar{\Omega}(P, -n)$. This formal substitution of $-n$ for n can easily be justified in several ways. For instance, $(\zeta - 1)^{p+1}(\varnothing, P) = 0$, so for any $n \in \mathbf{Z}$,

$$\zeta^n(\zeta - 1)^{p+1}(\varnothing, P) = 0.$$

Hence

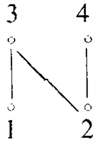
$$\sum_{i=0}^{p+1} (-1)^{p+1-i} \binom{p+1}{i} \zeta^{n+i}(\varnothing, P) = 0$$

for all $n \in \mathbf{Z}$. This is the recursion satisfied by a polynomial in n of degree $\leqslant p$, so the values of $\zeta^n(\varnothing, P)$ fit this polynomial for *all* $n \in \mathbf{Z}$. ∎

Second Proof. Let ω be any bijective order-preserving map $P \to \mathbf{p}$. We denote the elements of P by $x_1, x_2, \ldots, x_p$, where $\omega(x_i) = i$. List all permutations $\pi = (i_1, i_2, \ldots, i_p)$ of $\mathbf{p}$ with the property that if $x < y$ in P, then $\omega(x)$ precedes $\omega(y)$ in π. Put a "$\leqslant$" sign between two consecutive terms i_j and i_{j+1} of π if $i_j < i_{j+1}$; otherwise put a "$<$" sign.

Denote the array thus obtained by $\mathscr{L}(P, \omega)$. Denote by $\bar{\mathscr{L}}(P, \omega)$ the array obtained by changing all "$<$" signs to "$\leqslant$" signs and "$\leqslant$" signs to "$<$" signs. We say a map σ: $P \to \mathbf{n}$ is *compatible* with a permutation $\pi = (i_1, i_2, \ldots, i_p)$ appearing in $\mathscr{L}$ or $\bar{\mathscr{L}}$ if $\sigma(x_{i_1}) \leqslant \sigma(x_{i_2}) \leqslant \cdots \leqslant \sigma(x_{i_p})$ and $\sigma(x_{i_j}) < \sigma(x_{i_{j+1}})$ whenever a "$<$" sign appears in π between i_j and i_{j+1}.

EXAMPLE. Let P and ω be given by



Then $\mathscr{L}(P, \omega)$ and $\bar{\mathscr{L}}(P, \omega)$ are given by:

$1 \leqslant 2 \leqslant 3 \leqslant 4$	$1 < 2 < 3 < 4$
$2 < 1 \leqslant 3 \leqslant 4$	$2 \leqslant 1 < 3 < 4$
$1 \leqslant 2 \leqslant 4 < 3$	$1 < 2 < 4 \leqslant 3$
$2 < 1 \leqslant 4 < 3$	$2 \leqslant 1 < 4 \leqslant 3$
$2 \leqslant 4 < 1 \leqslant 3$	$2 < 4 \leqslant 1 < 3$
$\mathscr{L}(P, \omega)$	$\bar{\mathscr{L}}(P, \omega)$

Thus, for instance, σ: $P \to \mathbf{n}$ is compatible with the second row of $\mathscr{L}(P, \omega)$ if $\sigma(x_2) < \sigma(x_1) \leqslant \sigma(x_3) \leqslant \sigma(x_4)$.

The crucial lemma which we require (whose proof may be found in [32, Theorem 6.2]) is the following.

LEMMA 2.2. (i) *Every order-preserving map σ: $P \to \mathbf{n}$ is compatible with exactly one $\pi \in \mathscr{L}(P, \omega)$. Conversely, if a map σ: $P \to \mathbf{n}$ is compatible with some $\pi \in \mathscr{L}(P, \omega)$, then σ is order preserving.*

(ii) *Every strict order-preserving map τ: $P \to \mathbf{n}$ is compatible with exactly one $\pi \in \bar{\mathscr{L}}(P, \omega)$. Conversely, if a map τ: $P \to \mathbf{n}$ is compatible with some $\pi \in \bar{\mathscr{L}}(P, \omega)$, then τ is strict order-preserving.*

It follows that $\Omega(P, n)$ (respectively $\bar{\Omega}(P, n)$) is obtained by summing the contributions coming from each permutation π in $\mathscr{L}(P, \omega)$ (respectively $\bar{\mathscr{L}}(P, \omega)$). If exactly s "$<$" signs appear in π, then π is easily seen to contribute a term $\binom{n+p-1-s}{p}$ to $\Omega(P, n)$ or $\bar{\Omega}(P, n)$, as the case may be. Hence if w_s (respectively $\bar{w}_s$) denotes the number of $\pi \in \mathscr{L}(P, \omega)$ (respectively $\pi \in \bar{\mathscr{L}}(P, \omega)$) with exactly s "$<$" signs, then

$$\Omega(P, n) = \sum_{s=0}^{p-1} w_s \binom{p+n-1-s}{p},$$

$$\bar{\Omega}(P, n) = \sum_{s=0}^{p-1} \bar{w}_s \binom{p+n-1-s}{p}. \tag{3}$$

But clearly $\bar{w}_s = w_{p-1-s}$. Substituting into (3) and comparing the resulting expression for $\bar{\Omega}(P, n)$ with the expression for $\Omega(P, n)$, we obtain the desired result. ∎

We can now ask for what partially ordered sets P do results analogous to (1) and (2) hold. For instance, we will have

$$\Omega(P, n) = (-1)^p \, \Omega(P, -l-n) \tag{4}$$

for some $l \geqslant 0$ if we can construct a bijection between order-preserving maps $\sigma: P \to \mathbf{n}$ and strict order-preserving maps $\tau: P \to \boldsymbol{l} + \mathbf{n}$. If every maximal chain of P has length l (or cardinality $l + 1$), then such a bijection is given by $\tau(x) = \sigma(x) + \nu(x)$, where $\nu(x)$ is the height of x in P. Conversely, it can be shown [32, Proposition 19.3] that if $\Omega(P, n) = (-1)^p \, \Omega(P, -l-n)$ for all $n \in \mathbf{Z}$, then every maximal chain of P has length l. Moreover, for *any* P with longest chain of length l,

$$\Omega(P, -1) = \Omega(P, -2) = \cdots = \Omega(P, -l) = 0. \tag{5}$$

Further results along these lines may be found in [32, Section 19].

Besides being of theoretical interest, results such as (4) and (5) are also useful in reducing the effort in computing $\Omega(P, n)$ for particular P. Ordinarily one must compute $p + 1$ values to determine a polynomial of degree p. However, if (4) and (5) hold, then one need only compute the $m = [\frac{1}{2}(p - l - 1)]$ values $\Omega(P, 2), \Omega(P, 3), \ldots, \Omega(P, m + 1)$ (since $\Omega(P, 0) = 0$ and $\Omega(P, 1) = 1$ by definition).

3. Zeta Polynomials

In the first proof of Proposition 2.1, we saw that if L is a distributive lattice with bottom $\varnothing$, top P, and zeta function ζ, then $\zeta^n(\varnothing, P)$ is a polynomial function of $n \in \mathbf{Z}$. This fact had nothing to do with the structure of L. Indeed, if Q is any finite partially ordered set with bottom $\hat{0}$, top $\hat{1}$, zeta function ζ, and longest chain of length l, then $(\zeta - 1)^n(\hat{0}, \hat{1}) = 0$ for $n > l$, so

$$\zeta^n(\hat{0}, \hat{1}) = (1 + (\zeta - 1))^n (\hat{0}, \hat{1}) = \sum_{s=0}^{l} \binom{n}{s} (\zeta - 1)^s (\hat{0}, \hat{1}). \qquad (6)$$

From this it follows that $\zeta^n(\hat{0}, \hat{1})$ is a polynomial function of $n \in \mathbf{Z}$ of degree l. We denote this function by $Z(Q, n)$, i.e.,

$$Z(Q, n) = \zeta^n(\hat{0}, \hat{1}),$$

and call it the *zeta polynomial* of Q. G. Kreweras [15; 16] has explicitly computed this polynomial for certain partially ordered sets Q.

For the sake of completeness, we shall put (6) in a more general setting. Recall [29] that the zeta function ζ of a finite partially ordered set Q can be represented by an invertible matrix A whose rows and columns are indexed by the elements of Q. Thus if $x, y \in Q$, then $\zeta^n(x, y) = (A^n)_{x,y}$ for any $n \in \mathbf{Z}$. Let A be *any* nonsingular $p \times p$ matrix (over a field K) whose rows and columns are indexed by $\mathbf{p}$. If $(i, j) \in \mathbf{p} \times \mathbf{p}$, define a function W of $n \in \mathbf{Z}$ by

$$W(n) = (A^n)_{ij}\,.$$

Let

$$f(\lambda) = \lambda^m + a_{m-1}\lambda^{m-1} + \cdots + a_0$$

be the minimal polynomial of A. Since $f(A) = 0$, we have for all $n \in \mathbf{Z}$,

$$\begin{aligned} 0 &= (f(A)\, A^n)_{ij} \\ &= (A^{m+n} + a_{m-1}A^{m+n-1} + \cdots + a_0 A^n)_{ij} \\ &= W(m + n) + a_{m-1}W(m + n - 1) + \cdots + a_0 W(n). \end{aligned}$$

Hence W satisfies a homogeneous linear difference equation with constant coefficients, and we have the "reciprocity theorem"

$$W(-n) = (B^n)_{ij}\,, \qquad B = A^{-1}. \qquad (7)$$

In the case of the zeta function, $f(\lambda) = (\lambda - 1)^{l+1}$, which implies $Z(Q, n)$ is a polynomial of degree $\leq l$ (actually equal to l). Here (7) becomes

$$Z(Q, -n) = \mu^n(\hat{0}, \hat{1}). \qquad (8)$$

Although (7) and (8) may be regarded as "reciprocity theorems," in general they are not very interesting because they have no combinatorial significance. One case which does have combinatorial significance occurs when every maximal chain of Q has length l and for all intervals $[x, y]$ in Q of length k, either $\mu(x, y) = 0$ or $\mu(x, y) = (-1)^k$ (the choice depending on $[x, y]$). In this case $(-1)^l \mu^n(\hat{0}, \hat{1})$ is equal to the number of chains $\hat{0} = x_0 \leqslant x_1 \leqslant \cdots \leqslant x_n \leqslant \hat{1}$ such that for each $i \in \mathbf{n}$, $\mu(x_{i-1}, x_i) \neq 0$.

One class of such partially ordered sets Q consists of the finite distributive lattices, and (8) reduces to Proposition 2.1. A more general example consists of the finite *locally meet-distributive lattices* L, defined by the condition that for any $x, y \in L$, if x is a meet of elements which y covers, then the interval $[x, y]$ is a boolean algebra. Here $\mu(x, y) = 0$ unless $[x, y]$ is a boolean algebra, and $\mu(x, y) = (-1)^k$ for a boolean algebra $[x, y]$ of length k. Hence we get:

PROPOSITION 3.1. *Let L be a finite locally meet-distributive lattice of length l. If $n \in \mathbf{N}$, let $Z(L, n)$ equal the number of chains $\hat{0} = x_0 \leqslant x_1 \leqslant \cdots \leqslant x_n = \hat{1}$ in L, and let $\bar{Z}(L, n)$ equal the number of such chains with the property that for all $i \in \mathbf{n}$, $[x_{i-1}, x_i]$ is a boolean algebra. Then*

$$\bar{Z}(L, n) = (-1)^l Z(L, -n). \quad \blacksquare$$

a. *Natural Partial Orders.* Let $\mathfrak{N}(m)$ denote the lattice of all natural partial orders on $\mathbf{m}$ [6]. Hence the elements of $\mathfrak{N}(m)$ consist of all partial orderings P of $\mathbf{m}$ such that if $i < j$ in P, then $i < j$ as integers. Moreover, $P \leqslant Q$ in $\mathfrak{N}(m)$ if $i \leqslant j$ in P implies $i \leqslant j$ in Q. $\mathfrak{N}(m)$ is known to be a locally meet-distributive lattice of length $\binom{m}{2}$. Furthermore, an interval $[P, Q]$ of $\mathfrak{N}(m)$ is a boolean algebra of rank k if and only if P is obtained from Q by removing k covering relations from Q. Since every interval of a locally distributive lattice is locally distributive, we obtain the following special case of Proposition 3.1.

COROLLARY 3.2. *Let P and Q be partial orders on a finite set S such that $P \leqslant Q$, i.e., if $x \leqslant y$ in P, then $x \leqslant y$ in Q. Let $Z(n)$ denote the number of chains $P = P_0 \leqslant P_1 \leqslant \cdots \leqslant P_n = Q$ of partial orders between P and Q, and let $\bar{Z}(n)$ denote the number of such chains with the property that if $x < y$ in P_i but not in P_{i-1}, then y covers x in P_i. Then $Z(n)$ and $\bar{Z}(n)$ are polynomials in n of degree d satisfying*

$$\bar{Z}(n) = (-1)^d Z(-n),$$

where d is the number of pairs (x, y) such that $x < y$ in Q but not in P. $\blacksquare$

For example, if P is the trivial ordering of a 3-element set S (i.e., $x \leqslant y$ in P implies $x = y$) and Q is a linear ordering of S, then

$$Z(n) = 4\binom{n}{3} + 5\binom{n}{2} + n$$

and

$$\bar{Z}(n) = 4\binom{n}{3} + 3\binom{n}{2}.$$

If P is trivial on a 4-element set S and Q is an ordinal sum $(\mathbf{1} + \mathbf{1}) \oplus \mathbf{2}$ (notation as in [2]), then

$$Z(n) = 56\binom{n}{5} + 128\binom{n}{4} + 94\binom{n}{3} + 23\binom{n}{2} + n$$

and

$$\bar{Z}(n) = 56\binom{n}{5} + 94\binom{n}{4} + 44\binom{n}{3} + 5\binom{n}{2}.$$

Finally, if P is trivial on a 4-element set S and Q is a boolean algebra of rank 2 (so $Q = \mathbf{1} \oplus (\mathbf{1} + \mathbf{1}) \oplus \mathbf{1}$), then

$$Z(n) = 64\binom{n}{5} + 140\binom{n}{4} + 98\binom{n}{3} + 23\binom{n}{2} + n$$

and

$$\bar{Z}(n) = 64\binom{n}{5} + 116\binom{n}{4} + 62\binom{n}{3} + 9\binom{n}{2}.$$

b. *Convex Polytopes.* Another class of partially ordered sets for which $\mu(x, y) = 0$ or $\mu(x, y) = (-1)^k$ for all intervals $[x, y]$ of length k are the lattices of faces of convex polytopes, ordered by inclusion. Specifically, let $\mathscr{P}$ be a convex polytope of dimension d, and let $L(\mathscr{P})$ denote the set of faces of $\mathscr{P}$, including the void face $\varnothing$ and $\mathscr{P}$ itself, ordered by inclusion. Then $L(\mathscr{P})$ is a lattice for which every maximal chain has length $l = d + 1$, and for which $\mu(x, y) = (-1)^k$ for *every* interval $[x, y]$ of length k. (This follows from [17, Theorem 2]; see also [28, Theorem 4]). It follows that $\mu^n(\varnothing, \mathscr{P}) = (-1)^{d+1} \zeta^n(\varnothing, \mathscr{P})$ for all $n \in \mathbf{N}$. Hence from (8) we obtain:

PROPOSITION 3.3. *Let $\mathscr{P}$ be a d-dimensional convex polytope, and let $Z(\mathscr{P}, n)$ denote the number of chains $\varnothing = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n = \mathscr{P}$ of faces of $\mathscr{P}$ between $\varnothing$ and $\mathscr{P}$. Then $Z(\mathscr{P}, n)$ is a polynomial in n of degree $d + 1$ satisfying $Z(\mathscr{P}, -n) = (-1)^{d+1} Z(\mathscr{P}, n)$.* ▮

For instance, if $\mathscr{P}$ is a d-simplex, then $Z(\mathscr{P}, n) = n^{d+1}$. If $\mathscr{P}$ is a polygon with v vertices, then $Z(\mathscr{P}, n) = \frac{1}{3}(vn^3 + (3 - v)n)$. If $\mathscr{P}$ is a d-dimensional cube or octahedron, then

$$Z(\mathscr{P}, n) = 1 + 3^d + 5^d + 7^d + \cdots + (2n - 1)^d.$$

Finally, if $\mathscr{P}$ is an icosahedron or dodecahedron, then $Z(\mathscr{P}, n) = 5n^4 - 4n^2$.

c. *The Dehn–Sommerville Equations.* Suppose $\mathscr{P}$ is a *simplicial* convex d-polytope, so every facet of $\mathscr{P}$ is a simplex. Let f_i be the number of i-faces of $\mathscr{P}$, with $f_{-1} = 1$. Now given a chain

$$\varnothing = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n = \mathscr{P}$$

of faces of $\mathscr{P}$, there is a unique $i < d$ and a unique $j < n$ such that $\dim F_j = i$ and $F_{j+1} = F_{j+2} = \cdots = F_n = \mathscr{P}$. Hence

$$Z(\mathscr{P}, n) = \sum_{i=-1}^{d-1} f_i[Z(\mathscr{S}_i, n - 1) + Z(\mathscr{S}_i, n - 2) + \cdots + Z(\mathscr{S}_i, 0)],$$

where $\mathscr{S}_i$ denotes an i-simplex. But $Z(\mathscr{S}_i, m) = m^{i+1}$, so

$$Z(\mathscr{P}, n) = \sum_{i=-1}^{d-1} f_i((n - 1)^{i+1} + (n - 2)^{i+1} + \cdots + 0^{i+1}),$$

with the convention $0^0 = 1$. More concisely,

$$\Delta Z(\mathscr{P}, n) = \sum_{i=-1}^{d-1} f_i n^{i+1}, \qquad Z(\mathscr{P}, 0) = 0. \tag{9}$$

Thus the relation $Z(\mathscr{P}, -n) = (-1)^{d+1} Z(\mathscr{P}, n)$ of Proposition 3.3 imposes certain conditions on the f_i's. These conditions turn out to be equivalent to the well-known Dehn–Sommerville equations [13, 9.2]. (We will omit the verification of this fact.) Hence Proposition 3.3 may be regarded as a generalization of the Dehn–Sommerville equations to arbitrary convex polytopes.

Equations analogous to (9) can be given for certain other classes of polytopes. For instance, if $\mathscr{P}$ is a *cubical* convex d-polytope (so every facet is a $(d - 1)$-cube) with f_i i-faces, then

$$\Delta^2 Z(\mathscr{P}, n) = \sum_{i=-1}^{d-1} f_i(2n + 1)^{i+1}, \qquad Z(\mathscr{P}, 0) = 0, \qquad Z(\mathscr{P}, 1) = 1.$$

Proposition 3.3 now is equivalent to the equations of [13, Section 9.4].

4. Chromatic Polynomials

Having seen the examples of Sections 1–3, it is natural to ask for what other classes of polynomials occurring in combinatorics do reciprocity theorems hold. Although we have no intention of being exhaustive in this matter, nevertheless there is one well-known class of polynomials which seems worthwhile singling out; namely, the *chromatic polynomials* of graphs (defined, e.g., in [14, Chap. 12]). Let G be a finite graph, which we may assume is without loops and multiple edges, and let $\chi(G, n)$ denote the chromatic polynomial of G. Thus if G has p vertices, then $\chi(G, n)$ is a polynomial in n of degree p. Let V denote the set of vertices of G.

Define two functions $\Omega(G, n)$ and $\bar{\Omega}(G, n)$ of the positive integer n as follows. $\Omega(G, n)$ (resp. $\bar{\Omega}(G, n)$) is equal to the number of pairs $(\sigma, \mathcal{O})$, where σ is any map $\sigma\colon V \to \mathbf{n}$ and $\mathcal{O}$ is an orientation of G (i.e., an assignment of a direction to each edge of G), subject to the two conditions:

(a) The orientation $\mathcal{O}$ is acyclic, i.e., contains no directed cycles;

(b) If $u, v \in V$ are connected by an edge and $u \to v$ in the orientation $\mathcal{O}$, then $\sigma(u) \geqslant \sigma(v)$ (resp. $\sigma(u) > \sigma(v)$).

We shall simply state without proof the reciprocity theorem connecting $\Omega(G, n)$ and $\bar{\Omega}(G, n)$, and its connection with $\chi(G, n)$. For two proofs, one based on Proposition 2.1 and the other a direct proof, see [34].

Proposition 4.1. *$\Omega(G, n)$ and $\bar{\Omega}(G, n)$ are polynomial functions of n of degree p satisfying $\bar{\Omega}(G, n) = (-1)^p\, \Omega(G, -n)$. Moreover, $\bar{\Omega}(G, n) = \chi(G, n)$ for all $n \in \mathbf{Z}$. In particular, $(-1)^p\, \chi(G, -1)$ is equal to the number of acyclic orientations of G.* ∎

Readers familiar with characteristic polynomials of finite geometric lattices will recall that $\chi(G, n) = n^b p(L, n)$, where b is the number of blocks of G and $p(L, n)$ is the characteristic polynomial of the lattice L of contractions of G [27, Section 9]. Hence it is natural to ask to what extent Proposition 4.1 generalizes to the characteristic polynomial $p(L, n)$ of an arbitrary finite geometric lattice L. Brylawski and Lucas [4] have considered this problem and have given a combinatorial interpretation to $(-1)^p\, p(L, -1)$ when G is an *orientable* geometry in the sense of Minty [21] (see also [5, Section 5] and [3, Section 12]).

It seems likely that this result can be extended to $(-1)^p p(L, -n)$ for arbitrary $n \in \mathbf{P}$, but no attempt has yet been made to do so.

5. (P, ω)-Partitions and Generating Functions

We have considered generalizations of Proposition 2.1 suggested by the first proof we gave of this result. Now let us consider generalizations suggested by the second proof. We could consider the arrays $\mathscr{L}(P, \omega)$ and $\bar{\mathscr{L}}(P, \omega)$ used in the second proof not only for order-preserving bijections $\omega: P \to \mathbf{p}$, but for *any* bijections $\omega: P \to \mathbf{p}$. Moreover, we could try to deal with functions more discriminating than $\Omega(P, n)$ by taking into account not merely the number of "$<$" signs in each permutation in $\mathscr{L}(P, \omega)$ and $\bar{\mathscr{L}}(P, \omega)$, but rather the actual permutations themselves.

The preceding suggested generalizations form the subject matter of the monograph [32]. We shall discuss the parts relevant to reciprocity theorems. We shall also for the sake of uniformity continue to deal with order-preserving maps $\sigma: P \to \mathbf{N}$, although [32] uses order-reversing maps. (The two concepts are of course equivalent since P can be replaced by its dual.)

Let ω be any bijection $P \to \mathbf{p}$, where P is a finite partially ordered set of cardinality p. A *(P, ω)-partition* is an order-preserving map $\sigma: P \to \mathbf{N}$ such that if $x < y$ in P and $\omega(x) > \omega(y)$, then $\sigma(x) < \sigma(y)$. Thus if ω is order-preserving, then any order-preserving map $\sigma: P \to \mathbf{N}$ is a (P, ω)-partition. If ω is order-reversing, then only strict order-preserving maps $\tau: P \to \mathbf{N}$ are (P, ω)-partitions.

Given (P, ω), define the generating function (formal power series) $F(P, \omega; X_1, X_2, \ldots, X_p)$ in the variables $X_1, X_2, \ldots, X_p$ by

$$F(P, \omega; X_1, \ldots, X_p) = \sum_{\sigma} X_1^{\sigma(x_1)} X_2^{\sigma(x_2)} \cdots X_p^{\sigma(x_p)}, \tag{10}$$

where the sum is over all (P, ω)-partitions σ, and where $x_1, x_2, \ldots, x_p$ are the elements of P (in some arbitrary but fixed order). Also define the *complementary labeling* $\bar{\omega}: P \to \mathbf{p}$ by

$$\bar{\omega}(x) = p + 1 - \omega(x).$$

Hence $\omega(x) < \omega(y)$ if and only if $\bar{\omega}(x) > \bar{\omega}(y)$.

PROPOSITION 5.1. *$F(P, \omega; X_1, \ldots, X_p)$ and $F(P, \bar{\omega}; X_1, \ldots, X_p)$ are rational functions of the X_i's related by*

$$(X_1 \cdots X_p) F(P, \omega; X_1, \ldots, X_p) = (-1)^p F(P, \omega; 1/X_1, \ldots, 1/X_p). \quad ▮$$

The proof of Proposition 5.1 can be found in [32, Section 10]. It is basically a generalization of the second proof of Proposition 2.1.

The previous proposition deals with transforming variables X_i of a generating function to $1/X_i$, while our other reciprocity theorems have dealt with the relationship between functions evaluated at $+n$ and $-n$. The next proposition clarifies the connection between these two types of reciprocity theorems.

PROPOSITION 5.2. *Let $\{H(i)\}$, $i \in \mathbf{Z}$, be a doubly-infinite sequence of complex numbers satisfying for all $N \in \mathbf{Z}$ a recurrence*

$$H(N + m) + \alpha_{m-1} H(N + m - 1) + \cdots + \alpha_0 H(N) = 0, \tag{11}$$

where m is a fixed non-negative integer, and $\alpha_0, \alpha_1, \ldots, \alpha_{m-1}$ are fixed complex numbers. Define

$$F(X) = \sum_{r=0}^{\infty} H(r) X^r, \qquad \bar{F}(X) = \sum_{r=1}^{\infty} H(-r) X^r.$$

Then $F(X)$ and $\bar{F}(X)$ are rational functions of X related by $F(X) = -\bar{F}(1/X)$. ▮

Ehrhart [8, p. 21] attributes this result to Popoviciu [23]. Ehrhart himself gives a proof in [10]. The most direct proof consists of expanding $F(X)$ by partial fractions into sums of terms of the form $\beta X^s/(1 - \gamma X)^t$ ($\beta, \gamma \in \mathbf{C}$) and verifying the result for each such term separately. Observe that any polynomial $H(i)$ of degree d satisfies a recurrence (11), with $m = d + 1$ and $\alpha_j = (-1)^{d+1-j}\binom{d+1}{j}$.

We remark that it is not hard (though not completely trivial) to derive Proposition 2.1 from Propositions 5.1 and 5.2. Hence we may regard Proposition 5.1 as a generalization of Proposition 2.1.

6. THE EHRHART–MACDONALD LAW OF RECIPROCITY

In this section we shall discuss a "loi de réciprocité" conjectured by E. Ehrhart in 1959–1960, proved by him in [9], and proved in a somewhat improved form by I. G. Macdonald in [19]. In the next section we shall discuss a "homogeneous reciprocity theorem" related to both this section and the previous section.

Let $\mathscr{P}$ be a rational cell complex in s-dimensional Euclidean space $\mathbf{R}^s$ whose underlying topological space $|\mathscr{P}|$ is a manifold (possibly with boundary) of dimension d. Hence by definition $\mathscr{P}$ is a finite rectilinear cell complex (whose cells are convex polytopes) whose vertices have rational coordinates. For $n \in \mathbf{P}$, let $j(\mathscr{P}, n)$ denote the number of rational points $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_s)$ belonging to $|\mathscr{P}|$ such that $n\boldsymbol{\alpha}$ has integer coordinates, and let $i(\mathscr{P}, n)$ denote the number of such points belonging to the (relative) interior of $|\mathscr{P}|$ (i.e., not belonging to the boundary $|\partial\mathscr{P}|$ of $|\mathscr{P}|$). For instance, if $|\mathscr{P}|$ is a line segment in the plane connecting (0, 0) and (2, 2), then $j(\mathscr{P}, n) = 2n + 1$ and $i(\mathscr{P}, n) = 2n - 1$.

It is known that the functions $j(\mathscr{P}, n)$ and $i(\mathscr{P}, n)$ satisfy a linear homogeneous difference equation with constant coefficients, i.e., a recurrence (11) for $N \in \mathbf{P}$. A proof is given by Ehrhart [8, Theorem 4.1]. Ehrhart actually determines the recurrence (11) explicitly, but we do not need this fact here (though it is interesting to note that $i(\mathscr{P}, n)$ and $j(\mathscr{P}, n)$ are polynomials if the vertices of $\mathscr{P}$ have *integer* coordinates). An even stronger result than [8, Theorem 4.1] appears in [35, Theorem 2.5]. At any rate, it is thus possible to extend the values of $j(\mathscr{P}, n)$ and $i(\mathscr{P}, n)$ to all $n \in \mathbf{Z}$.

PROPOSITION 6.1. *The functions* $j(\mathscr{P}, n)$ *and* $i(\mathscr{P}, n)$ *are related by* $j(\mathscr{P}, n) = (-1)^d i(\mathscr{P}, -n)$. *Moreover,* $j(\mathscr{P}, 0) = \chi(\mathscr{P})$, *where* $\chi(\mathscr{P})$ *is the Euler characteristic of* $\mathscr{P}$.

Sketch of proof. Without loss of generality, we may assume that $\mathscr{P}$ is *simplicial*, i.e., the cells of $\mathscr{P}$ are simplices, since any cell complex can be refined to a simplicial complex without introducing new vertices. The proofs of Ehrhart and Macdonald mentioned above deal with simplicial complexes and can be divided into two parts. First, the proposition is proved when $\mathscr{P}$ is a rational simplex. Then it is shown that the simplices of $\mathscr{P}$ "fit together" in the proper way for the proposition to hold for any $\mathscr{P}$.

For the case when $\mathscr{P}$ is a rational simplex, see the elegant proof of Ehrhart [8] (also given in [19]). To prove Proposition 6.1 for arbitrary rational polyhedra $\mathscr{P}$, we invoke the following result of Macdonald [19, Proposition 1.1]. Let φ be any function on $\mathscr{P}$ with values in a real vector space V (for our purposes, V will be the space $\mathbf{R}[t]$ of polynomials in one variable over $\mathbf{R}$). For any subset $\mathscr{P}'$ of $\mathscr{P}$, define

$$S(\mathscr{P}', \varphi) = \sum_{\mathscr{S} \in \mathscr{P}'} (-1)^{1+\dim \mathscr{S}} \varphi(\mathscr{S}),$$

where the dimension of the empty set $\varnothing$ is taken to be -1. Also define $\varphi^*: \mathscr{P} \to V$ by

$$\varphi^*(\mathscr{S}) = \sum_{\mathscr{S}' \leqslant \mathscr{S}} (-1)^{1+\dim \mathscr{S}'} \varphi(\mathscr{S}'),$$

where $\mathscr{S}' \leqslant \mathscr{S}$ means that $\mathscr{S}'$ is a face of $\mathscr{S}$. Finally, define $\tilde{\chi}(\mathscr{P})$ to be $S(\mathscr{P}, \zeta)$, where $\zeta(\mathscr{S}) = 1$ for all $\mathscr{S} \in \mathscr{P}$. Hence $\tilde{\chi}(\mathscr{P}) = 1 - \chi(\mathscr{P})$, where $\chi(\mathscr{P})$ is the Euler characteristic of $\mathscr{P}$.

Proposition 6.2 (I. G. Macdonald [19, Proposition 1.1]). *We have*

$$S(\mathscr{P}, \varphi^*) + (-1)^d S(\mathscr{P} - \partial\mathscr{P}, \varphi) = \tilde{\chi}(\mathscr{P}) \varphi(\varnothing). \quad \blacksquare$$

It is now an easy matter to prove Proposition 6.1 (assuming its validity for rational simplices). Take $\varphi(\mathscr{S})$ to be the polynomial $(-1)^{1+\dim \mathscr{S}} i(\mathscr{S}, t)$. Then clearly $S(\mathscr{P}, \varphi) = j(\mathscr{P}, t)$, while

$$\varphi^*(\mathscr{S}) = \sum_{\mathscr{S}' \subseteq \mathscr{S}} \varphi(\mathscr{S}') = j(\mathscr{S}, t) = (-1)^{\dim \mathscr{S}} i(\mathscr{S}, -t),$$

since the proposition is assumed true for simplicies $\mathscr{S}$. Hence

$$S(\mathscr{P}, \varphi^*) = -\sum_{\mathscr{S} \in \mathscr{P}} i(\mathscr{S}, -t) = -j(\mathscr{P}, -t).$$

By Proposition 6.2,

$$-j(\mathscr{P}, -t) + (-1)^d j(\mathscr{P} - \partial\mathscr{P}, t) = \tilde{\chi}(\mathscr{P}) i(\varnothing, t).$$

Since $j(\mathscr{P} - \partial\mathscr{P}, t) = i(\mathscr{P}, t)$ and $i(\varnothing, t) = 0$, the proof follows. $\blacksquare$

The volume of polytopes. Let $\mathscr{P}$ be a polygon in the plane $\mathbf{R}^2$ such that the vertices of $\mathscr{P}$ have integer coordinates. $\mathscr{P}$ need not be convex, but self-intersections are not allowed. Thus the boundary $\partial\mathscr{P}$ of $\mathscr{P}$ is a simple closed polygonal curve. Let A be the area of $|\mathscr{P}|$, let j be the number of integer points belonging to $|\mathscr{P}|$, and let p be the number of integer points belonging to $|\partial\mathscr{P}|$. A well-known classical result states that

$$A = \frac{1}{2}(2j - p - 2). \tag{12}$$

Proposition 6.1 allows this result to be extended to higher dimensions. Let $\mathscr{P}$ be an *integral* cell complex in $\mathbf{R}^d$ such that $|\mathscr{P}|$ is a manifold (possibly with boundary) of dimension d (so the vertices of $\mathscr{P}$ have *integer* coordinates). Let $j(n) = j(\mathscr{P}, n)$ and $i(n) = i(\mathscr{P}, n)$ be as above, i.e., $j(n)$ is the number of points $\boldsymbol{\alpha} \in |\mathscr{P}|$ such that $n\boldsymbol{\alpha}$ has integer coordinates, while $i(n)$ is the number of such $\boldsymbol{\alpha} \in |\mathscr{P} - \partial\mathscr{P}|$. Hence if $p(n) = p(\mathscr{P}, n)$ is the number of $\boldsymbol{\alpha} \in |\partial\mathscr{P}|$ such that $n\boldsymbol{\alpha}$ has integer coordinates, then $p(n) + i(n) = j(n)$.

As mentioned above, it is known that $p(n)$, $i(n)$, and $j(n)$ are polynomials in n, with $\deg j(n) = \deg i(n) = d$ and $\deg p(n) = d - 1$. This fact was shown by Ehrhart [8, Theorem 5.1] and also follows from [35, Theorem 2.5]. Now if we take all $\boldsymbol{\alpha} \in |\mathscr{P}|$ such that $n\boldsymbol{\alpha}$ has integer coordinates and form d-dimensional hypercubes of side $1/n$ and center $\boldsymbol{\alpha}$, these hypercubes will form a region "approaching" $|\mathscr{P}|$ as $n \to \infty$. Hence (modulo some easily supplied rigor) $j(n) = Vn^d + o(n^d)$, where V is the volume of $|\mathscr{P}|$. Thus the leading coefficient of $j(n)$ is V. Hence V can be determined if we know $j(n)$ for any $d + 1$ values of n. By Proposition 6.1, $j(-n) = (-1)^d\, i(n) = (-1)^d\,(j(n) - p(n))$ for all $n \in \mathbf{P}$. Moreover, it follows from the proof of Proposition 6.1 that $j(0) = \chi(\mathscr{P})$ and $p(0) = \chi(\partial\mathscr{P}) = \chi(\mathscr{P})(1 - (-1)^d)$. In particular, if $|\mathscr{P}|$ is homeomorphic to a solid ball, then $j(0) = 1$ and $p(0) = 1 - (-1)^d$. This provides various generalizations of (12).

For example, to obtain (12) we have that $j(-1) = j - p$, $j(0) = 1$, and $j(1) = j$ are values of a quadratic polynomial with leading coefficient $A = \frac{1}{2}(j - 2 + (j - p)) = \frac{1}{2}(2j - p - 2)$. If $|\mathscr{P}|$ is homeomorphic to a solid 3-dimensional ball (embedded in $\mathbf{R}^3$), then we have $j(-1) = -(j(1) - p(1))$, $j(0) = 1$, $j(1)$, $j(2)$, $j(3)$ are values of a cubic polynomial with leading coefficient

$$V = \frac{1}{6}(j(2) - 3j(1) + 3 + (j(1) - p(1)))$$

$$= \frac{1}{6}(j(2) - 2j(1) - p(1) + 3)$$

or

$$V = \frac{1}{6}(j(3) - 3j(2) + 3j(1) - 1).$$

In general, for a solid d-ball in $\mathbf{R}^d$, we have

$$V = \frac{1}{d!}\left(\sum_{n=0}^{d-1} (-1)^n \binom{d}{n} j(d - n) + (-1)^d\right).$$

Similarly, a formula for V can be given depending on $j(n)$ and $p(m)$ for $1 \leqslant n \leqslant [(d+1)/2]$ and $1 \leqslant m \leqslant [d/2]$. More generally, V can be determined from any d values of $j(n)$ and $p(m)$ for $m, n \geqslant 1$. The first person to obtain results of this generality was Ehrhart [8, footnote on p. 19], while earlier less general results were obtained by Reeve [24; 25] and Ehrhart [37, Theorem 5]. This latter result was also found by I. G. Macdonald [18].

Abstract manifolds. Proposition 6.2 implies as a special case an "abstract" version of Proposition 6.1. Let Δ be a finite simplicial complex with vertex set V. Let $f_i = f_i(\Delta)$ be the number of $(i+1)$-sets contained in Δ. Hence $f_{-1} = 1$ and $f_0 = |V|$. Define a polynomial $\Lambda(\Delta, n)$ by

$$\Lambda(\Delta, n) = \sum_{i \geqslant 0} f_i \binom{n-1}{i}.$$

Note that $\Lambda(\Delta, 0) = f_0 - f_1 + f_2 - \cdots = \chi(\Delta)$, the Euler characteristic of Δ.

Now suppose that the underlying topological space $|\Delta|$ of Δ is homeomorphic to a d-dimensional manifold with boundary. Hence $\deg \Lambda(\Delta, n) = d$. Denote by $\partial\Delta$ the boundary complex of Δ, so $|\partial\Delta| = \partial|\Delta|$. Hence $\partial\Delta$ is itself a simplicial complex, with vertex set contained in V. It follows from Proposition 6.2 that

$$(-1)^d \Lambda(\Delta, -n) = \Lambda(\Delta, n) - \Lambda(\partial\Delta, n).$$

In particular, knowing the numbers f_i for Δ is sufficient for determining the corresponding numbers for $\partial\Delta$. If $\partial\Delta = \varnothing$ (i.e., if $|\Delta|$ is a closed manifold), then $(-1)^d \Lambda(\Delta, -n) = \Lambda(\Delta, n)$. This equation imposes certain linear constraints on the numbers f_i, which in the case when $|\Delta|$ is a sphere are equivalent to the Dehn–Sommerville equations discussed in Section 3c.

7. Homogeneous Linear Equations

In this section we shall consider a reciprocity theorem generalizing parts of Sections 5 and 6. By way of motivation, let P be a finite partially ordered set with elements $y_1, y_2, \ldots, y_p$. An order-preserving map $\sigma: P \to \mathbf{N}$ is equivalent to a solution in non-negative integers to a system of inequalities $x_i \leqslant x_j$, where (i, j) ranges over all pairs such that $y_i < y_j$. Here the value of x_i corresponds to $\sigma(y_i)$. Similarly, a *convex* polytope $\mathscr{P}$ is determined by a system of equalities and inequalities and is a special case of the polyhedra considered in the previous section. This suggests that we try to extend our results to more general classes of equalities and inequalities. First note that any inequality $E(x_1, x_2, ..., x_s) \geqslant 0$ can be rendered into an equality by introducing a slack variable z, viz., $E(x_1, x_2, ..., x_s) = z$, where it is to be assumed that $z \in \mathbf{N}$. Hence we need only consider equalities, and we shall be concerned here with a finite system

$$\begin{aligned} E_1(x_1, x_2, ..., x_s) &= 0 \\ E_2(x_1, x_2, ..., x_s) &= 0 \\ &\vdots \\ E_p(x_1, x_2, ..., x_s) &= 0, \end{aligned} \tag{13}$$

where each $E_i(x_1, x_2, ..., x_s)$ is a *homogeneous linear form with integral coefficients.*

The following notation and terminology will be associated with such a system of equations. The "vector" $(x_1, x_2, ..., x_s)$ will be denoted by $\mathbf{x}$ and the "vector" $(E_1, E_2, ..., E_p)$ of forms by $\mathbf{E}$. Hence the system (13) may be symbolically written as

$$\mathbf{E}(\mathbf{x}) = \mathbf{0}, \qquad \text{or just} \quad \mathbf{E} = \mathbf{0},$$

where $\mathbf{0}$ denotes a vector of p zeros. A system $\mathbf{E} = \mathbf{0}$ of linear homogeneous equations with integer coefficients will be called an *LHD-system* (D stands for "diophantine"). For brevity's sake, we shall often refer to the "LHD-system $\mathbf{E}$," it being understood that this means $\mathbf{E} = \mathbf{0}$.

By the *corank* $\kappa = \kappa(\mathbf{E})$ of the LHD-system $\mathbf{E}$ we mean the number of variables appearing in $\mathbf{E}$ minus the rank of $\mathbf{E}$. In other words, $\kappa(\mathbf{E})$ is equal to the dimension of the null space of the matrix of coefficients of $\mathbf{E}$. If the equations (13) are linearly independent, then $\kappa(\mathbf{E}) = s - p$.

Vectors of length s whose components are integers will be denoted by lower-case boldface Greek letters, such as $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_s) \in \mathbf{Z}^s$, and will be called $\mathbf{Z}$-*vectors*. If the components of $\boldsymbol{\alpha}$ all belong to $\mathbf{N}$ or $\mathbf{P}$, then $\boldsymbol{\alpha}$ will be called an $\mathbf{N}$-*vector* or $\mathbf{P}$-*vector*, as the case may be. Similarly, if $\boldsymbol{\alpha}$ is a solution in integers to the LHD-system $\mathbf{E}$, then $\boldsymbol{\alpha}$ will be called a $\mathbf{Z}$-*solution*. If $\boldsymbol{\alpha}$ is a solution in non-negative or positive integers, then $\boldsymbol{\alpha}$ is called an $\mathbf{N}$-*solution* or $\mathbf{P}$-*solution*.

Given an LHD-system $\mathbf{E}$, define the generating functions (formal power series)

$$\begin{aligned} F(\mathbf{E}; X_1, ..., X_s) &= \sum_{(\alpha_1, ..., \alpha_s)} X_1^{\alpha_1} \cdots X_s^{\alpha_s}, \\ \bar{F}(\mathbf{E}; X_1, ..., X_s) &= \sum_{(\beta_1, ..., \beta_s)} X_1^{\beta_1} \cdots X_s^{\beta_s}, \end{aligned} \tag{14}$$

where $(\alpha_1, ..., \alpha_s)$ ranges over all $\mathbf{N}$-solutions to $\mathbf{E}$ (i.e., to $\mathbf{E} = \mathbf{0}$), and where $(\beta_1, ..., \beta_s)$ ranges over all $\mathbf{P}$-solutions to $\mathbf{E}$. We shall denote the vector $(X_1, ..., X_s)$ symbolically by $\mathbf{X}$, and the monomial $X_1^{\alpha_1} \cdots X_s^{\alpha_s}$ by $\mathbf{X}^{\boldsymbol{\alpha}}$, where $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_s)$. Hence (14) may be rewritten as

$$F(\mathbf{E}; \mathbf{X}) = \sum_{\boldsymbol{\alpha}} \mathbf{X}^{\boldsymbol{\alpha}}, \qquad \bar{F}(\mathbf{E}; \mathbf{X}) = \sum_{\boldsymbol{\beta}} \mathbf{X}^{\boldsymbol{\beta}}.$$

If no confusion will result, we shall omit the symbol $\mathbf{E}$ from the notation, so $F(\mathbf{X})$ is short for $F(\mathbf{E}; \mathbf{X})$ etc. If $F(\mathbf{X}) = F(X_1, ..., X_s)$ is a rational function of $\mathbf{X}$ (in the algebra of formal power series), then $F(1/\mathbf{X})$ will denote the rational function $F(1/X_1, ..., 1/X_s)$. For instance, if $F(\mathbf{X}) = \sum_{0 \leqslant i < j < \infty} X_1^i X_2^j X_3^{2i+j}$, then $F(\mathbf{X}) = X_2X_3/(1 - X_2X_3)(1 - X_1X_2X_3^3)$ and $F(1/\mathbf{X}) = X_1X_2X_3^3/(1 - X_2X_3)(1 - X_1X_2X_3^3)$.

Example. Let $\mathbf{E}$ consist of the single equation $x_1 - x_2 = 0$. Then

$$F(\mathbf{X}) = \sum_{n=0}^{\infty} X_1^n X_2^n = 1/(1 - X_1X_2),$$

$$\bar{F}(\mathbf{X}) = \sum_{n=1}^{\infty} X_1^n X_2^n = X_1X_2/(1 - X_1X_2).$$

Observe that $\bar{F}(\mathbf{X}) = -F(1/\mathbf{X})$ (as rational functions). It is this result which we wish to extend to more general situations.

Proposition 7.1. *Let* $\mathbf{E}$ *be an* LHD-*system. Then* $F(\mathbf{X})$ *and* $\bar{F}(\mathbf{X})$ *are rational functions of* $\mathbf{X}$. *Moreover, a necessary and sufficient condition that* $\bar{F}(\mathbf{X}) = \pm F(1/\mathbf{X})$ (*as rational functions*) *is that* $\mathbf{E}$ *possesses a* $\mathbf{P}$-*solution. In this case,*

$$\bar{F}(\mathbf{X}) = (-1)^{\kappa} F(1/\mathbf{X}), \tag{15}$$

where κ *is the corank of* $\mathbf{E}$.

Sketch of Proof. There are several ways to see that $F(\mathbf{X})$ and $\bar{F}(\mathbf{X})$ are rational functions. For instance, an algorithm of Elliott and MacMahon [12; 20] (see also [35, Section 3]) can be used to compute $F(\mathbf{X})$ and $\bar{F}(\mathbf{X})$ explicitly. A further proof based on the Hilbert syzygy theorem appears in [35, Section 2].

The necessity for **E** to possess a **P**-solution is trivial. For if **E** does not possess a **P**-solution, then $\bar{F}(\mathbf{X}) = 0$ but $F(\mathbf{X}) \neq 0$ since $x_1 = x_2 = \cdots = x_s = 0$ is always an **N**-solution.

A proof that the existence of a **P**-solution to **E** implies (15) first appeared in [35, Theorem 4.1]. However, the methods which Ehrhart and Macdonald used to prove Proposition 6.1 can also be used to prove Proposition 7.1. In both the proof based on Ehrhart–Macdonald and the proof in [35, Theorem 4.1], the idea is to decompose the **N**-solutions to **E** into very simple structures for which Proposition 7.1 can be proved directly. Then an Euler characteristic argument is used, just as we gave in the previous section, to show that these structures fit together properly. Using the method of Ehrhart–Macdonald, the "simple structures" are *simplicial cones*. Here it is trivial to decompose the **N**-solutions to **E** properly, but it is not so trivial to prove Proposition 7.1 for simplicial cones. On the other hand, the proof given by [35, Theorem 7.1] decomposes the solutions into so-called *lattice cones*. Here it is trivial to prove Proposition 7.1 for lattice cones, but not so trivial to show that an appropriate decomposition into lattice cones exists. We refer the reader to [8; 19; 35] for further details. ▮

Remark. Even if there does not exist a **P**-solution to **E**, one can still obtain from Proposition 6.1 a reciprocity theorem, as follows. Since a sum of solutions to **E** is a solution, it follows that if **E** does not possess a **P**-solution, then there are always variables x_i which are equal to 0 in any **N**-solution. If these variables are deleted from **E**, then the resulting system possesses a **P**-solution. Hence for any LHD-system **E**, the rational function $(-1)^{\iota} F(1/\mathbf{X})$ is the generating function for all **Z**-solutions $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_s)$ to **E** such that $\alpha_i > 0$ whenever there exists some **N**-solution $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_s)$ with $\beta_i > 0$. Here ι is the corank of the system obtained from **E** by deleting all variables x_i which equal 0 in any **N**-solution.

a. *Self-Reciprocal Systems.* We have seen in Sections 1–3 (esp. (4)) how reciprocity theorems sometimes imply certain symmetry properties of polynomials. We now give an analogous result corresponding to the generating function $F(\mathbf{E}; \mathbf{X})$ of an LHD-system **E**.

COROLLARY 7.2. *Let* $\mathbf{E} = \mathbf{E}(\mathbf{x})$ *be an* LHD-*system of corank* κ *in the variables* $\mathbf{x} = (x_1, x_2, \ldots, x_s)$. *Suppose* $\mathbf{E}$ *possesses a* $\mathbf{P}$-*solution. A necessary and sufficient condition that* $F(1/\mathbf{X}) = \pm \mathbf{X}^{\boldsymbol{\beta}} F(\mathbf{X})$ *for some monomial* $\mathbf{X}^{\boldsymbol{\beta}}$ *is that* $\boldsymbol{\beta}$ *be a* $\mathbf{P}$-*solution to* $\mathbf{E}$ *with the property that for any* $\mathbf{P}$-*solution* $\boldsymbol{\gamma}$, *the vector* $\boldsymbol{\gamma} - \boldsymbol{\beta}$ *has non-negative coordinates. In this case the correct sign is* $(-1)^\kappa$.

Proof. By Proposition 7.1, $F(1/\mathbf{X}) = (-1)^\kappa \bar{F}(\mathbf{X})$. Hence $F(1/\mathbf{X})$ will equal $\pm \mathbf{X}^{\boldsymbol{\beta}} F(\mathbf{X})$ if and only if the sign is $(-1)^\kappa$ and the map $\boldsymbol{\alpha} \mapsto \boldsymbol{\alpha} + \boldsymbol{\beta}$ is a bijection between $\mathbf{N}$-solutions $\boldsymbol{\alpha}$ and $\mathbf{P}$-solutions $\boldsymbol{\alpha} + \boldsymbol{\beta}$. From this the proof is immediate. ▮

In general, it does not seem easy to determine when an LHD-system $\mathbf{E}$ possesses a solution $\boldsymbol{\beta}$ with the property of Corollary 7.2. An obvious sufficient condition that $\boldsymbol{\beta}$ exist is that $x_1 = x_2 = \cdots = x_s = 1$ should be a solution to $\mathbf{E}$, in which case this solution is equal to $\boldsymbol{\beta}$. Somewhat more general sufficient conditions for $\boldsymbol{\beta}$ to exist can be given, but we do not do so here.

b. *Connection with Partially Ordered Sets.* Let us see how Proposition 7.1 is related to Proposition 5.1 when ω is an order-preserving map $\omega: P \to \mathbf{p}$. In this case, a (P, ω)-partition $\sigma: P \to \mathbf{N}$ is just an order-preserving map, while a $(P, \bar{\omega})$-partition $\tau: P \to \mathbf{N}$ is a strict order-preserving map. As mentioned at the beginning of this section, σ corresponds to an $\mathbf{N}$-solution to a system of inequalities $x_i \leqslant x_j$, where (i, j) ranges over all pairs such that $y_i, y_j \in P$ and $y_i < y_j$. (One can restrict oneself to pairs (i, j) such that y_j covers y_i, but it is not necessary to do so.) By introducing slack variables z_{ij}, we see that σ corresponds to an $\mathbf{N}$-solution of the LHD-system

$$x_j - x_i - z_{ij} = 0 \qquad (\text{where } y_i < y_j). \tag{16}$$

Introducing variables $\mathbf{X} = (X_1, \ldots, X_p)$ and $\mathbf{Z} = (Z_{i_1 j_1}, Z_{i_2 j_2}, \ldots)$, where $P = \{y_1, \ldots, y_p\}$ and where (i_r, j_r) ranges over pairs such that $y_{i_r} < y_{j_r}$, we can define the generating function $F(\mathbf{X}, \mathbf{Z})$ as in (14). Then $F(P, \omega; \mathbf{X}) = F(\mathbf{X}, \mathbf{1})$, where $F(P, \omega; \mathbf{X})$ is the generating function of (10) and $\mathbf{1}$ denotes the vector obtained by setting each $Z_{ij} = 1$.

Now the system (16) clearly possesses a $\mathbf{P}$-solution, and its corank κ is easily seen to equal p. Hence by Proposition 6.1, $(-1)^p F(1/\mathbf{X}, \mathbf{1})$ is the generating function for $\mathbf{P}$-solutions to (16). But a $\mathbf{P}$-solution to (16) corresponds to a strict order-preserving map $\tau: P \to \mathbf{P}$, i.e., to a

$(P, \bar{\omega})$-partition *with positive values*. In other words, $(-1)^p F(1/\mathbf{X}, 1) = (X_1 X_2 \cdots X_p) F(P, \bar{\omega}; \mathbf{X})$, so Proposition 5.1 in the case of order-preserving labelings ω is a special case of Proposition 6.1. We also see that the "natural" reciprocal notion to order-preserving maps σ: $P \to \mathbf{N}$ is strict order-preserving maps τ: $P \to \mathbf{P}$, not τ: $P \to \mathbf{N}$.

The reader may be wondering whether Proposition 7.1 can be generalized so that it includes Proposition 5.1 for *any* labeling ω. Such a generalization will be the subject of the next section (Section 8).

c. *Connection with Convex Polytopes.* We now sketch how Proposition 6.1 can be derived from Proposition 7.1 in the case where $\mathscr{P}$ is a convex polytope. Let $\mathscr{P}$ be a rational convex polytope of dimension d imbedded in $\mathbf{R}^s$. Hence $\mathscr{P}$ is determined by a system of linear inequalities and equalities with integer coefficients, the number of independent equalities being $s - d$. We transform this system of inequalities and equalities into an LHD-system as follows. Introduce a new variable t, called the *scale factor variable*. For each inequality $E(\mathbf{x}) \geqslant \alpha$ or $E(\mathbf{x}) \leqslant \alpha$ (where $E(\mathbf{x})$ is a homogeneous linear form with integer coefficients, and where $\alpha \in \mathbf{Z}$), introduce a further variable w_E. Now change each equality $E(\mathbf{x}) = \alpha$ (E homogeneous, $\alpha \in \mathbf{Z}$) into the equation $E(\mathbf{x}) - \alpha t = 0$, and change each inequality $E(\mathbf{x}) \geqslant \alpha$ or $E(\mathbf{x}) \leqslant \alpha$ into the equation $E(\mathbf{x}) - \alpha t - w_E = 0$ or $\alpha t - E(\mathbf{x}) - w_E = 0$, respectively. It is easy to see that the number of $\mathbf{N}$-solutions to this new LHD-system satisfying $t = n \in \mathbf{N}$ is just $j(\mathscr{P}, n)$, while the number of such $\mathbf{P}$-solutions is just $i(\mathscr{P}, n)$. Proposition 6.1 (for $\mathscr{P}$ convex) now follows easily from Propositions 7.1 and 5.2.

d. *Magic Squares.* A typical application of Proposition 7.1 is to the theory of "magic labelings of graphs," as developed in [35]. We shall give the reader some of the flavor of the subject by simply stating a special case of [35, Theorem 1.3]. Let $H_n(r)$ be the number of $n \times n$ matrices M of non-negative integers summing to r in every row and column. It can be shown [35, Theorem 1.2] that for fixed n, $H_n(r)$ is a polynomial function of r of degree $(n - 1)^2$. For instance, $H_2(r) = r + 1$. Now clearly such a matrix M can be considered as an $\mathbf{N}$-solution to a certain LHD-system $\mathbf{E}$ (with $n^2 + 1$ variables and $2n$ equations). Moreover, $\mathbf{E}$ possesses a $\mathbf{P}$-solution, as exhibited by a matrix of all 1's. Combining Proposition 7.1 with Proposition 5.2 yields the fact that for $r \in \mathbf{P}$, $(-1)^{n-1} H_n(-r)$ is equal to he number of $n \times n$ matrices of *positive* integers summing to r in every row and column.

There readily follows:

$$H_n(-1) = H_n(-2) = \cdots = H_n(-n+1) = 0,$$
$$H_n(r) = (-1)^{n-1} H_n(-n-r).$$

8. Reciprocal Domains

We have already mentioned in Section 7c the possibility of generalizing Proposition 7.1 so that Proposition 5.1 becomes a special case, for *any* labeling ω. Now the labeling ω determines whether $z_{ij} = \sigma(y_j) - \sigma(y_i)$ is allowed to equal 0, for each pair $y_i < y_j$ in P. This suggests that in an LHD-system $\mathbf{E}(\mathbf{x})$ we should specify which variables x_i must be positive and which are allowed to take the value 0.

Looking at it from the point of view of the Ehrhart–Macdonald law of reciprocity (Proposition 6.1), we wish to compare functions $j_T(\mathscr{P}, n)$ and $i_T(\mathscr{P}, n)$ defined as follows. Let $\mathscr{P}$ be a d-dimensional rational polyhedron imbedded in some Euclidean space (so the vertices of $\mathscr{P}$ have rational coordinates) such that $|\mathscr{P}|$ is homeomorphic to a manifold (possibly with boundary). Let S denote the set of all (closed) boundary facets of $\mathscr{P}$ (i.e., $(d-1)$-faces contained in $\partial\mathscr{P}$), and let T be any subset of S. Define

$$\mathscr{B} = \bigcup_{\mathscr{F} \in T} \mathscr{F}, \qquad \mathscr{B}' = \bigcup_{\mathscr{F} \in S-T} \mathscr{F}.$$

For $n \in \mathbf{P}$, define $j_T(\mathscr{P}, n)$ (resp. $i_T(\mathscr{P}, n)$) to be the number of rational points $\boldsymbol{\alpha} \in \mathscr{P} - \mathscr{B}$ (resp. $\boldsymbol{\alpha} \in \mathscr{P} - \mathscr{B}'$) such that $n\boldsymbol{\alpha}$ has integer coordinates. Thus $i_T(\mathscr{P}, n) = j_{S-T}(\mathscr{P}, n)$. Ehrhart calls the two sets $\mathscr{P} - \mathscr{B}$ and $\mathscr{P} - \mathscr{B}'$ *reciprocal domains*. If $T = \varnothing$, then $j_T(\mathscr{P}, n) = j(\mathscr{P}, n)$ and $i_T(\mathscr{P}, n) = i(\mathscr{P}, n)$ as defined in Section 6.

It is easily seen that for $n \in \mathbf{P}$, $j_T(\mathscr{P}, n)$ and $i_T(\mathscr{P}, n)$ satisfy a linear homogeneous difference equation with constant coefficients, so $j_T(\mathscr{P}, n)$ and $i_T(\mathscr{P}, n)$ can be defined as usual for all $n \in \mathbf{Z}$. In [8, Proposition 6.6], Ehrhart deduces from what in this paper is Proposition 6.1 that $j_T(\mathscr{P}, n) = (-1)^d i_T(\mathscr{P}, -n)$ when the set $\mathscr{B}$ is a "normal polytope," as defined in [8]. In [11] Ehrhart explains that he inadvertently omitted the hypothesis that $\mathscr{B}$ is normal. In [11] Ehrhart dispenses with the hypothesis of normality when $d = 3$ by introducing the concept of "multiple points." The following example shows why $\mathscr{B}$ cannot be arbitrary.

EXAMPLE 8.1. Let $\mathscr{P}$ be the 3-polytope (polyhedron) with vertices $\alpha = (0, 0, 0)$, $\beta = (1, 0, 0)$, $\gamma = (0, 1, 0)$, $\delta = (1, 1, 0)$, $\epsilon = (0, 0, 1)$. $\mathscr{P}$ is a (skew) pyramid with square base $\overline{\alpha\beta\gamma\delta}$ and with the five facets $\mathscr{F}_1 = \overline{\alpha\beta\gamma\delta}$, $\mathscr{F}_2 = \overline{\alpha\beta\epsilon}$, $\mathscr{F}_3 = \overline{\gamma\delta\epsilon}$, $\mathscr{F}_4 = \overline{\alpha\gamma\epsilon}$, $\mathscr{F}_5 = \overline{\beta\delta\epsilon}$ (an overbar denotes "convex hull"). Let $T = \{\mathscr{F}_4, \mathscr{F}_5\}$. Then it is not hard to compute that

$$
\begin{aligned}
j_T(\mathscr{P}, n) &= \frac{1}{6}(2n^3 + 3n^2 - 5n), \\
i_T(\mathscr{P}, n) &= \frac{1}{6}(2n^3 - 3n^2 - 5n + 6).
\end{aligned}
\tag{17}
$$

Hence $j_T(\mathscr{P}, n) \neq (-1)^d\, i_T(\mathscr{P}, -n)$.

As mentioned previously, a reciprocity theorem does hold for the reciprocal domains $\mathscr{P} - \mathscr{B}$ and $\mathscr{P} - \mathscr{B}'$ under suitable additional hypotheses.

PROPOSITION 8.2. *Let $\mathscr{P}$ be a rational d-dimensional cell complex imbedded in some Euclidean space such that $|\mathscr{P}|$ is homeomorphic to a d-manifold (possibly with boundary). Let T be a subset of the set of boundary facets of $\mathscr{P}$, and let $\mathscr{B} = \bigcup_{\mathscr{F}\in T} \mathscr{F}$. Suppose $|\mathscr{B}|$ is homeomorphic to a $(d - 1)$-manifold (possibly with boundary). Then*

$$j_T(\mathscr{P}, n) = (-1)^d\, i_T(\mathscr{P}, -n). \tag{18}$$

Moreover,

$$j_T(\mathscr{P}, 0) = \chi(\mathscr{P}) - \chi(\mathscr{B}) = (-1)^d\, i_T(\mathscr{P}, 0).$$

Proof. We have $i_T(\mathscr{P}, n) = i(\mathscr{P}, n) + i(\mathscr{B}, n)$. Hence by Proposition 6.1,

$$
\begin{aligned}
(-1)^d\, i_T(\mathscr{P}, -n) &= (-1)^d\, i(\mathscr{P}, -n) + (-1)^d\, i(\mathscr{B}, -n) \\
&= j(\mathscr{P}, n) - j(\mathscr{B}, n) \\
&= j_T(\mathscr{P}, n).
\end{aligned}
$$

If we put $n = 0$ in the above formula and recall $j(\mathscr{Q}, 0) = \chi(\mathscr{Q})$ from Proposition 6.1, we get $j_T(\mathscr{P}, 0) = \chi(\mathscr{P}) - \chi(\mathscr{B})$, completing the proof. ∎

Just as Proposition 7.1 is a generalization to LHD-systems of Proposition 6.1 in the case when $\mathscr{P}$ is convex, so Proposition 8.2 can be generalized to LHD-systems when $\mathscr{P}$ is convex. Such a generalization, however, is cumbersome to state and difficult to apply in practice, so we shall not give it here. However, there is a special case of Proposition 8.2 which leads to a surprising and elegant generalization to LHD-systems. Namely, we shall generalize the case when $\mathscr{P}$ is convex and $\mathscr{B}$ consists of all points on $\partial\mathscr{P}$ "visible" from some point outside $\mathscr{P}$ but in the affine subspace generated by $\mathscr{P}$.

PROPOSITION 8.3. *Let* $\mathbf{E} = \mathbf{E}(\mathbf{x}_1, \mathbf{x}_2)$ *be an* LHD-*system of corank* κ *in the variables* $\mathbf{x}_1 = (x_{11}, x_{12}, \ldots, x_{1g})$ *and* $\mathbf{x}_2 = (x_{21}, x_{22}, \ldots, x_{2h})$. *Define the generating functions*

$$F_1(\mathbf{X}) = F_1(\mathbf{X}_1, \mathbf{X}_2) = \sum \mathbf{X}_1^{\gamma_1}\mathbf{X}_2^{\gamma_2},$$

$$F_2(\mathbf{X}) = F_2(\mathbf{X}_1, \mathbf{X}_2) = \sum \mathbf{X}_1^{\delta_1}\mathbf{X}_2^{\delta_2},$$

where (γ_1, γ_2) *runs over all* $\mathbf{Z}$-*solutions* $(\mathbf{x}_1, \mathbf{x}_2) = (\gamma_1, \gamma_2)$ *to* $\mathbf{E}$ *with* $\gamma_1 \geqslant 0$, $\gamma_2 > 0$ *(i.e., every* $\gamma_{1i} \geqslant 0$ *and* $\gamma_{2j} > 0$*), while* (δ_1, δ_2) *runs over all* $\mathbf{Z}$-*solutions* $(\mathbf{x}_1, \mathbf{x}_2) = (\delta_1, \delta_2)$ *with* $\delta_1 > 0$, $\delta_2 \geqslant 0$. *Suppose that* $\mathbf{E}$ *possesses a* $\mathbf{Z}$-*solution* $\alpha = (\alpha_1, \alpha_2)$ *such that* $\alpha_1 > 0$ *and* $\alpha_2 < 0$. *Then* F_1 *and* F_2 *are rational functions of* $\mathbf{X}$ *related by*

$$F_2(\mathbf{X}) = (-1)^{\kappa} F_1(1/\mathbf{X}). \tag{19}$$

Proof. We can assume $g > 0$ and $h > 0$; otherwise we are in the case of Proposition 7.1. The theorem is clearly true if $F_1(\mathbf{X}) = F_2(\mathbf{X}) = 0$. Hence we may suppose that either: (a) there is at least one $\mathbf{Z}$-solution $\gamma = (\gamma_1, \gamma_2)$ with $\gamma_1 \geqslant 0$ and $\gamma_2 > 0$, or (b) there is at least one $\mathbf{Z}$-solution $\delta = (\delta_1, \delta_2)$ with $\delta_1 > 0$ and $\delta_2 \geqslant 0$. Let $\alpha = (\alpha_1, \alpha_2)$ be a $\mathbf{Z}$-solution with $\alpha_1 > 0$ and $\alpha_2 < 0$. In case (a), if we choose $n \in \mathbf{P}$ sufficiently large, then $n\gamma + \alpha$ is a $\mathbf{P}$-solution. Similarly in case (b), for n large $n\delta - \alpha$ is a $\mathbf{P}$-solution. Hence we may assume $\mathbf{E}$ possesses a $\mathbf{P}$-solution, which we denote by $\beta = (\beta_1, \beta_2)$.

We can regard real solutions to $\mathbf{E}$ as lying in $\mathbf{R}^{g+h}$. We now claim that there is a hyperplane H in $\mathbf{R}^{g+h}$ which does not contain the origin $\mathbf{0}$, and which contains both a solution to $\mathbf{E}$ in positive real numbers and a real solution $\epsilon = (\epsilon_1, \epsilon_2)$ with $\epsilon_1 > 0$ and $\epsilon_2 < 0$, or with $\epsilon_1 < 0$ and $\epsilon_2 > 0$. We may assume $\alpha \cdot \beta \neq 0$; otherwise replace α with $n\alpha + \beta$ for n large. Define

$$H = \{\zeta \in \mathbf{R}^{g+h}: \alpha \cdot \zeta = \alpha \cdot \alpha\}, \qquad \text{if } \alpha \cdot \beta > 0,$$
$$H = \{\zeta \in \mathbf{R}^{g+h}: \alpha \cdot \zeta = -\alpha \cdot \alpha\}, \qquad \text{if } \alpha \cdot \beta < 0.$$

If $\alpha \cdot \beta > 0$, then $\mathbf{H}$ contains α and $((\alpha \cdot \alpha)/(\alpha \cdot \beta))\beta$, as desired. If

$\boldsymbol{\alpha} \cdot \boldsymbol{\beta} < 0$, then $\mathbf{H}$ contains $-\boldsymbol{\alpha}$ and $((-\boldsymbol{\alpha} \cdot \boldsymbol{\alpha})/(\boldsymbol{\alpha} \cdot \boldsymbol{\beta}))\boldsymbol{\beta}$, as desired. This establishes the existence of H.

Let $\mathscr{C}$ be the set of all solutions to $\mathbf{E}$ in non-negative real numbers. Hence $\mathscr{C}$ is a convex polyhedral cone in $\mathbf{R}^{g+h}$ whose extreme point is at $\mathbf{0}$. Since $\mathbf{E}$ possesses a $\mathbf{P}$-solution, the (relative) interior $\mathscr{C}^o$ of $\mathscr{C}$ consists of the positive real solutions to $\mathbf{E}$. Moreover, it is not hard to see that the existence of a $\mathbf{P}$-solution implies that $\dim \mathscr{C} = \kappa$ [35, Proposition 5.1].

Define H' to be the affine subspace of $\mathbf{R}^{g+h}$ obtained by intersecting H with all real solutions to $\mathbf{E}$. Since $\mathbf{0} \notin H$, it follows that $\dim H' = \kappa - 1$. Moreover, since H' contains an interior point $\boldsymbol{\beta}$ of $\mathscr{C}$, it follows that $H' \cap \mathscr{C}$ is a nondegenerate cross-section $\mathscr{P}$ of $\mathscr{C}$. Hence $\mathscr{P}$ is a $(\kappa - 1)$-dimensional convex polytope.

Suppose for the sake of definiteness $\boldsymbol{\alpha} \cdot \boldsymbol{\beta} > 0$ (so $\boldsymbol{\alpha} \in H'$); the case $\boldsymbol{\alpha} \cdot \boldsymbol{\beta} < 0$ is handled symmetrically. Consider a ray R from $\boldsymbol{\alpha}$ passing through $\mathscr{P}$. If R intersects $\mathscr{P}^o$, then since $\boldsymbol{\alpha} \in H'$, $\mathscr{P} \subseteq H'$, and $\dim H' = \dim \mathscr{P}$, it follows that R intersects the boundary $\partial\mathscr{P}$ of $\mathscr{P}$ in exactly two points, say $\boldsymbol{\eta} = (\boldsymbol{\eta}_1, \boldsymbol{\eta}_2)$ and $\boldsymbol{\theta} = (\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$, where $\boldsymbol{\eta}$ is chosen to lie between $\boldsymbol{\alpha}$ and $\boldsymbol{\theta}$. Since $\boldsymbol{\alpha}_1 > 0$, $\boldsymbol{\alpha}_2 < 0$, and an element $\boldsymbol{\gamma} = (\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2) \in H'$ belongs to $\mathscr{P}^o$ if and only if $\boldsymbol{\gamma}_1 > 0$ and $\boldsymbol{\gamma}_2 > 0$, it follows that $\boldsymbol{\eta}_1 > 0$, $\boldsymbol{\eta}_2 \geqslant 0$, and some $\eta_{2i} = 0$, while $\boldsymbol{\theta}_1 \geqslant 0$, $\boldsymbol{\theta}_2 > 0$, and some $\theta_{1j} = 0$. If R does not intersect $\mathscr{P}^o$, then it intersects $\mathscr{P}$ in exactly one point, which we denote by both $\boldsymbol{\eta}$ and $\boldsymbol{\theta}$. (R cannot intersect $\partial\mathscr{P}$ in some line segment since this would imply every element of R has some 0 coordinate, contradicting $\boldsymbol{\alpha} \in R$.) Here $\boldsymbol{\theta} \geqslant 0$ (or $\boldsymbol{\eta} \geqslant 0$) and some $\theta_{1j} = 0$, $\eta_{2i} = 0$.

We now come to the crucial argument in the proof. Let $\mathscr{D}$ be a cross-section very near $\boldsymbol{\alpha}$ of the cone $\mathscr{K}$ from $\boldsymbol{\alpha}$ to $\mathscr{P}$ (so $\boldsymbol{\gamma} \in \mathscr{K}$ if and only if $\boldsymbol{\gamma}$ is on some ray from $\boldsymbol{\alpha}$ to an element of $\mathscr{P}$). Since $\mathscr{P}$ is convex and $\dim H' = \dim \mathscr{P} = \kappa - 1$, $\mathscr{D}$ is homeomorphic to a $(\kappa - 2)$-cell (solid sphere of dimension $\kappa - 2$). If $\boldsymbol{\gamma} \in \mathscr{D}$, define $P_1(\boldsymbol{\gamma}) = \boldsymbol{\eta}$ and $P_2(\boldsymbol{\gamma}) = \boldsymbol{\theta}$, where $\boldsymbol{\eta}$ and $\boldsymbol{\theta}$ are defined as above with respect to the ray R from $\boldsymbol{\alpha}$ passing through $\boldsymbol{\gamma}$. Hence P_1 establishes a homeomorphism between $\mathscr{D}$ and all elements $\boldsymbol{\eta} \in \mathscr{P}$ such that some $\eta_{2i} = 0$, while P_2 establishes a homeomorphism between $\mathscr{D}$ and all elements $\theta \in \mathscr{P}$ such that some $\theta_{1i} = 0$. Thus we have proved:

(*) The set of points $\boldsymbol{\delta} \in \mathscr{P}$ such that some $\delta_{2i} = 0$ (resp. some $\delta_{1i} = 0$) is homeomorphic to a $(\kappa - 2)$-cell D_1 (resp. D_2). Moreover, $D_1 \cup D_2 = \partial\mathscr{P}$ and $D_1 \cap D_2 = \partial D_1 = \partial D_2$.

If S is a subset of $\mathscr{P}$, let $F(S, \mathbf{X}) = \sum \mathbf{X}^{\epsilon}$, where ϵ ranges over all $\mathbf{N}$-solutions to $\mathbf{E}$ such that the line in $\mathbf{R}^{g+h}$ between $\mathbf{0}$ and ϵ passes through S (by convention $\epsilon = \mathbf{0}$ is included). It follows from (*) that

$$\begin{aligned} F_1(\mathbf{X}) &= F(\mathscr{P}, \mathbf{X}) - F(\mathscr{D}_1, \mathbf{X}), \\ F_2(\mathbf{X}) &= F(\mathscr{P}, \mathbf{X}) - F(\mathscr{D}_2, \mathbf{X}). \end{aligned} \tag{20}$$

We claim that

$$F(\mathscr{D}_1, \mathbf{X}) = \sum (-1)^{\kappa - \dim \mathscr{F}} F(\mathscr{F}, \mathbf{X}), \tag{21}$$

where $\mathscr{F}$ runs over all faces of $\mathscr{P}$ contained in $\mathscr{D}_1$ but not contained entirely within $\partial\mathscr{D}_1$. To prove (21), first note that if $\epsilon \neq \mathbf{0}$ is an $\mathbf{N}$-solution to $\mathbf{E}$ such that the line in $\mathbf{R}^{g+h}$ between $\mathbf{0}$ and ϵ does not pass through $\mathscr{D}_1$, then $\mathbf{X}^{\epsilon}$ appears on neither side of (21) (since each $\mathscr{F} \subseteq \mathscr{D}_1$). Hence we need to prove that if ϵ is an $\mathbf{N}$-solution to $\mathbf{E}$ such that the line in $\mathbf{R}^{g+h}$ between $\mathbf{0}$ and ϵ passes through $\mathscr{D}_1$ (say at λ), then the coefficient of $\mathbf{X}^{\epsilon}$ in the right-hand side of (21) is 1. Now if $\epsilon \neq \mathbf{0}$, the coefficient of $\mathbf{X}^{\epsilon}$ in the right-hand side of (21) is just

$$f_{\kappa-2}^{o}(\lambda) - f_{\kappa-3}^{o}(\lambda) + \cdots + (-1)^{\kappa-2} f_0^{o}(\lambda), \tag{22}$$

where $f_i^{o}(\lambda)$ denotes the number of i-faces of $\mathscr{D}_1$ containing λ and not contained entirely within $\partial\mathscr{D}_1$. A straightforward argument involving Euler characteristics shows that since $\mathscr{D}_1$ is a cell, the sum (22) indeed is equal to 1 (cf., e.g., [25, Lemma 3.4]). Finally, if $\epsilon = \mathbf{0}$, then the coefficient of $\mathbf{X}^{\mathbf{0}}$ (i.e., the constant term) in the right-hand side of (21) is just

$$f_{\kappa-2}^{o} - f_{\kappa-3}^{o} + \cdots + (-1)^{\kappa-2} f_0^{o}, \tag{23}$$

where f_i^{o} denotes the total number of i-faces of $\mathscr{D}_1$ not contained in $\partial\mathscr{D}_1$. Again by an Euler characteristic argument (also included in [35, Lemma 3.4]), this sum is 1.

Now it follows from Proposition 7.1 (more precisely, from the remark following its proof) that for any face $\mathscr{F}$ of $\mathscr{P}$,

$$(-1)^{1+\dim \mathscr{F}} F(\mathscr{F}, 1/\mathbf{X}) = F(\mathscr{F}^{o}, \mathbf{X}) - 1,$$

where $\mathscr{F}^{o}$ denotes the interior of $\mathscr{F}$ (the term -1 appears because of our convention that $\epsilon = \mathbf{0}$ is always included in any $F(S, \mathbf{X})$). Hence from (21),

$$F(\mathscr{D}_1, 1/\mathbf{X}) = \sum (-1)^{\kappa - \dim \mathscr{F}} [(-1)^{1+\dim \mathscr{F}} (F(\mathscr{F}^{o}, \mathbf{X}) - 1)],$$

so from (20),

$$F_1(1/\mathbf{X}) = (-1)^\kappa [(F(\mathscr{P}^o \mathbf{X}) - 1) + \sum (F(\mathscr{F}^o, \mathbf{X}) - 1)]. \tag{24}$$

But the expression in brackets in (24) is equal to $\sum \mathbf{X}^\epsilon$, where ϵ ranges over all $\mathbf{N}$-solutions to $\mathbf{E}$ such that $\epsilon_1 > 0$, i.e., $\sum \mathbf{X}^\epsilon = F_2(\mathbf{X})$. Thus the proof follows from (24). ∎

At this point it is natural to ask to what extent the converse to Proposition 8.3 holds. There are two extreme possibilities: (a) Eq. (19) holds whenever $\mathbf{E}$ possesses a $\mathbf{P}$-solution, and (b) Eq. (19) holds if and only if $\mathbf{E}$ possesses a $\mathbf{Z}$-solution $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2)$ with $\boldsymbol{\alpha}_1 > 0$ and $\boldsymbol{\alpha}_2 < 0$. We shall give examples to show that neither of these possibilities holds.

Example 8.4. In view of Example 8.1, it is not surprising that Proposition 8.3 fails if we merely assume that $\mathbf{E}$ possesses a $\mathbf{P}$-solution. In fact, we shall use Example 8.1 to construct an LHD-system $\mathbf{E}(\mathbf{x}_1, \mathbf{x}_2)$ possessing a $\mathbf{P}$-solution but for which (19) fails. Let $\mathscr{P}$ be the polyhedron of Example 8.1. $\mathscr{P}$ is determined by five inequalities corresponding to its five facets as follows (the coordinate system is given by (x, y, z)):

$$\begin{array}{lrcl}
\mathscr{F}_1: & & z & \geqslant 0 \\
\mathscr{F}_2: & y & & \geqslant 0 \\
\mathscr{F}_3: & y + & z & \leqslant 1 \\
\mathscr{F}_4: & x & & \geqslant 0 \\
\mathscr{F}_5: & x & + z & \leqslant 1
\end{array}$$

As in Section 7c, introduce slack variables w_3 and w_5 and a "scale factor variable" t. Then $\mathscr{P}$ corresponds to the LHD-system $\mathbf{E}$ given by:

$$t - y - z - w_3 = 0$$

$$t - x - z - w_5 = 0.$$

There is a one-to-one correspondence between elements $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ of $\mathscr{P}$ such that $n\boldsymbol{\alpha}$ has integer coordinates and $\mathbf{N}$-solutions to $\mathbf{E}$ with $t = n$, viz., $\boldsymbol{\alpha}$ corresponds to $t = n$, $x = n\alpha_1$, $y = n\alpha_2$, $z = n\alpha_3$, $w_3 = n(1 - \alpha_2 - \alpha_3)$, $w_5 = n(1 - \alpha_1 - \alpha_3)$. An element of $\mathscr{P}$ enumerated by $j_T(\mathscr{P}, n)$ (resp. $i_T(\mathscr{P}, n)$) corresponds to an $\mathbf{N}$-solution of $\mathbf{E}$ with $x > 0$ and $w_5 > 0$ (resp. $z > 0$, $y > 0$, $w_3 > 0$). Hence take $\mathbf{x}_1 = (x, w_5, t)$ and $\mathbf{x}_2 = (y, z, w_3)$. (It makes no difference whether we include t with $\mathbf{x}_1$ or $\mathbf{x}_2$, since we automatically have $t > 0$ in all solutions enumerated by $F_1(\mathbf{X})$ and $F_2(\mathbf{X})$.) The fact that $\mathscr{P}$ is a counterexample to Ehrhart's "reciprocal domain theorem" implies that (19) fails for $\mathbf{E}(\mathbf{x}_1, \mathbf{x}_2)$, or more generally that $F_2(\mathbf{X}) - (-1)^\kappa F_1(1/\mathbf{X})$ is not even a constant. Indeed, it can be shown that

$$F_2(\mathbf{X}) - (-1)^\kappa F_1(1/\mathbf{X}) = TZ/(1 - TZ)$$

(where T, Z are the indeterminates corresponding to the variables t, z of $\mathbf{E}$).

Example 8.5. Here we shall show that the converse to Proposition 8.3 fails, i.e., we shall construct an LHD-system $\mathbf{E}(\mathbf{x}_1, \mathbf{x}_2)$ satisfying (19) although it possesses a $\mathbf{P}$-solution $\boldsymbol{\beta}$ but *no* solution $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2)$ with $\boldsymbol{\alpha}_1 > 0$ and $\boldsymbol{\alpha}_2 < 0$.

First consider the LHD-system $\mathbf{E}(\mathbf{x}_1, \mathbf{x}_2)$, where $\mathbf{x}_1 = (x_{11}, x_{12})$ and $\mathbf{x}_2 = (x_{21}, x_{22})$, as follows:

$$\begin{aligned} x_{11} - x_{21} \qquad &= 0, \\ x_{11} - x_{12} - x_{22} &= 0. \end{aligned}$$

We claim (19) holds for this system, although clearly there does not exist an $\mathbf{N}$-solution $\boldsymbol{\alpha}$ with $\boldsymbol{\alpha}_1 > 0$ and $\boldsymbol{\alpha}_2 < 0$ (even $\alpha_{11} > 0$ and $\alpha_{21} < 0$). Now an $\mathbf{N}$-solution $\boldsymbol{\beta} = (\boldsymbol{\beta}_1, \boldsymbol{\beta}_2)$ satisfies $\boldsymbol{\beta}_1 \geqslant 0$ and $\boldsymbol{\beta}_2 > 0$ if and only if $\beta_{11} > 0$, $\beta_{12} \geqslant 0$, $\beta_{21} > 0$, $\beta_{22} > 0$. Similarly, an $\mathbf{N}$-solution $\boldsymbol{\gamma} = (\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2)$ satisfies $\boldsymbol{\gamma}_1 > 0$ and $\gamma_2 \geqslant 0$ if and only if $\gamma_{11} \geqslant 0$, $\gamma_{12} > 0$, $\gamma_{21} \geqslant 0$, $\gamma_{22} \geqslant 0$. Now $(x_{11}, x_{12}, x_{21}, x_{22}) = (-1, 1, -1, -2)$ is a $\mathbf{Z}$-solution. Hence (19) holds since $F_1(\mathbf{X})$ and $F_2(\mathbf{X})$ are equal to generating functions to which Proposition 8.3 applies.

We would like to give a less trivial example, i.e., one which cannot be deduced from Proposition 8.3.

Example 8.6. The proof of Proposition 8.3 implies that an example of the type we seek can be obtained by considering a convex d-dimensional polytope $\mathscr{P}$ and a union $\mathscr{B}$ of some of its facets such that $\mathscr{B}$ is not homeomorphic to a $(d-1)$-cell, yet (22) and (23) hold for all $\boldsymbol{\lambda} \in \mathscr{B}$. Now (22) and (23) hold if $\mathscr{B}$ is a disjoint union $\mathscr{B}_1 \cup \mathscr{B}_2$, where $\mathscr{B}_1$ consists of four squares arranged cyclically in $\mathbf{R}^3$ to form a cylinder, while $\mathscr{B}_2$ is a single square. Such a cellular complex $\mathscr{B}$ exists on the boundary of the polyhedron $\mathscr{P}$ in $\mathbf{R}^3$ with the following twelve vertices:

$$\begin{array}{llll} \gamma_1 = (0,0,0), & \gamma_2 = (1,0,0), & \gamma_3 = (0,0,3), & \gamma_4 = (1,0,3) \\ \gamma_5 = (0,3,0), & \gamma_6 = (1,3,0), & \gamma_7 = (0,3,3), & \gamma_8 = (1,3,3) \\ \gamma_9 = (2,1,1), & \gamma_{10} = (2,1,2), & \gamma_{11} = (2,2,1), & \gamma_{12} = (2,2,2). \end{array}$$

$\mathscr{P}$ then has the following ten facets (writing $\overline{i_1, i_2, \ldots, i_j}$ as short for $\gamma_{i_1}\gamma_{i_2} \cdots \gamma_{i_j}$, an overbar denoting as usual "convex hull"):

$$\begin{array}{lll} \mathscr{F}_1 = \overline{1,3,5,7} & : & x \geqslant 0 \\ \mathscr{F}_2 = \overline{1,2,3,4} & : & y \geqslant 0 \\ \mathscr{F}_3 = \overline{1,2,5,6} & : & z \geqslant 0 \\ \mathscr{F}_4 = \overline{3,4,7,8} & : & z \leqslant 3 \\ \mathscr{F}_5 = \overline{5,6,7,8} & : & y \leqslant 3 \\ \mathscr{F}_6 = \overline{6,8,11,12} & : & x + y \leqslant 4 \\ \mathscr{F}_7 = \overline{2,4,9,10} & : & x - y \leqslant 1 \\ \mathscr{F}_8 = \overline{2,6,9,11} & : & x - z \leqslant 1 \\ \mathscr{F}_9 = \overline{4,8,10,12} & : & x + z \leqslant 4 \\ \mathscr{F}_{10} = \overline{9,10,11,12} & : & x \leqslant 2 \end{array}$$

Thus, as in Example 8.4, this corresponds to the LHD-system $\mathbf{E}$ given by:

$$\begin{array}{rl} 3t \qquad\qquad - z - w_4 &= 0 \\ 3t \qquad - y \qquad - w_5 &= 0 \\ 4t - x - y \qquad - w_6 &= 0 \\ t - x + y \qquad - w_7 &= 0 \\ t - x + z \qquad - w_8 &= 0 \\ 4t - x - z \qquad - w_9 &= 0 \\ 2t - x \qquad\qquad - w_{10} &= 0. \end{array}$$

Let $\mathscr{B}_1 = \mathscr{F}_2 \cup \mathscr{F}_3 \cup \mathscr{F}_4 \cup \mathscr{F}_5$, $\mathscr{B}_2 = \mathscr{F}_{10}$. Then $\mathscr{B}_1$ and $\mathscr{B}_2$ have the desired topological properties. The facets $\mathscr{F}_2$, $\mathscr{F}_3$, $\mathscr{F}_4$, $\mathscr{F}_5$, and $\mathscr{F}_{10}$ correspond to the variables y, z, w_4, w_5, w_{10}, respectively. Hence if we put $\mathbf{x}_1 = (t, y, z, w_4, w_5, w_{10})$ and $\mathbf{x}_2 = (x, w_6, w_7, w_8, w_9)$, then $F_1(\mathbf{X}) = +F_2(1/\mathbf{X})$. However, this fact cannot be deduced from Proposition 8.3 in the manner of Example 8.5, since $\mathscr{B} = \mathscr{B}_1 \cup \mathscr{B}_2$ is not homeomorphic to a cell. In particular, of course, there cannot be a $\mathbf{Z}$-solution to $\mathbf{E}$ with $\mathbf{x}_1 > 0$ and $\mathbf{x}_2 < 0$. This can be easily seen to be the case by subtracting the second equation of $\mathbf{E}$ from the third, giving $t - x + w_5 - w_6 = 0$.

Connection with partially ordered sets. We shall briefly indicate how Proposition 5.1 follows from Proposition 8.3. Let $P = \{y_1, y_2, \ldots, y_p\}$ be a finite partially ordered set of cardinality p and $\omega: P \to \mathbf{p}$ a bijection. Let $\mathbf{E}$ be the LHD-system

$$x_j - x_i - z_{ij} = 0 \qquad (\text{for all } y_i < y_j \text{ in } P).$$

A (P, ω)-partition (as defined in Section 5) is equivalent to an $\mathbf{N}$-solution to $\mathbf{E}$ satisfying $z_{ij} > 0$ whenever $\omega(y_i) > \omega(y_j)$. Hence to obtain a reciprocity theorem for (P, ω)-partitions from Proposition 8.3, we need to show the existence of an $\mathbf{N}$-solution to $\mathbf{E}$ satisfying $x_i > 0$, $x_j > 0$; $z_{ij} > 0$ if $\omega(y_i) < \omega(y_j)$; $z_{ij} < 0$ if $\omega(y_i) > \omega(y_j)$. But there is an obvious solution with this property, viz., $x_i = \omega(y_i)$ and $z_{ij} = \omega(y_j) - \omega(y_i)$. Hence Proposition 8.3 applies, and it can be seen without difficulty that Proposition 5.1 is a consequence.

9. Inhomogeneous Systems

Let us consider two possible ways of generalizing the results of Sections 7 and 8.

G1. Proposition 8.3 gives a reciprocity theorem between $\mathbf{Z}$-solutions $\boldsymbol{\gamma}$ and $\boldsymbol{\delta}$ of an LHD-system $\mathbf{E}$, where some fixed set of coordinates γ_i of $\boldsymbol{\gamma}$ must satisfy $\gamma_i \geqslant 0$ and the other coordinates γ_j must satisfy $\gamma_j > 0$, while the corresponding coordinates of $\boldsymbol{\delta}$ must satisfy $\delta_i > 0$ and $\delta_j \geqslant 0$. Equivalently, $\gamma_i \geqslant 0$ and $\gamma_j \geqslant 1$, while $\delta_i \geqslant 1$ and $\delta_j \geqslant 0$. This suggests we consider $\mathbf{Z}$-solutions $\boldsymbol{\gamma}$ to $\mathbf{E}$ satisfying $\boldsymbol{\gamma} \geqslant \boldsymbol{\beta}$ (i.e., $\gamma_i \geqslant \beta_i$ for all i) for some arbitrary vector $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_s)$ of integers β_i.

G2. Proposition 7.1 suggests generalizing our results to *inhomogeneous* linear equations

$$\mathbf{E}(\mathbf{x}) = \boldsymbol{\alpha}, \tag{25}$$

where $\mathbf{E}$ is a system of homogeneous linear forms and $\boldsymbol{\alpha}$ is a fixed vector of integers.

We claim that generalization G2 includes G1. For solving the system $\mathbf{E}(\mathbf{x}) = \mathbf{0}$ in integers $\mathbf{x} \geqslant \boldsymbol{\beta}$ is equivalent to solving the system $\mathbf{E}(\mathbf{y}) = -\mathbf{E}(\boldsymbol{\beta})$ in integers $\mathbf{y} \geqslant \mathbf{0}$ $(\mathbf{x} = \mathbf{y} + \boldsymbol{\beta})$. Hence we shall only consider generalization G2 in what follows. (To be strictly accurate, G2 does not include G1 when G1 has a variable which does not actually appear in any of the equations. This degenerate case offers no difficulties, however, so it does not hurt us to ignore it.)

What should be the correct "reciprocal system" to the system $\mathbf{E}(\mathbf{x}) = \boldsymbol{\alpha}$, $\mathbf{x} \geqslant 0$? The most natural guess is $\mathbf{E}(\mathbf{x}) = \boldsymbol{\alpha}$, $\mathbf{x} > \mathbf{0}$, but this is incorrect. By considering simple examples (such as Proposition 8.3), one is led to take the reciprocal system to be $\mathbf{E}(\mathbf{x}) = -\boldsymbol{\alpha}$, $\mathbf{x} > 0$. With these ideas in mind, we make the following definitions.

A *linear inhomogeneous diophantine system*, called for short an LID-system, is a pair $(\mathbf{E}, \boldsymbol{\alpha})$, where $\mathbf{E} = \mathbf{E}(\mathbf{x})$ is a set of p homogeneous linear forms in s variables $\mathbf{x}$ with integer coefficients, and where $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_p)$ is a vector of p integers. A $\mathbf{Z}$-*solution* (respectively, $\mathbf{N}$-*solution*, $\mathbf{P}$-*solution*) to $(\mathbf{E}, \boldsymbol{\alpha})$ is just a $\mathbf{Z}$-solution (respectively, $\mathbf{N}$-solution, $\mathbf{P}$-solution) to the system of equations $\mathbf{E}(\mathbf{x}) = \boldsymbol{\alpha}$. Clearly, when studying solutions to $(\mathbf{E}, \boldsymbol{\alpha})$, there is no loss of generality in assuming the equations of $\mathbf{E}$ are linearly independent. We then call $(\mathbf{E}, \boldsymbol{\alpha})$ an *ILID-system*, short for "independent linear inhomogeneous diophantine system." In an ILID-system, the corank $\kappa = \kappa(\mathbf{E})$ is equal to $s - p$, where s is the number of variables and p the number of equations.

Given an LID-system $(\mathbf{E}, \boldsymbol{\alpha})$, define the generating functions $F(\mathbf{E}, \boldsymbol{\alpha}; \mathbf{X})$ and $\bar{F}(\mathbf{E}, \boldsymbol{\alpha}; \mathbf{X})$ by

$$\begin{aligned} F(\mathbf{E}, \boldsymbol{\alpha}; \mathbf{X}) &= \sum_{\boldsymbol{\gamma}} \mathbf{X}^{\boldsymbol{\gamma}}, \\ \bar{F}(\mathbf{E}, \boldsymbol{\alpha}; \mathbf{X}) &= \sum_{\boldsymbol{\delta}} \mathbf{X}^{\boldsymbol{\delta}}, \end{aligned} \tag{26}$$

where $\boldsymbol{\gamma}$ ranges over all $\mathbf{N}$-solutions to $(\mathbf{E}, \boldsymbol{\alpha})$ and $\boldsymbol{\delta}$ over all $\mathbf{P}$-solutions to $(\mathbf{E}, -\boldsymbol{\alpha})$. If there is no possibility of confusion, we will write simply

$$\begin{aligned} F(\mathbf{X}) &= F(\mathbf{E}, \boldsymbol{\alpha}; \mathbf{X}), \\ \bar{F}(\mathbf{X}) &= \bar{F}(\mathbf{E}, \boldsymbol{\alpha}; \mathbf{X}). \end{aligned}$$

Observe that a **P**-solution of the system $\mathbf{E}(\mathbf{x}) = -\boldsymbol{\alpha}$ is equivalent to an **N**-solution to the system $\mathbf{E}(\mathbf{1} + \mathbf{y}) = -\boldsymbol{\alpha}$, where $\mathbf{x} = \mathbf{1} + \mathbf{y}$ (**1** denotes a vector of s ones). There follows the fundamental formula:

$$\bar{\mathbf{F}}(\mathbf{E}, \boldsymbol{\alpha}; \mathbf{X}) = \mathbf{X}^{\mathbf{1}} F(\mathbf{E}, \bar{\boldsymbol{\alpha}}; \mathbf{X}), \tag{27}$$

where

$$\bar{\boldsymbol{\alpha}} = -\mathbf{E}(\mathbf{1}) - \boldsymbol{\alpha}.$$

As usual, there is little difficulty in seeing that the generating functions (27) represent rational functions. We want to determine conditions under which the LID-system $(\mathbf{E}, \boldsymbol{\alpha})$ satisfies

$$\bar{F}(\mathbf{E}, \boldsymbol{\alpha}; \mathbf{X}) = (-1)^{\kappa} F(\mathbf{E}, \boldsymbol{\alpha}; 1/\mathbf{X}), \tag{28}$$

or in our briefer notation,

$$\bar{F}(\mathbf{X}) = (-1)^{\kappa} F(1/\mathbf{X}).$$

We shall say that an LID-system satisfying (28) has the *R-property*.

For instance, using the above notation, Proposition 8.3 can be restated as follows:

PROPOSITION 8.3 (rephrased). *Let* $\mathbf{E}(\mathbf{x})$ *be a system of* p *homogeneous linear forms with integer coefficients in the* s *unknowns* $\mathbf{x} = (x_1, x_2, \ldots, x_s)$, *and let* $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_s)$ *be a vector of* 0's *and* 1's *of length* s. *Suppose* $\mathbf{E}$ *possesses a* **Z**-*solution* γ *satisfying*

$$\gamma_t > 0 \quad \text{if } \beta_t = 0,$$
$$\gamma_t < 0 \quad \text{if } \beta_t = 1.$$

Then the LID-*system* $(\mathbf{E}, -\mathbf{E}(\boldsymbol{\beta}))$ *has the* R-*property, i.e.*,

$$\bar{F}(\mathbf{E}, -\mathbf{E}(\boldsymbol{\beta}); \mathbf{X}) = (-1)^{\kappa} F(\mathbf{E}, -\mathbf{E}(\boldsymbol{\beta}); 1/\mathbf{X}),$$

where κ *is the corank of* $\mathbf{E}$. ∎

Self-reciprocal systems. Let us call an LID-system $(\mathbf{E}, \boldsymbol{\alpha})$ *self-reciprocal* if $\boldsymbol{\alpha} = \bar{\boldsymbol{\alpha}}$, i.e., if $2\boldsymbol{\alpha} = -\mathbf{E}(\mathbf{1})$. The following result is similar in spirit to Corollary 7.2 and follows immediately from (27) and the definition (28) of the R-property.

PROPOSITION 9.1. *Let* $(\mathbf{E}, \boldsymbol{\alpha})$ *be a self-reciprocal* LID-*system of corank* κ *satisfying the* R-*property. Then*

$$\mathbf{X}^{1}F(\mathbf{X}) = (-1)^{\kappa}F(1/\mathbf{X}). \quad \blacksquare$$

It turns out that in the context of ILID-systems $(\mathbf{E}, \boldsymbol{\alpha})$, determinants of certain minors of $\mathbf{E}$ arise naturally. We shall require a simple notation for such determinants, as follows. Suppose the equations $\mathbf{E} = \boldsymbol{\alpha}$ of the ILID-system are given explicitly by

$$\begin{aligned} E_1(\mathbf{x}) &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1s}x_s = \alpha_1 \\ E_2(\mathbf{x}) &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2s}x_s = \alpha_2 \\ &\vdots \\ E_p(\mathbf{x}) &= a_{p1}x_1 + a_{p2}x_2 + \cdots + a_{ps}x_s = \alpha_p . \end{aligned} \tag{29}$$

If $i_1, i_2, \ldots, i_k, j_1, j_2, \ldots, j_k \in \mathbf{s}$, where $0 \leqslant k \leqslant p$, define $[j_1 j_2 \cdots j_k\colon i_1 i_2 \cdots i_k]$ to be the $k \times k$ determinant

$$[j_1 j_2 \cdots j_k\colon i_1 i_2 \cdots i_k] = \begin{vmatrix} a_{j_1 i_1} & a_{j_1 i_2} & \cdots & a_{j_1 i_k} \\ a_{j_2 i_1} & a_{j_2 i_2} & \cdots & a_{j_2 i_k} \\ & & \vdots & \\ a_{j_k i_1} & a_{j_k i_2} & \cdots & a_{j_k i_k} \end{vmatrix}. \tag{30}$$

Thus the rows are indexed by the j_r's and the columns by the i_r's. For instance, $[j\colon i] = a_{ji}$. By convention, when $k = 0$, the determinant (30) equals 1. If $j_1 = 1, j_2 = 2, \ldots, j_k = k$, we abbreviate

$$[12 \cdots k\colon i_1 i_2 \cdots i_k] = [i_1 i_2 \cdots i_k].$$

For instance, $[i] = a_{1i}$.

If in the determinant (30), the rth column is replaced by $(\alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_k})$, we denote the resulting determinant by $[j_1 j_2 \cdots j_k\colon i_1 i_2 \cdots i_{r-1} \hat{i}_r i_{r+1} \cdots i_k]$. Thus we write

$$[12 \cdots k\colon i_1 i_2 \cdots \hat{i}_r \cdots i_k] = [i_1 i_2 \cdots \hat{i}_r \cdots i_k].$$

For instance, $[\hat{i}_1] = \alpha_1$, $[\hat{i}_1 t] = \alpha_1 a_{2t} - \alpha_2 a_{1t}$, etc.

An even more concise notation for determinants will be needed. If $i_1, i_2, \ldots, i_k$ are regarded as fixed, then we write

$$
\begin{aligned}
D_k(m \to j, n \to t) &= [12 \cdots (m-1) j (m+1) \cdots k \colon i_1 i_2 \cdots i_{n-1} t i_{n+1} \cdots i_k], \\
D_k(m \to j, \hat{n}) &= [12 \cdots (m-1) j (m+1) \cdots k \colon i_1 i_2 \cdots i_{n-1} \hat{i}_n i_{n+1} \cdots i_k] \\
D_k(j, t) &= D_k(k \to j, k \to t) \\
\hat{D}_k(j) &= D_k(k \to j, \hat{k}) \\
D_k(n \to t) &= [i_1 i_2 \cdots i_{n-1} t i_{n+1} \cdots i_k] \\
&= D_k(m \to m, n \to t) \quad \text{(for any } m) \\
D_k(\hat{n}) &= [i_1 i_2 \cdots \hat{i}_n \cdots i_k] \\
D_k(t) &= D_k(k \to t) = [i_1 i_2 \cdots i_{k-1} t] \\
\hat{D}_k &= D_k(\hat{k}) \\
D_k &= [i_1 i_2 \cdots i_k] \\
&= D_k(m \to m, n \to i_n) \quad \text{(for any } m, n).
\end{aligned}
$$

We shall require one additional piece of terminology. Given an ILID-system (29), let $i_1, i_2, \ldots, i_{k+1} \in \mathbf{s}$, with $0 \leqslant k < p$. We shall say that the sequence $i_1, i_2, \ldots, i_{k+1}$ is a *pole sequence* (for a reason which will become clear in the proof of Theorem 10.2) if for all $j = 0, 1, \ldots, k$ we have

$$D_j D_{j+1} \left(D_j - \sum_{r=1}^{j} D_j(r \to i_{j+1}) \right) > 0.$$

For $j = 0$, this reduces to $D_1 > 0$. A simple computation shows that the expression $D_j - \sum_{r=1}^{j} D_j(r \to i_{j+1})$ can be expressed as a single $j \times j$ determinant whose (r, t) entry is $a_{ri_t} - a_{ri_{j+1}}$.

We now indicate a method for computing the generating function $F(\mathbf{E}, \boldsymbol{\alpha}; \mathbf{X})$ of (26) explicitly. Although this method has little practical value, it has considerable interest with regard to reciprocity theorems. Suppose $(\mathbf{E}, \boldsymbol{\alpha})$ is given by (29). Introduce new variables $\lambda_1, \lambda_2, \ldots, \lambda_p$. Consider the expression

$$G(\mathbf{E}, \boldsymbol{\alpha}) = \frac{\lambda_1^{\alpha_1} \lambda_2^{\alpha_2} \cdots \lambda_p^{\alpha_p}}{\prod_{t=1}^{s} (1 - \lambda_1^{-a_{1t}} \lambda_2^{-a_{2t}} \cdots \lambda_p^{-a_{pt}} X_t)}. \tag{31}$$

When this expression is formally expanded in a Laurent expansion in the λ_i's in the "natural" way (such an expansion will converge if each

$|\lambda_i| = 1$ and $|X_i| < 1$), it is easily seen by inspection that the term free from λ_i's (i.e., the "constant term") is equal to $F(\mathbf{X}) = F(\mathbf{E}, \boldsymbol{\alpha}; \mathbf{X})$. By continually applying partial fraction decomposition to $G(\mathbf{E}, \boldsymbol{\alpha})$ in a systematic way, we will be able to obtain an explicit expression for $F(\mathbf{X})$. Substituting $-\mathbf{E}(1) - \boldsymbol{\alpha}$ for $\boldsymbol{\alpha}$, we will also have by (27) an explicit expression for $\mathbf{X}^1\bar{F}(\mathbf{X})$. By comparing these two expressions, we will be able to read off certain conditions when $\bar{F}(\mathbf{X}) = (-1)^\kappa F(1/\mathbf{X})$.

Let $K(\lambda_1, \lambda_2, ..., \lambda_p)$ be any (formal) power series in the variables $\lambda_1, \lambda_2, ..., \lambda_p$ with rational (or even complex) *exponents*, and with complex coefficients. Let $\Omega_j\, K(\lambda_1, \lambda_2, ..., \lambda_p)$ denote those terms of $K(\lambda_1, \lambda_2, ..., \lambda_p)$ free from λ_j. (The notation is similar to MacMahon [20, Section 350].) Hence

$$F(\mathbf{X}) = \Omega_p\, \Omega_{p-1} \cdots \Omega_1\, G(\mathbf{E}, \boldsymbol{\alpha}; \lambda_1, ..., \lambda_p). \tag{32}$$

We shall evaluate this expression for $F(\mathbf{X})$ *iteratively*, by first applying Ω_1, then Ω_2, etc., up to Ω_p. We shall use the *residue theorem* of complex variable theory (e.g., [1, Theorem 19]) as a convenient bookkeeping device for carrying out this evaluation in a systematic manner. Thus, since we are interested in the Laurent expansion of $G(\mathbf{E}, \boldsymbol{\alpha})$ convergent for $|\lambda_j| = 1$, $|X_j| < 1$, we have that $\Omega_1\, G(\mathbf{E}, \boldsymbol{\alpha})$ is equal to the sum of the residues of $G(\mathbf{E}, \boldsymbol{\alpha})/\lambda_1$ inside the disk $|\lambda_1| \leqslant 1$, provided $G(\mathbf{E}, \boldsymbol{\alpha})/\lambda_1$ has only isolated singularities (as a function of λ_1) in $|\lambda_1| < 1$ and is analytic on $|\lambda_1| = 1$. We then evaluate $\Omega_2(\Omega_1\, G(\mathbf{E}, \boldsymbol{\alpha}))$ in a similar manner, etc. There are two technical complications which will arise. Firstly, in the course of carrying out this evaluation, we will encounter expressions involving λ_j's raised to nonintegral rational powers. In this case, the singularities need not be isolated—they can be branch points—so the residue theorem is inapplicable. We can get around this difficulty by the following trick. Clearly for any power series $K(\lambda)$ with rational (or even complex) exponents, $\Omega K(\lambda) = \Omega K(\lambda^\mu)$ for any rational (or complex) $\mu \neq 0$. Hence at any stage of our evaluation we are free to replace any λ_j by $\lambda_j^{\mu_j}$, for any rational $\mu_j \neq 0$. Thus, for instance, a term $\lambda_4^{2/3}$ can be changed to λ_4^2 by replacing λ_4 with λ_4^3 throughout. Such replacements will be made without further justification, and we shall not bother to distinguish $K(\lambda)$ from $K(\lambda^\mu)$. The second technical complication arises from the fact that we may encounter isolated singularities on the circles $|\lambda_j| = 1$ of integration. To avoid this difficulty, we replace the circle $|\lambda_j| = 1$ with $|\lambda_j| = 1 - \epsilon_j$, for very small $\epsilon_j > 0$. If the ϵ_j are suitably chosen (the exact condition will appear in the proofs of Lemma 9.2 and Theorem 10.2), the effect will be that we can simply ignore the singularities on the circles $|\lambda_j| = 1$. Loosely speaking, we will pick the ϵ_j so that $0 < \epsilon_p \ll \epsilon_{p-1} \ll \cdots \ll \epsilon_1 \ll 1$, where "$\ll$" means "much smaller than." We emphasize that despite the use of the residue theorem, this technique for evaluating $\mathbf{F}(X)$ is purely algebraic—the residue theorem applied to rational functions K is an *algebraic* result concerning the partial fraction decomposition of K.

Let $(\mathbf{E}, \boldsymbol{\alpha})$ be the ILID-system given by (29). If $0 \leqslant k \leqslant p$, define the rational function $I_k(\mathbf{E}, \boldsymbol{\alpha})$ in the variables $\lambda_{k+1}, \lambda_{k+2}, \ldots, \lambda_p$, $\mathbf{X}$ by

$$I_k(\mathbf{E}, \boldsymbol{\alpha}) = \frac{1}{\lambda_{k+1}} \left(\frac{1}{2\pi i}\right)^k \oint \left(\cdots \left(\oint \left(\oint G(\mathbf{E}, \boldsymbol{\alpha}) \frac{d\lambda_1}{\lambda_1}\right) \frac{d\lambda_2}{\lambda_2}\right) \cdots \frac{d\lambda_k}{\lambda_k}\right),$$

where we take $0 < |X_1| = |X_2| = \cdots = |X_s| = C \ll 1$, and where the jth integral from the inside is taken over one counterclockwise revolution of the circle $|\lambda_j| = 1 - \epsilon_j$. Suppose that for $0 \leqslant k \leqslant p$, the two rational functions $I_k(\mathbf{E}, \boldsymbol{\alpha})$ and $I_k(\mathbf{E}, \bar{\boldsymbol{\alpha}})$ (where $\bar{\boldsymbol{\alpha}} = -\mathbf{E}(1) - \boldsymbol{\alpha}$) have zero residues at $\lambda_{k+1} = 0$. We then say that $(\mathbf{E}, \boldsymbol{\alpha})$ has the *I-property* (short for "integration property").

We now come to the critical lemma involving the evaluation of the expression (32). Although the proof is straightforward, it is rather messy.

LEMMA 9.2. *Let* $(\mathbf{E}, \boldsymbol{\alpha})$ *be an* ILID-*system. If* $(\mathbf{E}, \boldsymbol{\alpha})$ *has the* I-*property, then it has the* R-*property.*

Proof. If we set $\lambda_{p+1} = 1$, then the iterated integral $I_p(\mathbf{E}, \boldsymbol{\alpha})$ is equal to $\Omega_p(\cdots \Omega_2(\Omega_1 G(\mathbf{E}, \boldsymbol{\alpha})) \cdots) = F(\mathbf{X})$. Hence by the residue theorem and the assumption that each $I_k(\mathbf{E}, \boldsymbol{\alpha})$ has zero residue at $\lambda_{k+1} = 0$, $F(\mathbf{X})$ may be computed as follows. Assume each $|X_t| < 1$. Pick a pole at λ_1, satisfying $0 < |\lambda_1| < 1 - \epsilon_1$, of T_0/λ_1, where $T_0 = G(\mathbf{E}, \boldsymbol{\alpha})(T_0/\lambda_1$ is regarded as a function of $\lambda_1)$. Let $T_1 = T_1(\mathbf{X}, \boldsymbol{\alpha}; \lambda_2, \lambda_3, \ldots, \lambda_p)$ be the residue. Pick a pole of T_1/λ_2 (regarded as a function of λ_2) satisfying $0 < |\lambda_2| < 1 - \epsilon_2$, and let $T_2 = T_2(\mathbf{X}, \boldsymbol{\alpha}; \lambda_3, \lambda_4, \ldots, \lambda_p)$ be the residue. Continue until we obtain $T_p = T_p(\mathbf{X}, \boldsymbol{\alpha})$. Then $F(\mathbf{E}, \boldsymbol{\alpha}; \mathbf{X})$ will be the sum of all such terms $T_p(\mathbf{X})$ obtained in this way.

We can now explain one of the conditions on the ϵ_i's. Each ϵ_i should be small enough so that in carrying out the above integration process, any pole of T_{i-1}/λ_i satisfying $|\lambda_i| < 1$ also satisfies $|\lambda_i| < 1 - \epsilon_i$. This insures that we are indeed dealing with the Laurent expansion convergent for $|\lambda_i| = 1$, $|X_j| < 1$, and that we will encounter no singularities on any of the circles $|\lambda_i| = 1 - \epsilon_i$ of integration. An additional condition on the ϵ_i's will be introduced in the proof of Theorem 10.2; for the present proof this condition is not needed.

Let us compute $T_p(\mathbf{X})$ for a given sequence of poles $\lambda_1, \lambda_2, \ldots, \lambda_p$, such that $0 < |\lambda_j| < 1 - \epsilon_j$ for each $j = 1, 2, \ldots, p$. At a pole of T_0/λ_1 satisfying $0 < |\lambda_1| < 1 - \epsilon_1$, we have from (31),

$$1 = \lambda_1^{-a_{1i_1}} \lambda_2^{-a_{2i_1}} \cdots \lambda_p^{-a_{pi_1}} X_{i_1},$$

for some i_1 such that $a_{1i_1} = [i_1] > 0$. Equivalently,

$$\lambda_1 = (\lambda_2^{-a_{2i_1}} \cdots \lambda_p^{-a_{pi_1}} X_{i_1})^{1/a_{1i_1}}, \tag{33}$$

for some fixed choice of the a_{1i_1}th root. We now replace each λ_j with $\lambda_j^{a_{1i_1}}$, $j = 2, \ldots, p$, as discussed earlier.

The residue at the pole (33), regardless of which a_{1i_1}th root was chosen to define λ_1, is easily computed by brute force to be

$$T_1 = \frac{X_{i_1}^{\alpha_1/[i_1]} \prod_{j=2}^{p} \lambda_j^{\hat{D}_2(j)}}{[i_1] \prod_{\substack{t=1 \\ t \neq i_1}}^{s} (1 - \lambda_2^{-D_2(2,t)} \lambda_3^{-D_2(3,t)} \cdots \lambda_p^{-D_2(p,t)} X_{i_1}^{-[t]/[i_1]} X_t)}. \tag{34}$$

Here i_1 is regarded as fixed (the value of i_2 has yet to be determined but is irrelevant in the above formula, since i_2 appears in $\hat{D}_2(j)$ and $D_2(j, t)$ only as a dummy variable).

We are now ready to find $T_2 = (1/2\pi i) \oint (T_1/\lambda_2)\, d\lambda_2 = \Omega_2 T_1$. Set $Y_t = X_{i_1}^{-[t]/[i_1]} X_t$. Then T_1 has exactly the same form as T_0 (except for the factor $X_{i_1}^{\alpha_1/[i_1]}/[i_1]$) with $\lambda_1, \lambda_2, \ldots, \lambda_p$ replaced with $\lambda_2, \lambda_3, \ldots, \lambda_p$, with X_t replaced with Y_t, and with a_{ji} replaced with $b_{ji} = D_2(j, i)$. Hence if i_2 is such that the roots λ_2 of

$$1 - \lambda_2^{-[i_1 i_2]} \lambda_3^{-[13:i_1 i_2]} \cdots \lambda_p^{-[1p:i_1 i_2]} Y_{i_2} = 0 \tag{35}$$

satisfy $|\lambda_2| < 1 - \epsilon_2$, then these values of λ_2 define poles of H_1/λ_2 inside the circle $|\lambda_2| = 1 - \epsilon_2$. (In the proof of Theorem 10.2, we will be concerned with determining exactly when the roots λ_2 of (35) satisfy $|\lambda_2| < 1 - \epsilon_2$. For the present proof this does not concern us.) Assume therefore that i_1, i_2 is such that the roots λ_2 of (35) satisfy

$|\lambda_2| < 1 - \epsilon_2$. Then from (34), the residue of H_1/λ_2 at the pole (35), for any choice of the $[i_1 i_2]$th root, is:

$$T_2 = \frac{X_{i_1}^{\alpha_1/[i_1]} Y_{i_2}^{\hat{D}_2/[i_1 i_2]} \prod_{j=3}^{p} \lambda_j^{\hat{E}_2(j)}}{D_1 D_2 \prod_{\substack{t=1 \\ t \neq i_1, i_2}}^{s} (1 - \lambda_3^{-c_{3t}} \lambda_4^{-c_{4t}} \cdots \lambda_p^{-c_{pt}} Y_{i_2}^{-D_2(2\to t)/D_2} Y_t)}, \tag{36}$$

where

$$c_{jt} = \begin{vmatrix} b_{2i_2} & b_{2t} \\ b_{ji_2} & b_{jt} \end{vmatrix},$$

$$\hat{E}_2(j) = \begin{vmatrix} b_{2i_2} & \hat{D}_2(2) \\ b_{ji_2} & \hat{D}_2(j) \end{vmatrix}.$$

A straightforward brute force computation shows that $c_{jt} = D_1 \cdot D_3(j, t)$ and $\hat{E}_2(j) = D_1 \cdot \hat{D}_3(j)$. Since D_1 is a constant, it can be absorbed into the λ_j's, by replacing λ_j with $\lambda_j^{D_1}$. Moreover, the exponent of X_{i_1} in the numerator of T_2 is

$$D_1^{-1}\alpha_1 - D_1^{-1} D_2^{-1} [i_2]\, \hat{D}_2 = D_2^{-1} \cdot D_2(\hat{1}) \qquad \text{(by computation)}.$$

Hence

$$T_2 = \frac{X_{i_1}^{D_2(\hat{1})/D_2} X_{i_2}^{D_2(\hat{2})/D_2} \prod_{j=3}^{p} \lambda_j^{\hat{D}_3(j)}}{D_1 D_2 \prod_{\substack{t=1 \\ t \neq i_1, i_2}}^{s} (1 - \lambda_3^{-D_3(3,t)} \lambda_4^{D_3(4,t)} \cdots \lambda_p^{D_3(p,t)} \cdot X_t X_{i_1}^{-D_2(1\to t)/D_2} X_{i_2}^{-D_2(2\to t)/D_2})}. \tag{37}$$

Equations (34) and (37) suggest the conjecture that for a sequence of poles (none at 0) indexed by $i_1, i_2, \ldots, i_k$ $(k \leqslant p)$, we have

$$T_k = \frac{\left(\prod_{r=1}^{k} X_{i_r}^{D_k(\hat{r})/D_k}\right) \prod_{j=k+1}^{p} \lambda_j^{\hat{D}_{k+1}(j)}}{D_1 D_2 \cdots D_k \prod_{t=1}^{s}{}' \left(1 - \left(\prod_{j=k+1}^{p} \lambda_j^{-D_{k+1}(j,t)}\right) \cdot X_t \cdot \prod_{r=1}^{k} X_{i_r}^{-D_k(r\to t)/D_k}\right)}, \tag{38}$$

where $\prod'$ indicates that the terms $t = i_1, i_2, \ldots, i_k$ are omitted. We prove (38) by induction on k. We have already established it for $k =$

0, 1, 2. Assume true for some $k < p$. Thus T_k has the form

$$T_k = \frac{\left(\prod_{r=1}^{k} X_{i_r}^{D_k(\hat{r})/D_k}\right) \prod_{j=k+1}^{p} \lambda_j^{\hat{D}_{k+1}(j)}}{D_1 D_2 \cdots D_k \prod_{t=1}^{s}{}' \left(1 - Z_t \prod_{j=k+1}^{p} \lambda_j^{-D_{k+1}(j,t)}\right)}.$$

This has the same form as T_0 (except for a constant factor) with the set $\lambda_1, \ldots, \lambda_p$ replaced with $\lambda_{k+1}, \ldots, \lambda_p$, with a_{jt} replaced with $D_{k+1}(j, t)$, with α_j replaced with $\hat{D}_{k+1}(j)$, and with X_t replaced with

$$Z_t = X_t \prod_{r=1}^{k} X_{i_r}^{-D_k(r \to t)/D_k}.$$

Hence by making these substitutions into T_1 we get

$$T_{k+1} = \frac{\left(\prod_{r=1}^{k} X_{i_r}^{D_k(\hat{r})/D_k}\right) Z_{i_{k+1}}^{\hat{D}_{k+1}/D_{k+1}(i_{k+1})} \prod_{j=k+2}^{p} \lambda_j^{\hat{M}_{k+1}(j)}}{D_1 D_2 \cdots D_k D_{k+1}(i_{k+1}) \prod_{t=1}^{s}{}' \left(1 - Z_t Z_{i_{k+1}}^{-D_{k+1}(t)/D_{k+1}(i_{k+1})} \prod_{j=k+2}^{p} \lambda_j^{-M(j,t)}\right)}, \tag{39}$$

where

$$M(j, t) = \begin{vmatrix} D_{k+1}(i_{k+1}) & D_{k+1}(t) \\ D_{k+1}(j, i_{k+1}) & D_{k+1}(j, t) \end{vmatrix},$$

$$\hat{M}_{k+1}(j) = \begin{vmatrix} D_{k+1}(i_{k+1}) & \hat{D}_{k+1}(k+1) \\ D_{k+1}(j, i_{k+1}) & \hat{D}_{k+1}(j) \end{vmatrix}.$$

We must show that the above expression (39) for T_{k+1} is equal to the expression obtained when $k + 1$ is substituted for k in (38). Let us check each case in turn.

1. *Constant in the denominator.* We need to show $D_{k+1}(i_{k+1}) = D_{k+1}$, which is immediate from the definition of $D_{k+1}(t)$.

2. *Exponent of λ_j in the numerator.* We need to show

$$\hat{M}_{k+1}(j) = A_j \hat{D}_{k+2}(j), \tag{40}$$

where A_j is an integer depending only on j and $i_1, i_2, \ldots, i_k$, so it can be absorbed into λ_j. Now a classical identity in the theory of determinants [22, Section 149] states that if $D = |d_{ij}|$ is a determinant (where i and j range over not necessarily equal index sets), then

$$D \frac{\partial^2 D}{\partial d_{hr}\, \partial d_{ls}} = \frac{\partial D}{\partial d_{hr}} \frac{\partial D}{\partial d_{ls}} - \frac{\partial D}{\partial d_{hs}} \frac{\partial D}{\partial d_{lr}}. \tag{41}$$

If we take $D = \hat{D}_{k+2}(j)$, $d_{hr} = \alpha_j$, $d_{ls} = a_{k+1,i_{k+1}}$, $d_{hs} = a_{ji_{k+1}}$, $d_{lr} = \alpha_{k+1}$, then (40) results with $A_j = D_k$ (independent of j).

3. *Exponent of* λ_j *in the* tth *factor of the product in the denominator.* This amounts to proving that

$$M(j, t) = A_j \cdot D_{k+2}(j, t), \tag{42}$$

where $A_j = D_k$ as above, so it can be absorbed into λ_j. Equation (42) is also a special case of (41), obtained by taking $D = D_{k+2}(j, t)$, $h = k + 1$, $r = i_{k+1}$, $l = j$, and $s = t$.

4. *Exponent of* X_{i_j} $(1 \leqslant j \leqslant k)$ *in the* tth *factor of the product in the denominator.* We need to show (using $D_{k+1}(i_{k+1}) = D_{k+1}$)

$$D_{k+1}^{-1} D_{k+1}(j \to t) = -D_k^{-1} D_k(j \to i_{k+1})\, D_{k+1}^{-1} D_{k+1}(t) + D_k^{-1} D_k(j \to t). \tag{43}$$

Equivalently, we need to show

$$D_{k+1}(j \to t)\, D_k = \begin{vmatrix} D_{k+1} & D_{k+1}(t) \\ D_k(j \to i_{k+1}) & D_k(j \to t) \end{vmatrix}. \tag{44}$$

This identity is once again a special case of (41), obtained by taking D to be the $(k+2) \times (k+2)$ determinant with rows indexed by $1, 2, \ldots, k+2$ and columns indexed by $i_1, i_2, \ldots, i_{k+1}, t$, defined as follows: The subdeterminant of D consisting of the first $k+1$ rows and columns is D_{k+1}. The last row of D consists of all 0's except for a 1 in the jth column (i.e., $d_{k+2,i_j} = 1$). The last column of D consists of $a_{1t}, a_{2t}, \ldots, a_{k+1,t}, 0$. Thus

$$D = -D_{k+1}(j \to t)$$

(from expanding by the last row). Let $h = k + 1$, $r = i_{k+1}$, $l = k + 2$, $s = t$ in (40). Then a straightforward computation shows that (40) reduces to (44).

5. *Exponent of* $X_{i_{k+1}}$ *in the* tth *factor of the product in the denominator.* Here we must prove

$$D_{k+1}^{-1} D_{k+1}(k + 1 \to t) = D_{k+1}(i_{k+1})^{-1}\, D_{k+1}(t),$$

which follows immediately from

$$D_{k+1}(i_{k+1}) = D_{k+1} \qquad \text{and} \qquad D_{k+1}(k+1 \to t) = D_{k+1}(t).$$

6. *Exponent of X_t in the* tth *factor of the product in the denominator.* Clearly, this exponent is equal to 1 in both (38) (with k replaced with $k+1$) and (39).

7. *Exponent of X_{i_j} $(1 \leqslant j \leqslant k)$ in the numerator.* The identity we wish to prove is

$$D_{k+1}^{-1} D_{k+1}(\hat{r}) = -D_k^{-1} D_k(r \to i_{k+1})\, D_{k+1}^{-1} \hat{D}_{k+1} + D_k^{-1} D_k(\hat{r}).$$

This is precisely the same as (43) after replacing a_{it} in (43) with α_i.

8. *Exponent of $X_{i_{k+1}}$ in the numerator.* We want to prove

$$D_{k+1}^{-1} D_{k+1}(\hat{\imath}_{k+1}) = D_{k+1}(i_{k+1})^{-1}\, \hat{D}_{k+1},$$

which is immediate from $D_{k+1} = D_{k+1}(i_{k+1})$ and $\hat{D}_{k+1} = D_{k+1}(\hat{\imath}_{k+1})$.

This completes the proof of (38).

Hence by our assumption that each $I_k(\mathbf{E}, \boldsymbol{\alpha})$ has zero residue at $\lambda_{k+1} = 0$, $F(\mathbf{X})$ will be a sum of expressions $T_p = T_p(\mathbf{E}, \boldsymbol{\alpha}; \mathbf{X})$ given by (38) (with $k = p$). Now since a $\mathbf{P}$-solution $\mathbf{y}$ to $\mathbf{E}(\mathbf{y}) = -\boldsymbol{\alpha}$ corresponds to a $\mathbf{N}$-solution $\mathbf{x} = \mathbf{y} - \mathbf{1}$ to $\mathbf{E}(\mathbf{x}) = -\mathbf{E}(\mathbf{1}) - \boldsymbol{\alpha}$, we can obtain $\bar{F}(\mathbf{X})$ simply by substituting $-\mathbf{E}(\mathbf{1}) - \boldsymbol{\alpha}$ for $\boldsymbol{\alpha}$ in $G(\mathbf{E}, \boldsymbol{\alpha})$, applying the integration procedure, and multiplying by $\mathbf{X}^{\mathbf{1}}$ (cf. (27)). Since the D_k's depend only on $\mathbf{E}$ and not $\boldsymbol{\alpha}$, the same T_p's arise when evaluating $\bar{F}(\mathbf{X})$ as when evaluating $F(\mathbf{X})$, with $\boldsymbol{\alpha}$ replaced with $-E(\mathbf{1}) - \boldsymbol{\alpha}$. Since by assumption $I_k(\mathbf{E}, \bar{\boldsymbol{\alpha}})$ has zero residue at $\lambda_{k+1} = 0$, only the terms $\mathbf{X}^{\mathbf{1}} \cdot T_p(\mathbf{E}, -\mathbf{E}(\mathbf{1}) - \boldsymbol{\alpha}; \mathbf{X})$ will arise in evaluating $\bar{F}(\mathbf{X})$. It therefore suffices to prove that

$$\mathbf{X}^{\mathbf{1}} \cdot T_p(\mathbf{E}, -\mathbf{E}(\mathbf{1}) - \boldsymbol{\alpha}; \mathbf{X}) = (-1)^{\kappa}\, T_p(\mathbf{E}, \boldsymbol{\alpha}; 1/\mathbf{X}). \tag{45}$$

Now by (38),

$$T_p(\mathbf{E}, \boldsymbol{\alpha}; \mathbf{X}) = \frac{K \prod_{r=1}^{p} X_{i_r}^{D(\hat{r})/D}}{\prod_{t=1}^{s}{}' \left(1 - X_t \prod_{r=1}^{p} X_{i_r}^{-D(r\to t)/D}\right)},$$

where K is a constant, and where $D(\hat{r})$, $D(r \to t)$, and D are short for

$D_p(\hat{r})$, $D_p(r \to t)$, and D_p, respectively. Clearly

$$T_p(\mathbf{E}, \boldsymbol{\alpha}; 1/\mathbf{X}) = \frac{(-1)^{s-p} K \left(\prod_{r=1}^{p} X_{i_r}^{-D(\hat{r})/D} \right) \prod_{t=1}^{s}{}' \left(X_t \prod_{r=1}^{p} X_{i_r}^{-D(r \to t)/D} \right)}{\prod_{t=1}^{s}{}' \left(1 - X_t \prod_{r=1}^{p} X_{i_r}^{-D(r \to t)/D} \right)}. \tag{46}$$

Since $D(r \to i_k) = 0$ if $k \neq r$ while $D(r \to i_r) = D$, we have that

$$\prod_{t=1}^{s}{}' \left(X_t \prod_{r=1}^{p} X_{i_r}^{-D(r \to t)/D} \right) = \left(\prod_{t=1}^{s}{}' X_t \right) \left(\prod_{r=1}^{p} X_{i_r} \prod_{t=1}^{s} X_{i_r}^{-D(r \to t)/D} \right)$$

$$= \mathbf{X}^1 \prod_{r=1}^{p} X_{i_r}^{-\Sigma}, \tag{47}$$

where

$$\Sigma = D^{-1} \sum_{t=1}^{s} D(r \to t).$$

By the linearity property of determinants, $\sum$ is just D^{-1} times the determinant obtained from D by replacing the rth column of D by $\mathbf{E}(1)$. Then $D(\hat{r}) + \sum$ is just the determinant obtained from D by replacing the rth column by $\mathbf{E}(1) + \boldsymbol{\alpha}$. Hence (45) follows from (46) and (47), so the proof is complete. ▮

10. The Monster Reciprocity Theorem

Lemma 9.2 gives a sufficient condition for an LID-system $(\mathbf{E}, \boldsymbol{\alpha})$ to have the R-property, but this condition is not very illuminating. (This is why Lemma 9.2 is not called a "theorem.") In this section we shall analyze the proof of Lemma 9.2 in order to obtain more explicit conditions for the R-property to hold.

The main theorem of this section, and indeed of the whole paper, states that an ILID-system $(\mathbf{E}, \boldsymbol{\alpha})$ has the R-property if certain linear combinations of its equations (considered as ILID-systems with one equation) have the R-property. This result is called the "Monster Reciprocity Theorem." We then complement the Monster Reciprocity Theorem by giving a necessary and sufficient condition, and also a simpler sufficient but not necessary condition, for a single equation to have the R-property.

We first need some additional terminology. Suppose $(\mathbf{E}, \boldsymbol{\alpha})$ is an ILID-system given by (29). Let $i_1 \in \mathbf{s}$ satisfy $D_1 \neq 0$ (i.e., $a_{1i_1} \neq 0$). Define a new ILID-system $(\mathbf{E}', \boldsymbol{\alpha}')$, called the $\langle i_1 \rangle$*-eliminated system* of $(\mathbf{E}, \boldsymbol{\alpha})$, as follows:

$$\begin{aligned} E_1'(\mathbf{x}) &= -a_{2i_1}E_1(\mathbf{x}) + a_{1i_1}E_2(\mathbf{x}) = \alpha_1' = -a_{2i_1}\alpha_1 + a_{1i_1}\alpha_2 \\ E_2'(\mathbf{x}) &= -a_{3i_1}E_1(\mathbf{x}) + a_{1i_1}E_3(\mathbf{x}) = \alpha_2' = -a_{3i_1}\alpha_1 + a_{1i_1}\alpha_3 \\ &\vdots \\ E_{p-1}'(\mathbf{x}) &= -a_{pi_1}E_1(\mathbf{x}) + a_{1i_1}E_p(\mathbf{x}) = \alpha_{p-1}' = -a_{pi_1}\alpha_1 + a_{1i_1}\alpha_p \,. \end{aligned}$$

Note that $(\mathbf{E}', \boldsymbol{\alpha}')$ is essentially the ILID-system obtained from $(\mathbf{E}, \boldsymbol{\alpha})$ by Gaussian elimination of the variable x_{i_1}. Next choose $i_2 \in \mathbf{s}$ such that $D_2 \neq 0$ and define the $\langle i_1 i_2 \rangle$*-eliminated system* $(\mathbf{E}'', \boldsymbol{\alpha}'')$ of $(\mathbf{E}, \boldsymbol{\alpha})$ to be the $\langle i_2 \rangle$-eliminated system of $(\mathbf{E}', \boldsymbol{\alpha}')$. Thus

$$E_t''(\mathbf{x}) = -D_2(t+2, i_2)\, E_1'(\mathbf{x}) + D_2 \cdot E_{t+1}'(\mathbf{x})$$

and

$$\alpha_t'' = -D_2(t+2, i_2)\, \alpha_1' + D_2 \cdot \alpha_{t+1}' \,, \qquad 1 \leqslant t \leqslant p-2.$$

In general, if $i_1, i_2, \ldots, i_k$ is a sequence from $\mathbf{s}$ $(0 \leqslant k < p)$ such that $D_1 \neq 0$, $D_2 \neq 0, \ldots, D_k \neq 0$, define the $\langle i_1 i_2 \cdots i_k \rangle$*-eliminated system* $(\mathbf{E}^{(k)}, \boldsymbol{\alpha}^{(k)})$ of $(\mathbf{E}, \boldsymbol{\alpha})$ to be the $\langle i_k \rangle$-eliminated system of $(\mathbf{E}^{(k-1)}, \boldsymbol{\alpha}^{(k-1)})$. Hence $(\mathbf{E}^{(k)}, \boldsymbol{\alpha}^{(k)})$ is essentially the system obtained from $(\mathbf{E}, \boldsymbol{\alpha})$ by Gaussian elimination of $x_{i_1}, x_{i_2}, \ldots, x_{i_k}$, in succession. Note that $(\mathbf{E}^{(k)}, \boldsymbol{\alpha}^{(k)})$ is an ILID-system of corank $\kappa(\mathbf{E}^{(k)}) = \kappa(\mathbf{E})$, with $s - k$ variables and $p - k$ equations. Finally, we use the convention that $(\mathbf{E}^{(0)}, \boldsymbol{\alpha}^{(0)}) = (\mathbf{E}, \boldsymbol{\alpha})$.

EXAMPLE. Suppose $(\mathbf{E}, \boldsymbol{\alpha})$ is given by

$$\begin{aligned} x_1 + 2x_2 - x_3 - x_4 &= 2, \\ 2x_1 - x_2 - 3x_3 + 2x_4 &= -4, \\ -x_1 - x_2 + 2x_3 - 3x_4 &= 1 \end{aligned}$$

Then the $\langle 1 \rangle$-eliminated system of $(\mathbf{E}, \boldsymbol{\alpha})$ is given by

$$\begin{aligned} -5x_2 - x_3 + 4x_4 &= -8 \\ x_2 + x_3 - 4x_4 &= 3, \end{aligned}$$

while the $\langle 1, 2\rangle$-eliminated system of $(\mathbf{E}, \boldsymbol{\alpha})$ is given by

$$-4x_3 + 16x_4 = -7.$$

PROPOSITION 10.1. *Let* $(\mathbf{E}, \boldsymbol{\alpha})$ *be an* ILID-*system given by* (29). *Let* $i_1, i_2, \dots, i_k$ $(0 \leqslant k < p)$ *be a sequence from* s *satisfying* $D_1 \neq 0$, $D_2 \neq 0, \dots, D_k \neq 0$. *Then the* $\langle i_1 i_2 \cdots i_k\rangle$-*eliminated system of* $(\mathbf{E}, \boldsymbol{\alpha})$ *is given by the equations*

$$C_k \sum_t D_{k+1}(j + k, t)\, x_t = C_k \cdot \hat{D}_{k+1}(j + k), \qquad j = 1, 2, \dots, p - k,$$

where

$$C_k = D_1^{2^{k-2}} D_2^{2^{k-3}} \cdots D_{k-1}$$

(*by convention* $C_0 = C_1 = 1$).

Proof. The proof is by induction on k. The proof reduces immediately to proving two identities involving determinants. These identities are just (40) and (42), so the proof follows. ∎

By Proposition 10.1, if we divide the equations of the $\langle i_1 i_2 \cdots i_k\rangle$-eliminated system $(\mathbf{E}^{(k)}, \boldsymbol{\alpha}^{(k)})$ by $C_k \neq 0$, we obtain the ILID-system

$$\sum_t D_{k+1}(j + k, t)\, x_t = \hat{D}_{k+1}(j + k), \qquad j = 1, 2, \dots, p - k. \tag{48}$$

We call (48) the *reduced* $\langle i_1 i_2 \cdots i_k\rangle$-eliminated system of $(\mathbf{E}, \boldsymbol{\alpha})$.

MONSTER RECIPROCITY THEOREM 10.2. *Let* $(\mathbf{E}, \boldsymbol{\alpha})$ *be an* ILID-*system with s variables and p equations. Suppose that for every pole sequence* $i_1, i_2, \dots, i_k$ *from* s $(0 \leqslant k < p)$ (*as defined in Section* 9), *the first equation*

$$\sum_t D_{k+1}(t)\, x_t = \hat{D}_{k+1} \tag{49}$$

of the reduced $\langle i_1 i_2 \cdots i_k\rangle$-*eliminated system of* $(\mathbf{E}, \boldsymbol{\alpha})$ *has the* R-*property* (*as an* ILID-*system with one equation*). *Then* $(\mathbf{E}, \boldsymbol{\alpha})$ *has the* R-*property.*

Remark. The statement that (49) has the R-property is of course equivalent to the statement that the equation $C_k \cdot \sum_t D_{k+1}(t) x_t = C_k \cdot \hat{D}_{k+1}$ has the R-property, since these two equations have the same $\mathbf{Z}$-solutions (note $C_k \neq 0$). This observation combined with Proposition 10.1 is useful for computational purposes, since it shows that (49) can be obtained via Gaussian elimination.

Theorem 10.2 (whose proof will be given later) suggests that we investigate when an ILID-system consisting of a single equation $E(\mathbf{x}) = a_1x_1 + a_2x_2 + \cdots + a_sx_s = \alpha$ has the R-property. The next result completely settles this question.

PROPOSITION 10.3. *Let* $(\mathbf{E}, \boldsymbol{\alpha})$ *be an* LID-*system consisting of the single equation*

$$E(\mathbf{x}) = a_1x_1 + a_2x_2 + \cdots + a_sx_s = \alpha.$$

The following three conditions are equivalent:

(i) *The rational functions*

$$H(\mathbf{E}, \boldsymbol{\alpha}; \mathbf{X}) = \lambda^{\alpha-1}/(1 - \lambda^{-a_1}X_1)(1 - \lambda^{-a_2}X_2) \cdots (1 - \lambda^{-a_s}X_s)$$

and

$$H(\mathbf{E}, \bar{\boldsymbol{\alpha}}; \mathbf{X}) = \lambda^{\bar{\alpha}-1}/(1 - \lambda^{-a_1}X_1)(1 - \lambda^{-a_2}X_2) \cdots (1 - \lambda^{-a_s}X_s)$$

have zero residues at $\lambda = 0$. *Here*

$$\bar{\alpha} = -E(1) - \alpha = -a_1 - a_2 - \cdots - a_s - \alpha.$$

(ii) *The following two conditions are both satisfied*:

(a) *There does not exist a* $\mathbf{Z}$-*solution* $\boldsymbol{\beta}$ *to* $(\mathbf{E}, \boldsymbol{\alpha})$ *such that*

$$\beta_t < 0 \quad \text{if} \quad a_t > 0 \quad \text{and} \quad \beta_t \geqslant 0 \quad \text{if} \quad a_t < 0. \tag{50}$$

(b) *There does not exist a* $\mathbf{Z}$-*solution* $\boldsymbol{\gamma}$ *to* $(\mathbf{E}, \boldsymbol{\alpha})$ *such that*

$$\gamma_t \geqslant 0 \quad \text{if} \quad a_t > 0 \quad \text{and} \quad \gamma_t < 0 \quad \text{if} \quad a_t < 0. \tag{51}$$

(iii) $(\mathbf{E}, \boldsymbol{\alpha})$ *has the* R-*property*.

Proof. Consider the expression (31) for $G(\mathbf{E}, \boldsymbol{\alpha})$. Writing λ for λ_1, we see that $H(\mathbf{E}, \boldsymbol{\alpha}; \mathbf{X}) = G(\mathbf{E}, \boldsymbol{\alpha}; \mathbf{X})/\lambda$. By the proof of Lemma 9.2, we can write

$$\begin{aligned} F(\mathbf{X}) &= Q(\mathbf{X}) + \operatorname*{Res}_{\lambda=0} H(\mathbf{E}, \boldsymbol{\alpha}; \mathbf{X}) \\ \bar{F}(\mathbf{X}) &= \bar{Q}(\mathbf{X}) + \operatorname*{Res}_{\lambda=0} H(\mathbf{E}, \bar{\boldsymbol{\alpha}}; \mathbf{X})\, \mathbf{X}^{\mathbf{1}}, \end{aligned} \tag{52}$$

where $Q(\mathbf{X})$ and $\bar{Q}(\mathbf{X})$ satisfy

$$Q(\mathbf{X}) = (-1)^{\kappa}\bar{Q}(1/\mathbf{X}). \tag{53}$$

Let us calculate $\operatorname{Res}_{\lambda=0} H(\mathbf{E}, \boldsymbol{\alpha}; \mathbf{X})$ explicitly. Let k be the number of t's for which $a_t > 0$. Let

$$A = \alpha + \sum_{t:a_t>0} a_t .$$

A straightforward computation gives

$$G(\mathbf{E}, \boldsymbol{\alpha}; \mathbf{X}) = (-1)^k \Big/ \Big(\prod_{t:a_t>0} X_t\Big) \lambda^{-A} \Big(\prod_{t:a_t<0} (1 - \lambda^{-a_t} X_t)\Big)\Big(\prod_{t:a_t>0} (1 - \lambda^{a_t} X_t^{-1})\Big). \tag{54}$$

Hence $\operatorname{Res}_{\lambda=0} H(\mathbf{E}, \boldsymbol{\alpha}; \mathbf{X})$ is equal to the coefficient of λ^0 (i.e., the constant term) in the Laurent expansion of (54) about $\lambda = 0$ (convergent in a deleted neighborhood of 0). Clearly this coefficient is equal to

$$(-1)^k \Big(\prod_{t:a_t>0} X_t^{-1}\Big) \sum \Big(\prod_{t:a_t<0} X_t^{B_t}\Big)\Big(\prod_{t:a_t>0} X_t^{-B_t}\Big), \tag{55}$$

where the sum ranges over all sequences $B_1, B_2, \ldots, B_s$ of non-negative integers satisfying

$$\sum_{t:a_t<0} a_t B_t - \sum_{t:a_t>0} a_t B_t = A. \tag{56}$$

Now (56) can be rewritten

$$\sum_{t:a_t<0} a_t B_t - \sum_{t:a_t>0} a_t(B_t + 1) = \alpha. \tag{57}$$

Let

$$\begin{aligned} \beta_t &= B_t && \text{if } a_t < 0, \\ \beta_t &= -(B_t + 1) && \text{if } a_t > 0. \end{aligned} \tag{58}$$

By (57) and (58), a sequence B_t of non-negative integers satisfying (57) corresponds to a $\mathbf{Z}$-solution $\boldsymbol{\beta}$ to $(\mathbf{E}, \boldsymbol{\alpha})$ satisfying (50). Hence (55) can be rewritten

$$(-1)^k \sum \Big(\prod_{t:a_t<0} X_t^{\beta_t}\Big)\Big(\prod_{t:a_t>0} X_t^{-1+(\beta_t+1)}\Big) = (-1)^k \sum \mathbf{X}^{\boldsymbol{\beta}},$$

where the sums range over all $\mathbf{Z}$-solutions $\boldsymbol{\beta}$ to $(\mathbf{E}, \boldsymbol{\alpha})$ satisfying (50).

Substituting $-\mathbf{E}(1) - \boldsymbol{\alpha}$ for $\boldsymbol{\alpha}$ in (50), we also get

$$\operatorname*{Res}_{\lambda=0} H(\mathbf{E}, \bar{\boldsymbol{\alpha}}; \mathbf{X})\, \mathbf{X}^{\mathbf{1}} = (-1)^k \sum \mathbf{X}^{-\boldsymbol{\gamma}},$$

where $\boldsymbol{\gamma}$ ranges over all $\mathbf{Z}$-solutions to $(\mathbf{E}, \boldsymbol{\alpha})$ satisfying (51). Thus in view of (52), to complete the proof it suffices to show that there cannot exist a $\mathbf{Z}$-solution $\boldsymbol{\beta}$ to $(\mathbf{E}, \boldsymbol{\alpha})$ satisfying (50) *and* a $\mathbf{Z}$-solution $\boldsymbol{\gamma}$ to $(\mathbf{E}, \boldsymbol{\alpha})$ satisfying (51). But if $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ satisfy (50) and (51) (regardless of whether or not they are solutions to $(\mathbf{E}, \boldsymbol{\alpha})$), then $E(\boldsymbol{\beta}) < E(\boldsymbol{\gamma})$. This completes the proof. ∎

The next result gives a simpler but weaker criterion than Proposition 10.3(ii) for a single equation to have the R-property. We shall need the following notation. If $a_1, a_2, \ldots, a_s$ is a sequence of real numbers, we write $\sum_{t+} a_t$ to denote the sum of all those a_t satisfying $a_t > 0$. Similarly, $\sum_{t-} a_t$ denotes the sum of all those a_t satisfying $a_t < 0$. For instance, if $a_1 = 2$, $a_2 = 0$, $a_3 = -1$, $a_4 = 5$, then $\sum_{t+} a_t = 7$ and $\sum_{t-} a_t = -1$.

PROPOSITION 10.4. *Let* $(\mathbf{E}, \boldsymbol{\alpha})$ *be an* LID-*system consisting of the single equation*

$$E(\mathbf{x}) = a_1 x_1 + a_2 x_2 + \cdots + a_s x_s = \alpha.$$

The following two conditions are equivalent:

(i) *The rational functions* $H(\mathbf{E}, \boldsymbol{\alpha}; \mathbf{X})$ *and* $H(\mathbf{E}, \bar{\boldsymbol{\alpha}}; \mathbf{X})$ *of Proposition* 10.3(i) *have no poles at* $\lambda = 0$.

(ii) $\sum_{t-} a_t < -\alpha < \sum_{t+} a_t$.

If, moreover, either of the two (*equivalent*) *conditions* (i) *or* (ii) *is satisfied, then* $(\mathbf{E}, \boldsymbol{\alpha})$ *has the* R-*property.*

Proof. The equivalence of (i) and (ii) is routine—one merely checks when $\lim_{\lambda\to 0} H(\mathbf{E}, \boldsymbol{\alpha}) = \infty$ and $\lim_{\lambda\to 0} H(\mathbf{E}, \bar{\boldsymbol{\alpha}}) = \infty$. Clearly, condition (i) of the present proposition implies condition (ii) of Proposition 10.3, so the rest of the proof follows from Proposition 10.3. (It is also easy to give a direct proof that Proposition 10.4(ii) implies Proposition 10.3(ii).) ∎

Remark. The proof of Proposition 10.3 leads easily to the following curious result, valid for *any* LID-system with one equation. This result may be regarded as a generalization of Proposition 10.3.

PROPOSITION 10.5. *Let* $(\mathbf{E}, \boldsymbol{\alpha})$ *be an* LID*-system consisting of the single equation*

$$E(\mathbf{x}) = a_1x_1 + a_2x_2 + \cdots + a_sx_s = \alpha.$$

Suppose for simplicity each $a_t \neq 0$. *Let* k *be the number of* t's *such that* $a_t > 0$. *Then*

$$\sum_{\boldsymbol{\delta}} \mathbf{X}^{\boldsymbol{\delta}} - (-1)^k \sum_{\boldsymbol{\beta}} \mathbf{X}^{\boldsymbol{\beta}} + (-1)^s \sum_{\boldsymbol{\epsilon}} \mathbf{X}^{\boldsymbol{\epsilon}} - (-1)^{s+k} \sum_{\boldsymbol{\gamma}} \mathbf{X}^{\boldsymbol{\gamma}} = 0,$$

where $\boldsymbol{\delta}$ *ranges over* $\mathbf{N}$*-solutions to* $(\mathbf{E}, \boldsymbol{\alpha})$, $\boldsymbol{\beta}$ *over all* $\mathbf{Z}$*-solutions satisfying* (50), $\boldsymbol{\epsilon}$ *over all* $\mathbf{Z}$*-solutions satisfying* $\epsilon_t < 0$ *for all* t, *and* $\boldsymbol{\gamma}$ *over all* $\mathbf{Z}$*-solutions satisfying* (51). *Each sum is regarded as a rational function. The sum over* $\boldsymbol{\delta}$ *converges for* $|X_i| < 1$, *over* $\boldsymbol{\epsilon}$ *for* $|X_i| > 1$, *while the sums over* $\boldsymbol{\beta}$ *and* $\boldsymbol{\gamma}$ *are finite.* ▮

We are now ready to prove the Monster Reciprocity Theorem.

Proof of Theorem 10.2. By the proof of Lemma 9.2, $(\mathbf{E}, \boldsymbol{\alpha})$ will have the R-property if for every function

$$T_k = T_k(\mathbf{E}, \boldsymbol{\alpha}, \mathbf{X}, \lambda_{k+1}, \lambda_{k+2}, \ldots, \lambda_p, i_1, i_2, \ldots, i_k)$$

and

$$\bar{T}_k = \bar{T}_k(\mathbf{E}, \bar{\boldsymbol{\alpha}}, \mathbf{X}, \lambda_{k+1}, \lambda_{k+2}, \ldots, \lambda_p, i_1, i_2, \ldots, i_k) \qquad (0 \leqslant k < p)$$

encountered in the proof of Lemma 9.2 (cf. (38)), the residues of T_k/λ_{k+1} and $\bar{T}_k/\lambda_{k+1}$ at $\lambda_{k+1} = 0$ are zero. We need to determine under what conditions on $i_1, i_2, \ldots, i_k$ will the functions T_k and $\bar{T}_k$ be encountered. We prove by induction on k that if we choose the ϵ_i in the proof of Lemma 9.2 so that $0 < \epsilon_p \lll \epsilon_{p-1} \lll \cdots \lll \epsilon_1 < 1$ (where "$\lll$" means "much less than"), then T_k and $\bar{T}_k$ will be encountered if and only if $i_1, i_2, \ldots, i_k$ is a pole sequence (as defined in Section 9).

We begin the induction on $k = 1$. By definition, the sequence i_1 is a pole sequence if and only if $D_1 > 0$, i.e., $a_{1i_1} > 0$. From (33), it is apparent that $|\lambda_1| < 1 - \epsilon_1$ if and only if $D_1 > 0$, since each $|\lambda_i|$ is near 1 and since $0 < |X_{i_1}| = C \lll 1$ by assumption. Hence T_1 and $\bar{T}_1$ will be encountered if and only if $D_1 > 0$, as desired.

Now assume the induction hypothesis for k. Hence we assume that we have a pole sequence $i_1, i_2, \ldots, i_k$ and have encountered T_k

and $\overline{T}_k$. By (38), we want to know the condition on i_{k+1} such that the roots λ_{k+1} of

$$\left(\prod_{j=k+1}^{p} \lambda_j^{-D_{k+1}(j,i_{k+1})}\right) X_{i_{k+1}} \prod_{r=1}^{k} X_{i_r}^{-D_k(r\to i_{k+1})/D_k} = 1$$

satisfy $|\lambda_{k+1}| < 1 - \epsilon_{k+1}$. Set

$$Y = X_{i_{k+1}} \prod_{r=1}^{k} X_{i_r}^{-D_k(r\to i_{k+1})/D_k}.$$

Since we are assuming $0 < |X_1| = |X_2| = \cdots = |X_s| = C \ll 1$, we have $|Y| = C^b$, where

$$b = 1 - \sum_{r=1}^{k} D_k(r \to i_{k+1})/D_k .$$

Case 1. $|Y| \neq 1$. Since each $|\lambda_i|$ is close to 1 and since $0 < C \ll 1$, we see that $|\lambda_{k+1}| < 1 - \epsilon_{k+1}$ if and only if $b/D_{k+1} > 0$. But $b/D_{k+1} > 0$ if and only if

$$D_k D_{k+1}\left(D_k - \sum_{r=1}^{k} D_k(r \to i_{k+1})\right) > 0.$$

Since we are assuming $i_1, i_2, ..., i_k$ is a pole sequence, the above inequality is the exact condition for $i_1, i_2, ..., i_{k+1}$ to be a pole sequence.

Case 2. $|Y| = 1$. In this case, by our assumption that ϵ_{k+1} is much larger than $\epsilon_{k+2}, ..., \epsilon_p$, then the roots λ_{k+1} of

$$\prod_{j=k+1}^{p} \lambda_j^{-D_{k+1}(j,i_{k+1})} = 1$$

will be much closer to 1 in absolute value than they will be to $1 - \epsilon_{k+1}$. Hence $|\lambda_{k+1}| > 1 - \epsilon_{k+1}$, so the poles at λ_{k+1} do not lead to terms T_{k+1} and $\overline{T}_{k+1}$. But $|Y| = 1$ if and only if

$$D_k D_{k+1}\left(D_k - \sum_{r=1}^{k} D_k(r \to i_{k+1})\right) = 0,$$

so $i_1, i_2, ..., i_{k+1}$ is not a pole sequence when $|Y| = 1$.

We have therefore proved that we will encounter terms T_k and $\overline{T}_k$ if and only if $i_1, i_2, ..., i_k$ is a pole sequence. (This explains the terminology "pole sequence.")

Set

$$Y_t = X_t \cdot \prod_{r=1}^{k} X_{i_r}^{-D_k(r\to t)/D_k} \cdot \prod_{j=k+2}^{p} \lambda_j^{-D_{k+1}(j,t)}.$$

Then

$$T_k = K \cdot \lambda_{k+1}^{\hat{D}_{k+1}} \Big/ \prod_{t=1}^{s}{}' (1 - \lambda_{k+1}^{-D_{k+1}(t)} Y_t), \tag{59}$$

where K is nonzero and independent of λ_{k+1}. Similarly

$$\bar{T}_k = \bar{K} \cdot \lambda_{k+1}^{D_{k+1}} \Big/ \prod_{t=1}^{s}{}' (1 - \lambda_{k+1}^{-D_{k+1}(t)} Y_t), \tag{60}$$

where $\hat{D}_{k+1}$ denotes the determinant obtained by replacing α_i with $-E_i(1) - \alpha_i$ in $\hat{D}_{k+1}$. Note that the Y_t's are algebraically independent indeterminates since the X_t's are. Hence by Proposition 10.3, T_k and $\bar{T}_k$ have zero residue at $\lambda_{k+1} = 0$ if (and only if) the equation

$$E(\mathbf{y}) = \sum_t D_{k+1}(t)\, y_t = \hat{D}_{k+1}$$

has the R-property. This completes the proof. ∎

It is possible to give a generalization of Theorem 10.2 analogous to Proposition 10.5, but this generalization is extremely messy and will be omitted.

Example. We give an explicit example for the sake of clarity. Consider the system $(\mathbf{E}, \boldsymbol{\alpha})$ given by

$$3x_1 - x_2 - 2x_3 = \alpha,$$

$$-x_1 + x_2 - x_3 = \beta.$$

Then $D_0 > 0$ (since $D_0 = 1$ by convention) and $[1] = 3 > 0$. Hence by Proposition 10.3 and Theorem 10.2, $(\mathbf{E}, \boldsymbol{\alpha})$ will satisfy the R-property if the following two conditions are met:

(i) There do not exist integers $m_1 < 0$, $m_2 \geqslant 0$, $m_3 \geqslant 0$ such that $3m_1 - m_2 - 2m_3 = \alpha$, and there do not exist integers $n_1 \geqslant 0$, $n_2 < 0$, $n_3 < 0$ such that $3n_1 - n_2 - 2n_3 = \alpha$.

(ii) There do not exist integers $m_2 < 0$, $m_3 \geqslant 0$ such that

$2m_2 - 5m_3 = \alpha + 3\beta$, and there do not exist integers $n_2 \geqslant 0$, $n_3 < 0$ such that $2n_2 - 5n_3 = \alpha + 3\beta$.

A simpler condition for $(\mathbf{E}, \boldsymbol{\alpha})$ to have the R-property is provided by Proposition 10.4 and Theorem 10.2. Namely, $(\mathbf{E}, \boldsymbol{\alpha})$ has the R-property if $-3 < -\alpha < 3$ and $-5 < -\alpha - 3\beta < 2$. These conditions hold precisely for the ten pairs $(\alpha, \beta) = (-2, 1)$, $(-2, 2)$, $(-1, 0)$, $(-1, 1)$, $(0, 0)$, $(0, 1)$, $(1, 0)$, $(1, 1)$, $(2, -1)$, and $(2, 0)$.

Example. We give an example to illuminate the integration process of Lemma 9.2. We take for our ILID-system the following:

$$\begin{aligned} x + y - 2z \qquad &= 0, \\ y - \quad z - w &= 0. \end{aligned}$$

Assume $0 < |X| = |Y| = |Z| = |W| \ll 1$ and that $|\lambda| = 1 - \epsilon_1$, $|\mu| = 1 - \epsilon_2$, where $0 < \epsilon_2 \ll \epsilon_1 \ll 1$. Then

$$\begin{aligned} F(X, Y, Z, W) &= \left(\frac{1}{2\pi i}\right)^2 \oint\oint \frac{d\lambda \cdot d\mu}{\lambda\mu(1 - \lambda^{-1}X)(1 - \lambda^{-1}\mu^{-1}Y)(1 - \lambda^2\mu Z)(1 - \mu W)} \\ &= \frac{1}{2\pi i} \oint \frac{d\mu}{\mu(1 - \mu^{-1}X^{-1}Y)(1 - \mu X^2 Z)(1 - \mu W)} \\ &\quad + \frac{1}{2\pi i} \oint \frac{d\mu}{\mu(1 - \mu X Y^{-1})(1 - \mu^{-1}Y^2 Z)(1 - \mu W)}, \end{aligned}$$

by summing the residues at $\lambda = X$ and $\lambda = \mu^{-1}Y$. Since $|\mu| < 1$ and $|X| = |Y|$, the only residue inside $|\mu| = 1 - \epsilon_2$ occurs in the second integrand when $\mu = Y^2 Z$, yielding

$$F(X, Y, Z, W) = 1/(1 - XYZ)(1 - Y^2 WZ).$$

What if, however, we had chosen X and Y to satisfy $0 < |X| < 1$, $0 < |Y| < 1$, and $|X^{-1}Y| < 1 - \epsilon_2$? We would then get residues at $\mu = X^{-1}Y$ in the first and second integrands. These residues are equal to $1/(1 - XYZ)(1 - X^{-1}YW)$ and $-1/(1 - XYZ)(1 - X^{-1}YW)$, respectively. Hence these two residues cancel out and we achieve the same answer for $F(X, Y, Z, W)$. This must be the case since the generating function defining $F(X, Y, Z, W)$ converges whenever $|X|$, $|Y|$, $|Z|$, $|W| < 1$.

11. Ramifications of the Monster Theorem

In this section we shall consider the following two questions suggested by Theorem 10.2: (a) To what extent is the converse to Theorem 10.2 true? (b) How is Theorem 10.2 related to our previous reciprocity theorems? We begin with three examples.

Example 11.1. Let $(\mathbf{E}, \boldsymbol{\alpha})$ consist of the single equation $x_1 + x_2 = -2$. Then $F(\mathbf{X}) = \bar{F}(\mathbf{X}) = 0$, so $(\mathbf{E}, \boldsymbol{\alpha})$ has the R-property. However, $(\mathbf{E}, \boldsymbol{\alpha})$ does not have the I-property and therefore does not satisfy the conditions of Propositions 10.3 and 10.4.

Example 11.2. Let $(\mathbf{E}, \boldsymbol{\alpha})$ be given by:

$$\begin{aligned} x_1 - x_2 \qquad\qquad &= -1, \\ x_1 \qquad - x_3 - x_4 &= 0 \end{aligned}$$

Then

$$F(\mathbf{X}) = X_2/(1 - X_1X_2X_3)(1 - X_1X_2X_4)$$

and

$$\bar{F}(\mathbf{X}) = X_1^2X_2X_3X_4/(1 - X_1X_2X_3)(1 - X_1X_2X_4),$$

so $(\mathbf{E}, \boldsymbol{\alpha})$ has the R-property. However, the first equation $x_1 - x_2 = -1$ does *not* have the R-property. Hence the converse to Theorem 10.2 is false.

Example 11.3. Let $(\mathbf{E}, \boldsymbol{\alpha})$ consist of the single equation

$$3x_1 - 2x_2 = -4.$$

Then $(\mathbf{E}, \boldsymbol{\alpha})$ satisfies the conditions of Proposition 10.3 but not Proposition 10.4. Hence Proposition 10.3 is strictly stronger than Proposition 10.4.

Although Examples 11.1 and 11.2 show that the I-property or the condition in Theorem 10.2 is not necessary for the R-property, it is possible to give a simple necessary (but by no means sufficient) condition for the R-property to hold.

Proposition 11.4. *Let* $(\mathbf{E}, \boldsymbol{\alpha})$ *be an* LID-*system satisfying the* R-*property. Then either* $F(\mathbf{X}) = \bar{F}(\mathbf{X}) = 0$ *or else the homogeneous system* $\mathbf{E}(\mathbf{x}) = \mathbf{0}$ *possesses a* $\mathbf{P}$-*solution.*

Proof. If one of $F(\mathbf{X})$ or $\bar{F}(\mathbf{X})$ is nonzero, then both are nonzero since $(\mathbf{E}, \boldsymbol{\alpha})$ has the R-property. Thus there is an $\mathbf{N}$-vector $\boldsymbol{\beta}$ and a $\mathbf{P}$-vector $\boldsymbol{\gamma}$ satisfying $\mathbf{E}(\boldsymbol{\beta}) = \boldsymbol{\alpha}$, $\mathbf{E}(\boldsymbol{\gamma}) = -\boldsymbol{\alpha}$. Then $\boldsymbol{\beta} + \boldsymbol{\gamma}$ is a $\mathbf{P}$-solution to $\mathbf{E}(\mathbf{x}) = \mathbf{0}$. ∎

Example 11.1 shows that the possibility $F(\mathbf{X}) = \bar{F}(\mathbf{X}) = 0$ can actually occur.

An LID-system $(\mathbf{E}, \boldsymbol{\alpha})$ with $\boldsymbol{\alpha} = \mathbf{0}$ is just an LHD-system. Hence we should expect some connection between Proposition 7.1 and Theorem 10.2 in the case $\boldsymbol{\alpha} = \mathbf{0}$. Such a connection is provided by the next result.

PROPOSITION 11.5. *Let* $\mathbf{E}$ *be an* LHD-*system consisting of the* p *linearly independent equations* (13). *Let* $\mathbf{0}$ *denote the vector of* s *zeroes. The following five conditions are equivalent.*

(i) $(\mathbf{E}, \mathbf{0})$ *has the* R-*property.*

(ii) *For any pole sequence* $i_1, i_2, \ldots, i_k$ *from* $\mathbf{s}$ $(0 \leqslant k < p)$ *there exist* $t, u \in \mathbf{s}$ *such that* $D_{k+1}(t) < 0$ *and* $D_{k+1}(u) > 0$.

(iii) *Same as condition* (ii), *except that we only require* $D_1 \neq 0$, $D_2 \neq 0, \ldots, D_k \neq 0$, *rather than* $i_1, i_2, \ldots, i_k$ *being a pole sequence.*

(iv) *There is no* $\mathbf{Z}$-*combination* $E'(x) = 0$ *of the equations of* E *such that every coefficient of* E' *is nonnegative and not all coefficients are zero.*

(v) $\mathbf{E}$ *possesses a* $\mathbf{P}$-*solution.*

Proof. (i) ⇔ (v). This is Proposition 7.1.

(iv) ⇔ (v). This is Stiemke's theorem [36], a forerunner of the duality theorems of linear programming.

(ii) ⇔ (i). Consider the single equation

$$\sum_t D_{k+1}(t)\, x_t = 0.$$

This equation will have a $\mathbf{P}$-solution if and only if there are some $t, u \in \mathbf{s}$ such that $D_{k+1}(t) < 0$ and $D_{k+1}(u) > 0$, i.e., such that

$$\sum_{t-} D_{k+1}(t) < 0 < \sum_{u+} D_{k+1}(u).$$

The proof follows from Theorem 10.2 and Proposition 10.4.

(v) ⇒ (ii). Assume (ii) fails. Thus for some $k = 0, 1, \ldots, p-1$ and some pole sequence $i_1, i_2, \ldots, i_k$ from $\mathbf{s}$, we have $D_{k+1}(t) \geqslant 0$

for all $t \in \mathbf{s}$ or $D_{k+1}(t) \leqslant 0$ for all $t \in \mathbf{s}$. For definiteness, assume $D_{k+1}(t) \geqslant 0$ for all $t \in \mathbf{s}$. Since $E_1, E_2, \ldots, E_{k+1}$ are linearly independent, at least one $D_{k+1}(t) > 0$. Let $\boldsymbol{\beta}$ be any $\mathbf{Z}$-solution to $\mathbf{E}$. By the linearity properties of determinants, the sum $\sum_t \beta_t \cdot D_{k+1}(t)$ can be reduced to a single determinant whose last column is $(E_1(\boldsymbol{\beta}), E_2(\boldsymbol{\beta}), \ldots, E_{k+1}(\boldsymbol{\beta})) = (0, 0, \ldots, 0)$. Thus $\sum_t \beta_t D_{k+1}(t) = 0$. Hence not every $\beta_t > 0$, so $\mathbf{E}$ does not possess a $\mathbf{P}$-solution.

(iii) $\Rightarrow$ (ii). Trivial.

(v) $\Rightarrow$ (iii). Suppose $\mathbf{E}$ has a $\mathbf{P}$-solution $\boldsymbol{\beta}$.

Let $i_1, i_2, \ldots, i_k$ be a sequence from $\mathbf{s}$ satisfying $D_1 \neq 0$, $D_2 \neq 0, \ldots, D_k \neq 0$. Define ϵ_j (for $1 \leqslant j \leqslant k$), as follows:

$$\epsilon_j = \begin{cases} +1, & \text{if } A_{j-1}A_j > 0 \\ -1, & \text{if } A_{j-1}A_j < 0, \end{cases}$$

where A_j is defined by

$$A_0 = 1; \quad A_1 = D_1; \quad A_j = D_{j-1}D_j \left(D_{j-1} - \sum_{r=1}^{j-1} D_{j-1}(r \to i_j)\right), \qquad \text{if } j \geqslant 2.$$

Let $\mathbf{E}'$ be the LHD-system obtained from $\mathbf{E}$ by replacing the equation $E_j = 0$ with $\epsilon_j E_j = 0$, $j \in \mathbf{k}$. Then the equations of $\mathbf{E}'$ are still linearly independent, and $\boldsymbol{\beta}$ is a $\mathbf{P}$-solution to $\mathbf{E}'$. Let a prime (′) always refer to the system $\mathbf{E}'$. Then $i_1, i_2, \ldots, i_k$ is a pole sequence in $\mathbf{E}'$. We have already proved (v) $\Rightarrow$ (ii) so there exist $t, u \in \mathbf{s}$ such that $D'_{k+1}(t) < 0$ and $D'_{k+1}(u) > 0$. Then $D_{k+1}(t)$ and $D_{k+1}(u)$ also have the opposite sign, completing the proof. ∎

Note that condition (ii) above is equivalent to the following statement: For any pole sequence $i_1, i_2, \ldots, i_k$ from $\mathbf{s}$ $(0 \leqslant k < p)$, neither of the two equations

$$\sum_t D_{k+1}(t)\, x_t = 0 \qquad \text{nor} \qquad -\sum_t D_{k+1}(t)\, x_t = 0$$

have all their coefficients non-negative. By Proposition 10.1, these two equations are just constant multiples of the first equation of the $\langle i_1 i_2 \cdots i_k \rangle$-eliminated system of $(\mathbf{E}, \mathbf{0})$. Hence condition (ii) is a strengthening of Stiemke's theorem (condition (iv)). Condition (ii) tells precisely *which* $\mathbf{Z}$-combinations of the equations of $(\mathbf{E}, \mathbf{0})$ must have positive and negative coefficients in order to conclude that *all* nonzero

Z-combinations have this property. The explicit condition (ii) for an LHD-system to possess a **P**-solution appears to be new, though some results of Dines [7] are somewhat similar. The proof of the equivalence of (ii) and (v) can be modified to allow real coefficients and solutions.

Proposition 11.5 has the following additional consequence. If (**E**, **0**) is an ILID-system satisfying the hypothesis of Theorem 10.2 (so that *certain* equations (49) dependent on (**E**, **0**) have the R-property), then *all* equations dependent on (**E**, **0**) have the R-property. This statement is false if **0** is replaced with an arbitrary **Z**-vector. For instance, the ILID-system

$$\begin{aligned} x_1 - x_2 - x_3 \qquad &= \ \ 0 \\ x_1 \qquad\qquad -x_4 &= -1 \end{aligned}$$

satisfies the hypothesis of Theorem 10.2 (and therefore has the R-property), yet the second equation $x_1 - x_4 = -1$ does not have the R-property. Observe that Proposition 11.5 also implies that condition (ii) of Proposition 10.3 is equivalent to condition (ii) of Proposition 10.4 in the case $\alpha = 0$ (this can easily be verified directly). These two conditions are not equivalent, however, for arbitrary $\alpha \in \mathbf{Z}$, as shown by Example 11.3.

We now turn to the relationship between Proposition 8.3 (in the form given in Section 9) and Theorem 10.2.

PROPOSITION 11.6. *Let* $(\mathbf{E}, \boldsymbol{\alpha})$ *be an* ILID-*system given by* (29). *Suppose* $\boldsymbol{\alpha} = -\mathbf{E}(\boldsymbol{\beta})$, *where* $\boldsymbol{\beta}$ *is a vector of* 0's *and* 1's *of length* s. *Suppose the homogeneous system* $\mathbf{E} = \mathbf{0}$ *possesses a* **Z**-*solution* $\boldsymbol{\gamma}$ *satisfying*

$$\begin{aligned} \gamma_t > 0 \quad &\text{if} \quad \beta_t = 0, \\ \gamma_t < 0 \quad &\text{if} \quad \beta_t = 1. \end{aligned} \tag{61}$$

Then for all sequences $i_1, i_2, \ldots, i_k$ $(0 \leqslant k < p)$ *from* $\mathbf{s}$ *such that* $D_1 \neq 0$, $D_2 \neq 0, \ldots, D_k \neq 0$, *we have*

$$\sum_{t-} D_{k+1}(t) < -\hat{D}_{k+1} < \sum_{t+} D_{k+1}(t), \tag{62}$$

or equivalently,

$$\sum_{t-} D_{k+1}(t) < \sum_{t:\beta_t=1} D_{k+1}(t) < \sum_{t+} D_{k+1}(t). \tag{63}$$

Hence, since $D_1 \neq 0$, $D_2 \neq 0, \ldots, D_k \neq 0$ *for a pole sequence* $i_1, i_2, \ldots, i_k$, *we have by Theorem* 10.2 *and Proposition* 10.4 *that* $(\mathbf{E}, \boldsymbol{\alpha})$ *has the* R-*property, in accordance with Proposition* 8.3.

Proof. First observe that (62) and (63) are indeed equivalent since $-\hat{D}_{k+1} = \sum_{t:\beta_t=1} D_{k+1}(t)$ is just the linearity property of determinants applied to the last column.

Suppose $\mathbf{E} = \mathbf{0}$ possesses a $\mathbf{Z}$-solution γ satisfying (61). Let $i_1, i_2, \ldots, i_k$ $(0 \leqslant k < p)$ be a sequence from $\mathbf{s}$ such that $D_1 \neq 0$, $D_2 \neq 0, \ldots,$ $D_k \neq 0$. Suppose that for exactly m values of i_j do we have $\beta_{i_j} = 1$. Let $\mathbf{E}'$ denote the LHD-system obtained from $\mathbf{E}$ first by changing a_{jt} to $-a_{jt}$ whenever $\beta_t = 1$, and then by multiplying the first m equations by -1. Hence $\mathbf{E}'$ possesses a $\mathbf{P}$-solution γ', viz., $\gamma_t' = \gamma_t$ if $\beta_t = 0$ and $\gamma_t' = -\gamma_t$ if $\beta_t = 1$. Let a prime (′) always refer to the system $\mathbf{E}'$. Now D_j' is obtained from D_j by multiplying the first m rows of D_j by -1 and a certain m columns by -1, so $D_j = D_j'$. Now by the equivalence of (ii) and (v) in Proposition 11.5, there exist $t, u \in \mathbf{s}$ such that $D'_{k+1}(t) < 0$ and $D'_{k+1}(u) > 0$. Then

$$\begin{aligned} D_{k+1}(t) &< 0 \quad \text{if} \quad \beta_t = 0 \\ &> 0 \quad \text{if} \quad \beta_t = 1, \\ D_{k+1}(u) &< 0 \quad \text{if} \quad \beta_u = 1 \\ &> 0 \quad \text{if} \quad \beta_u = 0. \end{aligned}$$

This means

$$\sum_{\substack{t- \\ \beta_t=0}} D_{k+1}(t) - \sum_{\substack{t+ \\ \beta_t=1}} D_{k+1}(t) < 0 < \sum_{\substack{u+ \\ \beta_u=0}} D_{k+1}(u) - \sum_{\substack{u- \\ \beta_u=1}} D_{k+1}(u).$$

Adding $\sum_{t:\beta_t=1} D_{k+1}(t)$ to this inequality yields (63), so the proof is complete. ∎

Note that Theorem 10.2, Proposition 10.4, and Proposition 11.6 provide a purely algebraic (rather than geometric) proof of Proposition 8.3.

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