COMBINATORIAL RECIPROCITY THEOREMS *)

R.P. STANLEY

Massachusetts Institute of Technology, Cambridge, Mass. 02139, USA

A *combinatorial reciprocity theorem* is a result which establishes a kind of duality between two related enumeration problems. This rather vague concept will become clearer as more and more examples of such theorems are given. We shall be content in this paper with explaining the meaning of various reciprocity theorems *via* mere statements of results, together with clarifying examples. A rigorous treatment with detailed proofs appears in [11].

1. POLYNOMIALS

A *polynomial reciprocity theorem* takes the following form. Two combinatorially defined sequences $S_1, S_2, \ldots$ and $\bar{S}_1, \bar{S}_2, \ldots$ of finite sets are given, such that the functions $f(n) = |S_n|$ and $\bar{f}(n) = |\bar{S}_n|$ are polynomials in n for all integers $n \geq 1$. One then concludes that $\bar{f}(n) = (-1)^d f(-n)$, where $d = \deg f$. Frequently the numbers $f(0)$ and $\bar{f}(0)$ will have a special significance.

EXAMPLE 1.1. Fix $p > 0$. Let $f(n)$ be the number of combinations *with repetitions* of n things taken p at a time. Let $\bar{f}(n)$ be the number of such combinations *without repetitions*. Thus $f(n) = \binom{n+p-1}{p}$ and $\bar{f}(n) = \binom{n}{p}$. Hence it can be verified by inspection that $f(n)$ and $\bar{f}(n)$ are polynomials in n of degree p, related by $\bar{f}(n) = (-1)^p f(-n)$.

EXAMPLE 1.2. (THE ORDER POLYNOMIAL). Let P be a finite partially ordered set of cardinality $p > 0$. Let $\omega\colon P \to [p]$ be a fixed bijection, where we use the "French notation" $[p] = \{1,2,\ldots,p\}$. Let $\Omega(n)$ denote the number of maps

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$\sigma: P \to [n]$ such that (i) $x \le y$ in P implies $\sigma(x) \le \sigma(y)$, and (ii) $x < y$ in P and $\omega(x) > \omega(y)$ implies $\sigma(x) < \sigma(y)$. Let $\overline{\Omega}(n)$ denote the number of maps $\tau: P \to [n]$ such that (i) $x \le y$ in P implies $\tau(x) \le \tau(y)$, and (ii) $x < y$ in P and $\omega(x) < \omega(y)$ implies $\tau(x) < \tau(y)$. Then it can be shown [8, Proposition 13.2(i)] that Ω and $\overline{\Omega}$ are polynomial functions in n of degree p related by $\overline{\Omega}(n) = (-1)^p\Omega(-n)$. We call Ω the *order polynomial* of (P,ω). There are several ways to prove this reciprocity relationship between Ω and $\overline{\Omega}$, perhaps the simplest by a judicious use of the Principle of Inclusion-Exclusion which we leave to the reader. Note that if P is a p-element chain and ω is order-preserving, then $\Omega(n) = \binom{n+p-1}{p}$ and $\overline{\Omega}(n) = \binom{n}{p}$, so example 1.1 is a special case.

Several interesting consequences of the reciprocity between Ω and $\overline{\Omega}$ are derived in [8, §19]. For instance, if ω is order-preserving then for some integer ℓ we have $\Omega(n) = (-1)^p\Omega(-\ell-n)$ for all n if and only if every maximal chain of P has length ℓ.

<u>EXAMPLE 1.3</u>. (CHROMATIC POLYNOMIALS). Let G be a finite graph without loops or multiple edges, with vertex set V of cardinality p. Let $\chi(n)$ denote the number of pairs $(\mathcal{O},\sigma)$, where (i) $\mathcal{O}$ is an acyclic orientation of the edges of G, and (ii) $\sigma: V \to [n]$ is any map $V \to [n]$ such that if $u \to v$ in $\mathcal{O}$ (so $u,v \in V$ and uv is an edge of G) then $\sigma(u) > \sigma(v)$. Let $\overline{\chi}(n)$ be the number of such maps with the condition $\sigma(u) > \sigma(v)$ replaced with $\sigma(u) \ge \sigma(v)$. It is easily seen that $\chi(n)$ is the chromatic polynomial of G. In [9] two proofs are given of the reciprocity theorem $\overline{\chi}(n) = (-1)^p\chi(-n)$. In particular, $(-1)^p\chi(-1)$ is the number of acyclic orientations of G.

<u>EXAMPLE 1.4</u>. (ABSTRACT MANIFOLDS). Let Δ be a finite simplicial complex with vertex set V, with $|V| = p$. Thus Δ is a collection of subsets S of V such that $\{v\} \in \Delta$ for all $v \in V$, and if $S \in \Delta$ and $T \subset S$, then $T \in \Delta$. Let $f_i = f_i(\Delta)$ be the number of $(i+1)$-sets contained in Δ. Hence $f_{-1} = 1$ and $f_0 = p$. Define the polynomial $\Lambda(\Delta,n)$ by

$$\Lambda(\Delta,n) = \sum_{i\ge 0} f_i\binom{n-1}{i} .$$

Note that $\Lambda(\Delta,0) = f_0-f_1+f_2-\dots = \chi(\Delta)$, the Euler characteristic of Δ.

Now suppose that the underlying topological space $|\Delta|$ of Δ is homeomorphic to a d-dimensional manifold with boundary. Hence $\deg \Lambda(\Delta,n) = d$. Denote by $\partial\Delta$ those elements of Δ such that $|\partial\Delta| = \partial|\Delta|$, in the obvious sense. Hence $\partial\Delta$ is itself a simplicial complex, with vertex set contained in V. It follows from a result of MACDONALD [5, Proposition 1.1] that

$$(1.1)\qquad (-1)^d \Lambda(\Delta,-n) = \Lambda(\Delta,n) - \Lambda(\partial\Delta,n) .$$

For instance, let Δ consist of ABCD, BCDE, and all their subsets (ABCD is short for $\{A,B,C,D\}$, etc.) Then $d = 3$, $|\Delta|$ is a 3-ball, and $\partial\Delta$ consists of ABC,ABD,ACD,BCE,CDE,BDE, and all their subsets. Moreover,

$$\Lambda(\Delta,n) = 5 + 9\binom{n-1}{1} + 7\binom{n-1}{2} + 2\binom{n-1}{3}$$

and

$$\Lambda(\partial\Delta,n) = 5 + 9\binom{n-1}{1} + 6\binom{n-1}{2}.$$

It follows from (1.1) that

$$-\Lambda(\Delta,-n) = \binom{n-1}{2} + 2\binom{n-1}{3} .$$

A special case of particular interest occurs when $\partial\Delta = \emptyset$, i.e., when $|\Delta|$ is a manifold. We then have from (1.1) that

$$(1.2)\qquad (-1)^d \Lambda(\Delta,-n) = \Lambda(\Delta,n) .$$

Now (1.2) imposes certain constraints on the numbers f_i which define Λ. When $|\Delta|$ is a sphere, these constraints are simply the well-known DEHN-SOMMERVILLE equations [4, Chapter 9] [6, Chapter 2.4].

EXAMPLE 1.5. (CONCRETE MANIFOLDS). Let M be a subset of the s-dimensional euclidean space with the following properties: (i) M is a union of finitely many convex polytopes, any two of which intersect in a common face of both, (ii) the vertices of these convex polytopes have integer coordinates, and (iii) M is homeomorphic to a d-dimensional manifold with boundary. If $n > 0$, then let $j(n)$ be the number of points $\alpha \in M$ such that $n\alpha$ has integer coordinates, and let $i(n)$ be the number of such points not belonging to ∂M. Then a result due essentially to E. EHRHART [2] (for the generality considered here, one also needs [5, Proposition 1.1]) states that $j(n)$ and $i(n)$ are polynomial functions of n of degree d satisfying

$$(1.3) \qquad j(0) = \chi(M),\ i(n) = (-1)^d j(-n) .$$

We remark that condition (ii) can be replaced by the requirement (ii') the vertices have *rational* coordinates. In this case i and j need no longer be polynomials, but instead there is some $N > 0$ and polynomials $j_0, j_1, \ldots, j_{N-1}$ and $i_0, i_1, \ldots, i_{N-1}$ such that $j(n) = j_a(n)$ and $i(n) = i_a(n)$ whenever $n \equiv a (\mathrm{mod}\ N)$. We then have in place of (1.3) that $j_0(0) = \chi(M)$ and $i_a(n) = = (-1)^d j_{-a}(-n)$, where the subscripts are taken modulo N.

An interesting application of (1.3) is to the problem of finding the volume V(M) of a subset M satisfying conditions (i), (ii), (iii), and the additional condition that $s = d$. It is easy to see that then the leading coefficient of j(n) is V(M). Hence from (1.3) we see that if we know any d+1 of the numbers $\chi(M)$, $j(n)$, $i(n)$, $n \geq 1$, then we can compute V(M). For a further discussion of this result (including references), see [11].

EXAMPLE 1.6. (MAGIC SQUARES). As a special case of example 1.5, take M to be the set of all doubly stochastic $N \times N$ matrices, so $s = N^2$ and $d = (N-1)^2$. It is well-known that M is a convex polytope whose vertices have integer coordinates, so j(n) and i(n) are polynomials in n of degree $(N-1)^2$. It is easy to see that j(n) is the number of $N \times N$ matrices of non-negative integers with every row and column sum equal to n, while i(n) is the number of such matrices with positive entries. Clearly $i(0) = i(1) = \ldots = i(N-1) = = 0$ and $i(N+n) = j(n)$ for $n \geq 0$. There follows from (1.3),

$$j(-1) = j(-2) = \ldots = j(-N+1) = 0 ,$$

$$j(n) = (-1)^{N-1} j(-N-n) .$$

These results were first obtained in [10]. Another proof is given in [3].

2. HOMOGENEOUS LINEAR EQUATIONS

Consider the homogeneous linear equation $x = y$. Let $F(X,Y) = \sum X^\alpha Y^\beta$, where the sum is over all solutions $(x,y) = (\alpha,\beta)$ to $x = y$ in non-negative integers α,β. Let $\overline{F}(X,Y)$ be the corresponding sum over all solutions in positive integers. Clearly $F(X,Y) = 1/(1-XY)$ and $\overline{F}(X,Y) = XY/(1-XY)$. Hence as rational functions we have $\overline{F}(X,Y) = -F(1/X,1/Y)$. It is this result we wish to extend to more general systems of equations.

THEOREM 2.1. [10, Theorem 4.1]. *Let* E *be a system of finitely many linear homogeneous equations with integer coefficients, in the variables* $x_1, x_2, \ldots, x_s$. *Define*

$$
\begin{aligned}
F(X_1, X_2, \ldots, X_s) &= \sum X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_s^{\alpha_s}, \\
\overline{F}(X_1, X_2, \ldots, X_s) &= \sum X_1^{\beta_1} X_2^{\beta_2} \cdots X_s^{\beta_s},
\end{aligned} \tag{2.1}
$$

where $(\alpha_1, \alpha_2, \ldots, \alpha_s)$ *ranges over all solutions* $x_i = \alpha_i$ *of* E *in non-negative integers* α_i, *while* $(\beta_1, \beta_2, \ldots, \beta_s)$ *ranges over all solutions in positive integers. Then* F *and* $\overline{F}$ *are rational functions of the* X_i*'s (in the algebra of formal power series, or for* $|X_i| < 1$*). A necessary and sufficient condition that*

$$\overline{F}(X_1, X_2, \ldots, X_s) = \pm F(1/X_1, 1/X_2, \ldots, 1/X_s),$$

as rational functions, is for E *to possess a solution in positive integers. In this case the correct sign is* $(-1)^\kappa$, *where* κ *is the corank (*= s-rank E*) of* E.

Many of the results in section 1 can be deduced from the above theorem. We require a connection between evaluating polynomials at +n and -n, and substituting $1/X_i$ for X_i in a rational function. Such a connection is provided by the next result, which EHRHART [1] attributes to POPOVICIU [7].

PROPOSITION 2.1. *Let* H(n) *be a function from the integers* Z *to the complex numbers* C *of the form*

$$H(n) = \sum_{i=1}^{r} P_i(n)\alpha_i^n,$$

where the α_i*'s are fixed non-zero complex numbers and each* P_i *is a polynomial in* n. *Define*

$$F(X) = \sum_{n=0}^{\infty} H(n)X^n, \quad \overline{F}(X) = \sum_{n=1}^{\infty} H(-n)X^n.$$

Then F *and* $\bar{F}$ *are rational functions of* X, *related by* $\bar{F}(X) = -F(1/X)$.

Theorem 2.1 suggests that we try to find "rational function analogues" of examples 1.4 and 1.5.

PROPOSITION 2.2. *Let* Δ *be a finite simplicial complex with vertices* $V_1, V_2, \dots, V_p$. *Suppose* $|\Delta|$ *is homeomorphic to a* d*-manifold with boundary. Define the generating functions*

$$F(V_1,V_2,\dots,V_p) = \sum V_1^{\delta_1} V_2^{\delta_2} \dots V_p^{\delta_p} + \chi(\Delta) - 1,$$

$$\bar{F}(V_1,V_2,\dots,V_p) = \sum V_1^{\varepsilon_1} V_2^{\varepsilon_2} \dots V_p^{\varepsilon_p},$$

where $(\delta_1,\delta_2,\dots,\delta_p)$ *ranges over all* p*-tuples of non-negative integers such that* $\{V_i \mid \delta_i > 0\} \in \Delta$, *while* $(\varepsilon_1,\varepsilon_2,\dots,\varepsilon_p)$ *ranges over all* p*-tuples of non-negative integers such that* $\emptyset \neq \{V_i \mid \varepsilon_i > 0\} \in \Delta - \partial\Delta$. *Then* F *and* $\bar{F}$ *are rational functions of the* V_i*'s related by*

$$\bar{F}(V_1,V_2,\dots,V_p) = (-1)^{d+1} F(1/V_1, 1/V_2, \dots, 1/V_p) .$$

Proposition 2.2 is a consequence of MACDONALD's result [5, Proposition 1.1] mentioned earlier. It is easily seen that

$$F(X,X,\dots,X) = \sum_{n=0}^{\infty} \Lambda(\Delta,n) X^n ,$$

$$\bar{F}(X,X,\dots,X) = \sum_{n=1}^{\infty} [\Lambda(\Delta,n) - \Lambda(\partial\Delta,n)] X^n ,$$

in the notation of example 1.4. Thus (1.1) follows from propositions 2.1 and 2.2.

PROPOSITION 2.3. *Let* M *satisfy properties* (i), (ii') *and* (iii) *of example 1.5. Define*

$$F(X_1,X_2,\dots,X_s,Y) = \chi(M) + \sum X_1^{\alpha_1} X_2^{\alpha_2} \dots X_s^{\alpha_s} Y^n ,$$

$$\bar{F}(X_1,X_2,\dots,X_s,Y) = \sum X_1^{\beta_1} X_2^{\beta_2} \dots X_s^{\beta_s} Y^n ,$$

where $(\alpha_1,\alpha_2,\dots,\alpha_s,n)$ *ranges over all* (s+1)*-tuples of non-negative integers*

α_i *and positive integers* n *such that* $(\alpha_1/n,\alpha_2/n,\ldots,\alpha_s/n) \in M$, *while* $(\beta_1,\beta_2,\ldots,\beta_s,n)$ *ranges over all such* $(s+1)$*-tuples with* $(\beta_1/n,\beta_2/n,\ldots,\beta_s/n) \in M-\partial M$. *Then* F *and* $\bar{F}$ *are rational functions related by*

$$\bar{F}(X_1,X_2,\ldots,X_s,Y) = (-1)^{d+1} F(1/X_1,1/X_2,\ldots,1/X_s,1/Y) .$$

If we put each $X_i = 1$ and apply proposition 2.1, then we get (1.3).

3. RECIPROCAL DOMAINS

In theorem 2.1, we considered solutions $\alpha_i \geq 0$ $(i=1,2,\ldots,s)$ and $\beta_j > 0$ $(j=1,2,\ldots,s)$ to a system of homogeneous linear equations. It is natural to consider the following generalization. Let E be a system of finitely many linear homogeneous equations with integer coefficients, in the variables $x_1,x_2,\ldots,x_s$ (as in theorem 2.1). Let $S \subset [s]$. Define

$$F_S(X_1,X_2,\ldots,X_s) = \sum X_1^{\alpha_1} X_2^{\alpha_2} \ldots X_s^{\alpha_s} ,$$

(3.1)

$$\bar{F}_S(X_1,X_2,\ldots,X_s) = \sum X_1^{\beta_1} X_2^{\beta_2} \ldots X_s^{\beta_s} ,$$

where $(\alpha_1,\alpha_2,\ldots,\alpha_s)$ ranges over all solutions to E in non-negative integers such that $\alpha_i > 0$ if $i \in S$, while $(\beta_1,\beta_2,\ldots,\beta_s)$ ranges over all solutions to E in non-negative integers with $\beta_i > 0$ if $i \notin S$. Thus $\bar{F}_S = F_{[s]-S}$. Note that $F_\emptyset = F$ and $\bar{F}_\emptyset = \bar{F}$, where F and $\bar{F}$ are given by (2.1).

We now ask under what conditions do we have

$$\bar{F}(X_1,X_2,\ldots,X_s) = (-1)^{\kappa} F(1/X_1,1/X_2,\ldots,1/X_s) , \tag{3.2}$$

where κ is the corank of E. It seems plausible that (3.2) will hold whenever E has a solution in positive integers, as in theorem 2.1. In [11], however, we show that this is not the case; and we show why it is likely that there are no simple necessary and sufficient conditions for (3.2) to hold. There is, however, an elegant and surprising sufficient condition.

THEOREM 3.1. [11, Proposition 8.3]. *A sufficient condition for (3.2) to hold is that there exists a solution* $(\gamma_1,\gamma_2,\ldots,\gamma_s)$ *to* E *in integers* γ_i *such that*

$\gamma_i > 0$ *if* $i \in S$ *and* $\gamma_i < 0$ *if* $i \notin S$.

The proof of theorem 3.1 depends on a rather complicated geometric argument suggested by a result of EHRHART [1, p.22] on "reciprocal domains". It is much easier, on the other hand, to give a necessary condition for (3.2) to hold.

PROPOSITION 3.1. *If (3.2) holds, then either* $F = \bar{F} = 0$, *or else* E *has a solution in positive integers.*

PROOF. Assume (3.2) holds but not $F = \bar{F} = 0$. Then $F \neq 0$ and $\bar{F} \neq 0$, so E has solutions $\alpha = (\alpha_1,\alpha_2,\ldots,\alpha_s)$ and $\beta = (\beta_1,\beta_2,\ldots,\beta_s)$ as given in (3.1). Then $\alpha + \beta$ is a solution to E in positive integers. □

4. INHOMOGENEOUS EQUATIONS

Another way of extending theorem 2.1 besides theorem 3.1 is to consider *inhomogeneous* linear equations. Suppose we have a system

$$(4.1) \qquad \sum_{i=1}^{s} a_{ij}x_i = b_j \ , \qquad j \in [p] \ ,$$

of p inhomogeneous linear equations with integer coefficients a_{ij} and integer constants b_j, in the variables $x_1,x_2,\ldots,x_s$. It turns out that the correct reciprocal notions to consider in this context are (i) solutions to (4.1) in non-negative integers, and (ii) solutions in positive integers to the "reciprocal system"

$$(4.2) \qquad \sum_{i=1}^{s} a_{ij}x_i = -b_j, \qquad j \in [p] \ .$$

Suppose, for example, that $S \subset [s]$ and that

$$b_j = -\sum_{i \in S} a_{ij}, \qquad j \in [p] \ .$$

Hence a solution $(\alpha_1,\ldots,\alpha_s)$ to (4.1) in non-negative integers corresponds to a solution $(\beta_1,\ldots,\beta_s)$ of the system $\sum a_{ij}x_i = 0$ in integers β_i satisfying $\beta_i \geq 0$ if $i \notin S$, $\beta_i > 0$ if $i \in S$ (set $\beta_i = \alpha_i$ if $i \notin S$, $\beta_i = \alpha_i+1$ if $i \in S$). Moreover, a solution $(\alpha_1,\ldots,\alpha_s)$ to (4.2) in positive integers corresponds to a solution $(\beta_1,\ldots,\beta_s)$ of the system $\sum a_{ij}x_i = 0$ in integers β_i satisfying

$\beta_i > 0$ if $i \notin S$, $\beta_i \geq 0$ if $i \in S$ (set $\beta_i = \alpha_i$ if $i \notin S$; $\beta_i = \alpha_i - 1$ if $i \in S$). Hence our notion of reciprocity for inhomogeneous systems includes the reciprocity of section 3 as a special case.

We therefore define

$$(4.3)\qquad \begin{aligned} F(X_1,X_2,\ldots,X_s) &= \sum X_1^{\alpha_1} X_2^{\alpha_2} \ldots X_s^{\alpha_s}, \\ \overline{F}(X_1,X_2,\ldots,X_s) &= \sum X_1^{\beta_1} X_2^{\beta_2} \ldots X_s^{\beta_s}, \end{aligned}$$

where $(\alpha_1,\alpha_2,\ldots,\alpha_s)$ ranges over all solutions to (4.1) in non-negative integers, while $(\beta_1,\beta_2,\ldots,\beta_s)$ ranges over all solutions to (4.2) in positive integers. As usual, we seek conditions when $\overline{F}(X_1,X_2,\ldots,X_s) = (-1)^{\kappa} F(1/X_1,1/X_2,\ldots,1/X_s)$, where κ is the corank of (4.1) or (4.2). We shall say that (4.1) has the *R-property* if $\overline{F}(X_1,X_2,\ldots,X_s) = (-1)^{\kappa} F(1/X_1,1/X_2,\ldots,1/X_s)$. The possibility of obtaining reasonable necessary and sufficient conditions for E to have the R-property appears hopeless, and even reasonably general sufficient conditions are rather complex and not very edifying. We shall now discuss the nature of the sufficient conditions obtained in [11].

Let $\{i_1,i_2,\ldots,i_k\}$ be a set of $k < p$ elements from $[s]$ such that the determinant of coefficients taken from the first k rows and from columns $i_1,i_2,\ldots,i_k$ of (4.1) is non-zero. Hence we can solve the first k equations (i.e., $j \in [k]$) of (4.1) for $x_{i_1},x_{i_2},\ldots,x_{i_k}$ in terms of the remaining x_i's and substitute these values in the remaining p-k equations, obtaining p-k equations in s-k unknowns. Let $E(i_1,i_2,\ldots,i_k)$ denote the first of these p-k equations (i.e., the equation resulting from making the above substitution into the (k+1)-st equation of (4.1)). Thus in particular $E(\emptyset)$ is just the first equation $\sum a_{i1}x_i = b_1$ of (4.1). Note that the equations $E(i_1,i_2,\ldots,i_k)$ are really determined only up to a non-zero multiplicative constant. This need not concern us since we will be interested only in solutions to these equations.

<u>EXAMPLE 4.1</u>. Consider the system

$$\begin{aligned} x_1 - x_2 + 3x_3 \phantom{{}- x_4} &= b_1 \\ 2x_2 - x_3 - x_4 &= b_2. \end{aligned}$$

Then we obtain the equations

$$\begin{array}{llcccccl}
E(\emptyset): & x_1 & - & x_2 & + & 3x_3 & & = b_1 \\
E(1): & & & 2x_2 & - & x_3 & - x_4 & = b_2 \\
E(2): & 2x_1 & & & + & 5x_3 & - x_4 & = 2b_1 + b_2 \\
E(3): & x_1 & + & 5x_2 & & & - 3x_4 & = -b_1 + 3b_2 .
\end{array}$$

THEOREM 4.1. *A sufficient condition that the system (4.1) has the R-property is the following. For every set* $\{i_1, i_2, \ldots, i_k\} \subset [s]$ *for which* $E(i_1, i_2, \ldots, i_k)$ *is defined, the single equation* $E(i_1, i_2, \ldots, i_k)$ *should possess the R-property.*

It should be mentioned that in [11] theorem 4.1 is strengthened so that only a special subset of the equations $E(i_1, i_2, \ldots, i_k)$ need be considered. However, the definition of this subset is rather complicated and will be omitted here. Theorem 4.1 is proved in [11] using iterated contour integration. Contour integration may seem like an unwarranted artifice for a result like theorem 4.1. While it is undoubtedly possible to dispense with contour integration, the next results show that it is not too unnatural in the present context. We would like to complement theorem 4.1 by obtaining conditions for a single equation to possess the R-property.

THEOREM 4.2. *Let* $a_1x_1 + a_2x_2 + \ldots + a_sx_s = b$ *be a single linear equation* E *with integer coefficients* a_i *and integer constant term* b. *Then the following three conditions are equivalent.*

(i) *The rational functions*

$$\lambda^{b-1} / (1-\lambda^{-a_1})(1-\lambda^{-a_2}) \ldots (1-\lambda^{-a_s}) \tag{4.4}$$

and

$$\lambda^{\bar{b}-1} / (1-\lambda^{-a_1})(1-\lambda^{-a_2}) \ldots (1-\lambda^{-a_s}) \tag{4.5}$$

have zero residues at $\lambda = 0$. *Here* $\bar{b} = -b-a_1-a_2-\ldots-a_s$.

(ii) *The following two conditions are both satisfied.*

(a) *There does not exist a solution* $(\alpha_1, \alpha_2, \ldots, \alpha_s)$ *to* E *in integers such that*

$\alpha_t < 0$ *if* $a_t > 0$, *and*
$\alpha_t \geq 0$ *if* $a_t < 0$.

(b) There does not exist a solution $(\beta_1, \beta_2, \ldots, \beta_s)$ *to* E *in integers such that*

$$\beta_t \geq 0 \quad \textit{if} \quad a_t > 0 \text{ , and}$$
$$\beta_t < 0 \quad \textit{if} \quad a_t < 0 \text{ .}$$

(Note: *It is clear that at least one of (a) or (b) always holds.*)

(iii) E *has the R-property.*

THEOREM 4.3. *With the hypotheses of theorem 4.2, the following two conditions are equivalent.*

(i) *The rational functions of (4.4) and (4.5) have no poles at* $\lambda = 0$.

(ii) $\sum_{t-} a_t < -b < \sum_{t+} a_t$, *where* $\sum_{t-} a_t$ *(resp.* $\sum_{t+} a_t$*) denotes the sum of all* a_t *satisfying* $a_t < 0$ *(resp.* $a_t > 0$*).*

If, moreover, either of the two (equivalent) conditions (i) *or* (ii) *is satisfied, then* E *has the* R*-property.*

EXAMPLE 4.2. Consider the system E of example 4.1. By theorems 4.1 and 4.3, we see that E has the R-property if

$$\begin{aligned}
-1 &< -b_1 &&< 4\\
-2 &< \quad\; -b_2 &&< 2\\
-1 &< -2b_1 - b_2 &&< 7\\
-3 &< b_1 - 3b_2 &&< 6 \text{ .}
\end{aligned}$$

These conditions hold if and only if $(b_1,b_2) = (0,-1),(0,0),(-1,-1),(-1,0),$ $(-2,-1)$ or $(-2,0)$.

Analogously to proposition 3.1, we have a simple necessary condition for a system (4.1) to have the R-property. The proof is essentially the same as the proof of proposition 3.1.

PROPOSITON 4.1. *Suppose the system (4.1) has the R-property. Then either* $F = \overline{F} = 0$, *or else the homogeneous system* $\sum_{i=1}^{s} a_{ij} x_i = 0$, $j \in [p]$, *has a solution in positive integers.*

We have given a sampling of what we believe to be the most interesting examples of combinatorial reciprocity theorems. Some additional types of reciprocity theorems are given in [11]. There are many other combinatorial relationships which can be viewed as reciprocity theorems and which we have not touched on. Examples include the inverse relationship between the Stirling numbers of the first and second kinds, and the MacWilliams identities of coding theory. We believe that many new interesting results and unifying principles are awaiting discovery in the field of combinatorial reciprocity.

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