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On the Number of Open Sets of Finite Topologies

RICHARD P. STANLEY

Department of Mathematics, Harvard University,

Cambridge, Massachusetts 02138

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ABSTRACT

Recent papers of Sharp [4] and Stephen [5] have shown that any finite topology with n points which is not discrete contains $\leqslant(3/4)2^n$ open sets, and that this inequality is best possible. We use the correspondence between finite T_0-topologies and partial orders to find all non-homeomorphic topologies with n points and $\geqslant(7/16)2^n$ open sets. We determine which of these topologies are T_0, and in the opposite direction we find finite T_0 and non-T_0 topologies with a small number of open sets. The corresponding results for topologies on a finite set are also given.

If X is a finite topological space, then X is determined by the minimal open sets U_x containing each of its points x. X is a T_0-space if and only if $U_x = U_y$ implies $x = y$ for all points x, y in X. If X is not T_0, the space $\hat{X}$ obtained by identifying all points $x, y \in X$ such that $U_x = U_y$, is a T_0-space with the same lattice of open sets as X. Topological properties of the operation $X \to \hat{X}$ are discussed by McCord [3]. Thus for the present we restrict ourselves to T_0-spaces.

If X is a finite T_0-space, define $x \leqslant y$ for $x, y \in X$ whenever $U_x \subseteq U_y$. This defines a partial ordering on X. Conversely, if P is any partially ordered set, we obtain a T_0-topology on P by defining $U_x = \{y/y \leqslant x\}$ for $x \in P$. The open sets of this topology are the *ideals* (also called semi-ideals) of P, i.e., subsets Q of P such that $x \in Q$, $y \leqslant x$ implies $y \in Q$.

Let P be a finite partially ordered set of order p, and define $\omega(P) = j(P)\, 2^{-p}$, where $j(P)$ is the number of ideals of P. If Q is another finite partially ordered set, let $P + Q$ denote the disjoint union (direct sum) of P and Q. Then $j(P + Q) = j(P) j(Q)$ and $\omega(P + Q) = \omega(P)\, \omega(Q)$. Let H_p denote the partially ordered set consisting of p disjoint points, so $\omega(H_p) = 1$.

THEOREM 1. *If $n \geqslant 5$, then up to homeomorphism there is one T_0-space with n points and 2^n open sets, one with $(3/4)\,2^n$ open sets, two with $(5/8)\,2^n$ open sets, three with $(9/16)\,2^n$, two with $(17/32)\,2^n$, two with $(1/2)\,2^n$, two with $(15/32)\,2^n$, five with $(7/16)\,2^n$, and for each $m = 6, 7, \ldots, n$, two with $(2^{m-1}+1)\,2^{n-m}$. All other T_0-spaces with n points have $<(7/16)\,2^n$ open sets, giving a total of $2n + 8$ with $\geqslant(7/16)\,2^n$ open sets.*

PROOF. Consider the 18 partially ordered sets $P_1, \ldots, P_{18}$ of order 5 in Figure 1. Any partial order P weaker than some P_i (meaning P_i can be

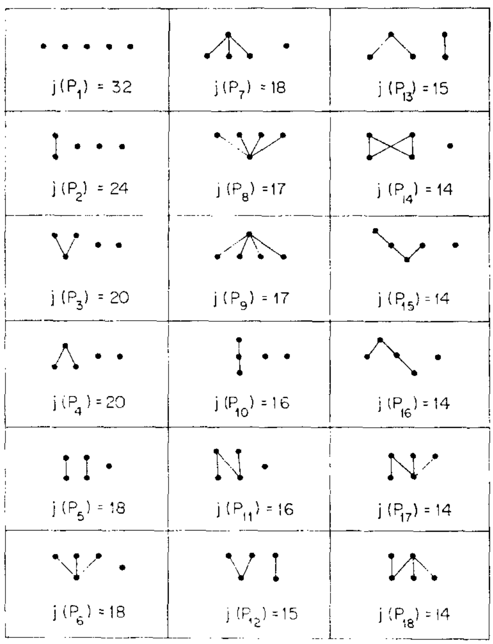

FIG. 1. Five-element partially ordered sets with maximal $j(P)$.

obtained from P by imposing additional relations $x \leqslant y$) is itself one of the P_j. Each P_i satisfies $\omega(P_i) \geqslant 7/16$. Suppose Q is obtained from P by adjoining more points to P and imposing additional relations. Then $\omega(Q) \leqslant \omega(P)$ with equality if and only if $Q = P + H_m$ for some m. By inspection of all possibilities it can be verified that:

(1) if P is obtained from one of $P_1, \ldots, P_{18}$ by imposing any additional relations $x \leqslant y$, then $\omega(P) < 7/16$, unless P is again one of $P_1, \ldots, P_{18}$,

(2) if P is obtained from one of $P_1, \ldots, P_{18}$ by adjoining one additional point and imposing any additional relations $x \leqslant y$ which do not give

$P_j + H_1$ for some $j = 1,..., 18$, then $\omega(P) < 7/16$, with the exception of adjoining a point above the minimal element of P_8 or below the maximal element of P_9, and

(3) if P is obtained by adjoining two points x, y to one of $P_1 ,..., P_{18}$ and the additional relation $x \leqslant y$ imposed, then $\omega(P) < 7/16$ unless $P = P_j + H_2$ for some $j = 1,..., 18$.

If any of the procedures (1), (2), (3) is iteratively applied to the two exceptions in (2), the resulting partially ordered sets P always satisfy $\omega(P) < 7/16$ unless P is the partial order obtained by adjoining either a minimal element or a maximal element to H_{m-1}. In this case $\omega(P) = (2^{m-1} + 1)\, 2^{-m}$. Thus any partially ordered set P of order $p \geqslant 5$ satisfying $\omega(P) \geqslant 7/16$ must be of the form $P_i + H_{p-5}$, $i = 1,..., 18$, or $\overline{H}_{m-1} + H_{p-m}$, where $\overline{H}_{m-1}$ is H_{m-1} with a minimal or maximal element adjoined. The proof of Theorem 1 now follows.

If X is not necessarily a T_0-space, of order n, and if the "T_0-quotient" $\hat{X}$ has order $m \leqslant n$, then X and $\hat{X}$ have the same number of open sets. From this observation we can deduce:

THEOREM 2. *If $n \geqslant 3$, then up to homeomorphism there is one* "non-T_0" *space with n points and* $(1/2)\, 2^n$ *open sets, three with* $(3/8)\, 2^n$ *open sets, and all the rest have* $<(3/8)\, 2^n$ *open sets.*

We omit the details of the proof. It is not difficult to use the partial orders of Figure 1 to refine Theorem 2, but we will not do this here. Theorem 2 suggests the following question: Given a T_0-space X of order n, how many spaces Y, up to homeomorphism, are there of order $n + r$, $r \geqslant 0$, satisfying $\hat{Y} = X$? The solution follows from a straightforward application of Pólya's theorem [1, Ch. 5, especially p. 174]; again we omit the details.

THEOREM 3. *Let X be a T_0-space of order n. Let $Z_X(x_1 , x_2 ,...)$ be the cycle index polynomial* [1, Ch. 5] *of Aut X, the group of homeomorphisms $X \to X$, regarded as a permutation group on X. Then the number of non-homeomorphic spaces Y of order $n + r$ satisfying $\hat{Y} = X$ is equal to the coefficient of x^r in the expansion of* $Z_X(1/(1 - x), 1/(1 - x^2),...)$.

EXAMPLES. (1) Let X be the three point T_0-space whose corresponding partial order is obtained by adjoining a minimal element to H_2. Then $Z_X(x_1 , x_2 ,...) = \frac{1}{2}(x_1^3 + x_1x_2)$, and

$$Z_X(1/(1 - x), 1/(1 - x^2),...) = \sum_{r=0}^{\infty} (1/8)(2r^2 + 8r + 7 + (-1)^r)\, x^r.$$

(2) If the open sets of Y are totally ordered by inclusion, then Y is called a *chain-topology* [5]. Suppose $X = \hat{Y}$ has order m, so that Y has m non-empty open sets. Then $Z_X(x_1, x_2, \ldots) = x_1{}^m$ and

$$Z_X(1/(1-x), 1/(1-x^2), \ldots) = \sum_{r=0}^{\infty} \binom{r+m-1}{r} x^r.$$

The total number of non-homeomorphic chain topologies with n points is

$$\sum_{m=1}^{n} \binom{n-m+m-1}{n-m} = 2^{n-1}.$$

The question of which T_0-spaces of order n have the least number of open sets can be treated similarly. If P is a partially ordered set with p points, then $\omega(P)$ is a minimum when P is a chain, in which case $\omega(P) = (p+1)\, 2^{-p}$. The next smallest value of $\omega(P)$ occurs when only one pair of points x, y of P are unrelated. The remaining $p-2$ points can be arranged so that m of them form a chain below x, y and $p-m-2$ of them a chain above x, y, for any $m = 0, 1, \ldots, p-2$. For each of these P, $\omega(P) = (p+2)\, 2^{-p}$. The next smallest value of $\omega(P)$ must occur when P has two pairs of incomparable points. This can occur in one of two ways:

(1) $x < y$, with z unrelated to both x and y. The remaining $p-3$ points can be arranged so m_1 of them form a chain above y and z and m_2 below x and z, with $m_1 + m_2 = p-3$. For $p > 1$, this yields a total of $p-2$ such P's with $\omega(P) = (p+3)\, 2^{-p}$.

(2) x and y unrelated, z and w unrelated, but each of x and y lying below each of z and w. The remaining $p-4$ points can be arranged so m_1 of them form a chain below x and y, m_2 above x and y but below z and w, and m_3 above z and w, with $m_1 + m_2 + m_3 = p-4$. For $p > 1$, this yields a total of $\frac{1}{2}(p-2)(p-3)$ such P's, again with $\omega(P) = (p+3)\, 2^{-p}$. This proves:

THEOREM 4. *If $n \geqslant 2$, then up to homeomorphism there is one T_0-space with n points and $n+1$ open sets, $n-1$ with $n+2$ open sets, $\frac{1}{2}(n-1)(n-2)$ with $n+3$ open sets, and all the rest have $> n+3$ open sets.*

The analog of Theorem 4 for non-T_0 spaces is obtained by applying Theorem 3 to partially ordered sets P with minimal $j(P)$.

THEOREM 5. *If $n \geqslant 5$, $r \geqslant 0$, then up to homeomorphism there is one* "non-T_0" *space with $n+r$ points and two open sets, $r+1$ with three open* *sets,* $\frac{1}{4}(2r + 3 + (-1)^r) + \binom{r+2}{2}$ *with four open sets,* $\frac{1}{4}(2r^2 + 8r + 7 + (-1)^r) + \binom{r+3}{3}$ *with five open sets, and all the rest have* >5 *open sets.*

Instead of considering finite spaces up to homeomorphism, we could consider *labeled* finite spaces, i.e., finite spaces on a given set. If X is an n-point space with Aut X of order g, then there are $n!/g$ ways of putting a topology on a set of order n homeomorphic to X. This observation allows us to state analogs of Theorems 1 – 5 for labeled spaces. We omit the proofs. We use the notation $(n)_k = n(n-1) \cdots (n-k+1)$.

THEOREM 1′. *If* $n \geqslant 5$, *then there is one labeled* T_0*-topology with* n *points and* 2^n *open sets,* $(n)_2$ *with* $(3/4)\,2^n$ *open sets,* $(n)_3$ *with* $(5/8)\,2^n$ *open sets,* $(5/6)(n)_4$ *with* $(9/16)\,2^n$, $(1/12)(n)_5$ *with* $(17/32)\,2^n$, $(n)_3 + (n)_4$ with $(1/2)\,2^n$, $(n)_5$ *with* $(15/32)\,2^n$, $(9/4)(n)_4 + (n)_5$ *with* $(7/16)\,2^n$, *and for each* $m = 6, 7, \ldots, n$, $2(n)_m/(m-1)!$ *with* $(2^{m-1} + 1)\,2^{n-m}$. *All the rest have* $<(7/16)\,2^n$ *open sets.*

THEOREM 2′. *If* $n \geqslant 3$, *then there are* $(1/2)(n)_2$ *labeled "non-*T_0*" topologies with n points and* $(1/2)\,2^n$ *open sets,* $(1/2)(n)_4 + (n)_3$ *with* $(3/8)\,2^n$ *open sets, and all the rest have* $<(3/8)\,2^n$ *open sets.*

THEOREM 3′. *Let* X *be a* T_0*-space of order* m, *with* Aut X *of order* g. *The number of labeled topologies* Y *of order* n *such that* $\hat{Y}$ *is homeomorphic to* X *is the coefficient of* $x^n/n!$ *in the expansion of* $(1/g)(e^x - 1)^m$.

EXAMPLES. (1′) Let X be the space of Example (1). Then $m = 3, g = 2$, and

$$\tfrac{1}{2}(e^x - 1)^3 = \sum_{n=3}^{\infty} \tfrac{1}{2}(3^n - 3 \cdot 2^n + 3)(x^n/n!).$$

(2′) Let Y be a chain topology, with Y having m points. Here $g = 1$, and the number of labeled chain topologies with n points and m non-empty open sets is the coefficient of $x^n/n!$ in the expansion of $(e^x - 1)^m$, an observation of Stephen [5]. The total number of labeled chain topologies with n points is the coefficient of $x^n/n!$ in the expansion of

$$\sum_{m=0}^{\infty} (e^x - 1)^m = 1/(2 - e^x).$$

A labeled chain topology may also be regarded as an *ordered set partition* or *preferential arrangement*. Preferential arrangements are discussed by Gross [2].

THEOREM 4′. *If $n \geqslant 2$, then there are $n!$ labeled T_0-topologies with n points and $n + 1$ open sets, $(1/2)(n - 1)\, n!$ with $n + 2$ open sets, $(1/8)(n - 2)(n + 5)\, n!$ with $n + 3$ open sets, and all the rest have $>n + 3$ open sets.*

THEOREM 5′. *If $n \geqslant 4$, then there is one labeled* "non-T_0" *topology with n points and two open sets, $2^n - 2$ with three open sets,* $(1/2)(2 \cdot 3^n - 5 \cdot 2^n + 4)$ with four open sets, $4^n - 3 \cdot 3^n + 2^n - 3$ *with five open sets, and all the rest have > 5 open sets.*

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