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MAGIC LABELINGS OF GRAPHS, SYMMETRIC MAGIC SQUARES, SYSTEMS OF PARAMETERS, AND COHEN-MACAULAY RINGS

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1. Introduction.

Let Γ be a finite graph allowing loops and multiple edges, so that Γ is a *pseudograph* in the terminology of [5]. Let $E = E(\Gamma)$ denote the set of edges of Γ and $\mathbf{N}$ the set of non-negative integers. A *magic labeling of* Γ *of index* r is an assignment $L : E \to \mathbf{N}$ of a non-negative integer $L(e)$ to each edge e of Γ such that for each vertex v of Γ, the sum of the labels of all edges incident to v is r (counting each loop at v once only). We will assume that we have chosen some fixed ordering $e_1, e_2, \cdots, e_q$ of the edges of Γ; and we will identify the magic labeling L with the vector $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_q) \in \mathbf{N}^q$, where $\alpha_i = L(e_i)$.

Let $H_\Gamma(r)$ denote the number of magic labelings of Γ of index r. It may happen that there are edges e of Γ that are always labeled 0 in any magic labeling. If these edges are removed, we obtain a pseudograph Δ satisfying the two conditions: (i) $H_\Gamma(r) = H_\Delta(r)$ for all $r \in \mathbf{N}$, and (ii) some magic labeling L of Δ satisfies $L(e) > 0$ for every edge e of Δ. We call a pseudograph Δ satisfying (ii) a *positive pseudograph*. By (i) and (ii), in studying the function $H_\Gamma(r)$ it suffices to assume that Γ is positive. A magic labeling L of Γ satisfying $L(e) > 0$ for all edges $e \in E(\Gamma)$ is called a *positive magic labeling*. Any undefined graph theory terminology used in this paper may be found in any textbook on graph theory, e.g., [5].

In [14] the following two theorems were proved.

THEOREM 1.1. [14, Thm. 1.1]. *Let Γ be a finite pseudograph. Then either $H_\Gamma(r) = \delta_{0r}$ (the Kronecker delta), or else there exist polynomials $P_\Gamma(r)$ and $Q_\Gamma(r)$ such that $H_\Gamma(r) = P_\Gamma(r) + (-1)^r Q_\Gamma(r)$ for all $r \in \mathbf{N}$.*

THEOREM 1.2 [14, Prop. 5.2]. *Let Γ be a finite positive pseudograph with at least one edge. Then* $\deg P_\Gamma(r) = q - p + b$, *where q is the number of edges of Γ, p the number of vertices, and b the number of connected components which are bipartite.*

For reasons which will become clear shortly, we define the *dimension* of Γ, denoted $\dim \Gamma$, by $\dim \Gamma = 1 + \deg P_\Gamma(r)$. In [14, p. 630], the problem was raised of obtaining a reasonable upper bound on $\deg Q_\Gamma(r)$. It is trivial that $\deg Q_\Gamma(r) \leq \deg P_\Gamma(r)$, and [14, Cor. 2.10] gives a condition for $Q_\Gamma(r) = 0$. Empirical evidence suggests that if Γ is a "typical" pseudograph, then $\deg Q_\Gamma(r)$

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will be considerably smaller than $\deg P_\Gamma(r)$. In this paper we will give a rigorous justification of this empirical fact. We will give an upper bound for $\deg Q_\Gamma(r)$ which we believe to be the best possible "theoretical" upper bound. (The degree of $Q_\Gamma(r)$ may be smaller than this upper bound because of "accidents" in the structure of Γ. See Example 3.2 for an illustration of what we mean by the term "accident.") The upper bound we obtain depends on analyzing a certain commutative ring R^Γ associated with Γ. We will try to provide a reasonable amount of ring-theoretic background for the reader unfamiliar with commutative algebra.

When Γ is the complete graph on n vertices with one loop at each vertex, $H_\Gamma(r)$ is the number $S_n(r)$ of $n \times n$ symmetric matrices of non-negative integers such that every row (and therefore every column) sums to r. Using a combinatorial argument whose basic idea was kindly supplied to this writer by Daniel Kleitman, we can transform our bound on $\deg Q_\Gamma(r)$, which depends in a rather complicated way on the structure of Γ, into an explicit integer. We obtain the result that $S_n(r) = P_n(r) + (-1)^r Q_n(r)$, where

$$\deg P_n(r) = \binom{n}{2} \quad \text{and} \quad \deg Q_n(r) \le \binom{n-1}{2} - 1$$

if n is odd, while

$$\deg Q_n(r) \le \binom{n-2}{2} - 1$$

if n is even. We conjecture that equality holds for all n. This conjecture is true for $n \le 5$.

It is more convenient to work with the generating function $F_\Gamma(\lambda) = \sum_{r=0}^{\infty} H_\Gamma(r)\lambda^r$ than with the function $H_\Gamma(r)$ itself. Using the fact that the ring R^Γ is a Cohen-Macaulay ring (which follows from a result of M. Hochster), we are able to obtain information on the coefficients of certain polynomials associated with $F_\Gamma(\lambda)$. For instance, we are able to show that $F_\Gamma(\lambda)(1 - \lambda^2)^d$ is a polynomial with non-negative integer coefficients, where $d = \dim \Gamma$.

2. Some ring theory background. Let Γ be a finite pseudograph with edge set $E = E(\Gamma) = \{x_1, x_2, \cdots, x_q\}$. Regard the x_i's as independent indeterminates and let R denote the polynomial ring $R = \mathbf{C}[x_1, \cdots, x_q]$, where $\mathbf{C}$ denotes the complex numbers. (We could use any infinite field in place of $\mathbf{C}$, but for definiteness we will use $\mathbf{C}$.) Let R^Γ denote the subring of R generated by all monomials $x_1^{\alpha_1} \cdots x_q^{\alpha_q}$, where $\alpha = (\alpha_1, \cdots, \alpha_q)$ is a magic labeling of Γ. For short we write $\mathbf{x}^\alpha = x_1^{\alpha_1} \cdots x_q^{\alpha_q}$. Thus since the sum $\alpha + \beta$ of two magic labelings α and β of Γ is also magic, it follows that the monomials $\mathbf{x}^\alpha$, where α is a magic labeling of Γ, form a vector space basis for R^Γ.

We want to investigate the structure of the ring R^Γ. First we will review certain relevant facts from commutative ring theory. Most of these facts are well-known and can be found in a number of references, of which [1], [3], [9], [11], [12], [13], [17], are a sample. Results which we shall need which can be found in these references we will merely state without proof; a few results which do not explicitly appear in these references we will prove. We shall restrict our attention to certain kinds of rings which we call "G-algebras", though some of our results on G-algebras are actually valid for more general classes of rings. There is a well-known analogy between the theory of G-algebras and the theory of local rings. Thus many of the references which we shall give for results on G-algebras actually do not refer to G-algebras as such, but to local rings. If one replaces "local ring" by "G-algebra", "the maximal ideal m of a local ring R" by "the irrelevant ideal $A_1 + A_2 + \cdots$ of a G-algebra $A = A_0 + A_1 + A_2 + \cdots$", "ideal" by "homogeneous ideal", etc., the theorems and their proofs remain valid.

We proceed to define the concept of a G-algebra. By a *graded ring*, we mean a commutative ring A with identity whose additive group has a direct sum decomposition $A = A_0 + A_1 + \cdots$ such that $A_iA_j \subset A_{i+j}$. If in addition A_0 is a field k, so that A is a k-algebra, and if A is finitely-generated as a k-algebra (so that A is Noetherian), then we say that A is a *G-algebra*. We can make the ring R^Γ defined above into a G-algebra by defining $R_r{}^\Gamma$ to be the vector space spanned by all monomials $\mathbf{x}^\alpha$ such that α is a magic labeling of index r.

If $A = A_0 + A_1 + \cdots$ is a G-algebra, we say that an element x of A is *homogeneous* if $x \in A_r$ for some $r \in \mathbf{N}$; and we say that x has *degree* r, written $\deg x = r$. In particular, deg 0 is arbitrary. An ideal I of A is said to be *homogeneous* if it is generated by homogeneous elements of A. The assumption that a G-algebra is finitely-generated implies that each A_r is a finite-dimensional vector space over $k = A_0$. The *Hilbert function* $H_A : \mathbf{N} \to \mathbf{N}$ of A is defined by $H_A(r) = \dim_k A_r$. Thus for the G-algebra structure we have defined on R^Γ, we have $H_{R^\Gamma}(r) = H_\Gamma(r)$, the number of magic labelings of Γ of index r.

If A is a G-algebra, the *Poincaré series* $F_A(\lambda)$ is a formal power series with integral coefficients in the variable λ defined by $F_A(\lambda) = \sum_{r=0}^{\infty} H_A(r)\lambda^r$. It is well-known that $F_A(\lambda)$ is a rational function of λ [1, Thm. 11.1] [13, Cor. 4.3]. If Γ is a finite positive pseudograph, we abbreviate $F_{R^\Gamma}(\lambda)$ to $F_\Gamma(\lambda)$. It follows from Theorem 1.1 that $F_\Gamma(\lambda)$ has the form

$$(1) \qquad F_\Gamma(\lambda) = \frac{W_\Gamma(\lambda)}{(1-\lambda)^d(1+\lambda)^s},$$

where $d, s \in \mathbf{N}$ and where $W_\Gamma(\lambda)$ is a polynomial in λ with integral coefficients satisfying (a) $W_\Gamma(1) \neq 0$ and (b) $W_\Gamma(-1) \neq 0$ if $s > 0$. Thus $d = 1 + \deg P_\Gamma(r) = \dim \Gamma$, and $s = 1 + \deg Q_\Gamma(r)$ (where we set the degree of the polynomial 0 to be -1). We call s the *subdimension* of Γ, denoted $s = sdm\ \Gamma$.

A fundamental result of commutative algebra [1, Thm. 11.14] [3, Thm. 2.3] [12, p. III-7, Thm. 1] [13, Thm. 5.5] states the following

PROPOSITION 2.1. *Let A be a G-algebra. Then the following three numbers are finite and equal:*

(i) *The length of a longest chain of prime ideals of A,*

(ii) *The maximum number of elements of A (which can be chosen to be homogeneous) which are algebraically independent over $k = A_0$, and*

(iii) *the order to which $\lambda = 1$ is a pole of the Poincaré series $F_A(\lambda)$.*

The integer defined by Proposition 2.1 is known as the *Krull dimension* of A and is denoted dim A. (The Krull dimension "dim" is not to be confused with the vector space dimension "$\dim_k$".) There follows immediately from Proposition 2.1 and our observations about $F_\Gamma(\lambda)$ the next result:

COROLLARY 2.2. *Let Γ be a finite pseudograph. Then* $\dim R^\Gamma = \dim \Gamma$. *Thus if Γ is positive with at least one edge, then* $\dim R^\Gamma = q - p + b + 1$, *in the notation of Theorem* 1.2.

Corollary 2.2 of course explains our reason for the notation "dim Γ". We now come to another basic result in commutative algebra [12, p. III-11] [13, §6].

PROPOSITION 2.3. *Let A be a G-algebra, and let $\theta_1, \theta_2, \cdots, \theta_d$ be homogeneous elements of A of positive degree. The following five conditions are equivalent:*

(i) $d = \dim A$ *and* $\dim A/(\theta_1, \cdots, \theta_d) = 0$.

(ii) $d = \dim A$ *and* $\dim_k A/(\theta_1, \cdots, \theta_d) < \infty$ *(recall that* $\dim_k$ *denotes dimension as a vector space over k, not Krull dimension),*

(iii) *For any subset* $\{\theta_{i_1}, \cdots, \theta_{i_j}\}$ *of* $\{\theta_1, \cdots, \theta_d\}$, $\dim A/(\theta_{i_1}, \cdots, \theta_{i_j}) = \dim A - j$; *and* $\dim A = d$.

(iv) *$\theta_1, \theta_2, \cdots, \theta_d$ are algebraically independent over k and A is a finitely-generated module over the polynomial subring $k[\theta_1, \cdots, \theta_d]$.*

(v) *$\theta_1, \theta_2, \cdots, \theta_d$ are algebraically independent over k and A is integral over the subring $B = k[\theta_1, \cdots, \theta_d]$ (i.e., every element of A satisfies a monic polynomial with coefficients in B).*

A set $\theta_1, \theta_2, \cdots, \theta_d$ of homogeneous elements of positive degree satisfying any one of the above five (equivalent) conditions is known as a *homogeneous system of parameters* (h.s.o.p.) for A. Every G-algebra A possesses an h.s.o.p. (e.g., [1, p. 69, Ex. 19] [12, p. III-20, Thm. 2] [13, Thm. 5.4]). If $\theta_1, \cdots, \theta_i$ belong to some h.s.o.p., we call $\theta_1, \cdots, \theta_i$ a *partial* h.s.o.p. If θ belongs to some h.s.o.p., then we call θ a *parameter*. A necessary and sufficient condition that a set $\theta_1, \cdots, \theta_i$ of homogeneous elements of A of positive degree be a partial h.s.o.p. is that $\dim A/(\theta_1, \cdots, \theta_i) = \dim A - i$ (e.g., [12, p. III-11, Prop. 6]).

PROPOSITION 2.4. *Let A be a G-algebra, and let $\theta_1, \cdots, \theta_d$ be an h.s.o.p., say with* $\deg \theta_i = e_i$. *Then the Poincaré series $F_A(\lambda)$ can be written in the form*

$$F_A(\lambda) = V_A(\lambda) \Big/ \prod_{i=1}^{d} (1 - \lambda^{e_i}),$$

where $V_A(\lambda)$ is a polynomial in λ with integer coefficients.

Proof. By Proposition 2.3, A is a finitely-generated (graded) module over the polynomial ring $k[\theta_1, \cdots, \theta_d]$. The proof now follows from [1, Thm. 11.1] or [13, Thm. 4.2].

If $\mathfrak{P}$ is a prime ideal of a G-algebra A, define the *height* of $\mathfrak{P}$ (sometimes called the *rank* of $\mathfrak{P}$), denoted $ht\ \mathfrak{P}$, to be the length of the longest chain of prime ideals of A whose maximum element is $\mathfrak{P}$. (Equivalently, $ht\ \mathfrak{P} = \dim A_{\mathfrak{P}}$, where $A_{\mathfrak{P}}$ is the localization of A at $\mathfrak{P}$.) Thus $ht\ \mathfrak{P} = 0$ if and only if $\mathfrak{P}$ is a minimal prime of A. If I is any ideal of A, define $ht\ I = \inf ht\ \mathfrak{P}$, where the inf is taken over all prime ideals $\mathfrak{P}$ of A which are minimal with respect to containing I.

Let I be a homogeneous ideal of a G-algebra A. Besides $ht\ I$, we wish to consider two other numerical invariants of I. Define quo I, the *quotient height* of I, by quo $I = \dim R - \dim R/I$. Also define par I to be the cardinality of the largest partial h.s.o.p. contained in I.

PROPOSITION 2.5. *Let I be a homogeneous ideal of a G-algebra A. Then* $ht\ I \leq$ quo $I =$ par I.

Proof. The inequality $ht\ I \leq$ quo I is well-known and easy to prove. Namely, let $\mathfrak{P}$ be a prime ideal containing I such that $\dim R/\mathfrak{P} = \dim R/I$ (such a $\mathfrak{P}$ exists since the primes in R/I are just the images of primes in R containing I). Then $\dim R/I + ht\ \mathfrak{P} = \dim R/\mathfrak{P} + ht\ \mathfrak{P} \leq \dim R$ (see [12, p. III-21]) and $ht\ \mathfrak{P} \geq ht\ I$. Thus $\dim R \geq \dim R/I + ht\ I$, which is equivalent to $ht\ I \leq$ quo I.

Suppose $\theta_1, \cdots, \theta_i$ is a partial h.s.o.p. contained in I. By Proposition 2.3, $\dim A/(\theta_1, \cdots, \theta_i) = \dim A - i$, so *a fortiori* $\dim A/I \leq \dim A - i$. Thus par $I \leq$ quo I.

It remains to show par $I \geq$ quo I. If quo $I = 0$ there is nothing to prove. Now suppose that quo $I \geq 1$ and par $I = 0$. Thus for all homogeneous $x \in I$, $\dim A/(x) = \dim A$. This means each homogeneous $x \in I$ is contained in a prime ideal $\mathfrak{P}$ of A, necessarily minimal, such that quo $\mathfrak{P} = 0$. Since a Noetherian ring contains only finitely many minimal primes (e.g., [9, Thm. 88]), we have that the set I_h of homogeneous elements of I is contained in a set union $\mathfrak{P}_1 \cup \mathfrak{P}_2 \cup \cdots \cup \mathfrak{P}_j$ of prime ideals $\mathfrak{P}_1, \mathfrak{P}_2, \cdots, \mathfrak{P}_j$. A straightforward modification of an argument in [9, Thm. 81] or [13, Lemma 5.1] shows that then I_h is contained in some $\mathfrak{P}_i$. Namely, we argue by induction on j. For every i we may assume $I_h \not\subset \mathfrak{P}_1 \cup \cdots \cup \hat{\mathfrak{P}}_i \cup \cdots \cup \mathfrak{P}_j$, where the notation $\hat{\mathfrak{P}}_i$ means that $\mathfrak{P}_i$ is omitted. Pick $y_i \in I_h$ but not in $\mathfrak{P}_1 \cup \cdots \cup \hat{\mathfrak{P}}_i \cup \cdots \cup \mathfrak{P}_j$. The desired result is trivial for $j = 1$. For $j \geq 2$, let $a = \deg y_1$ and $b = \deg y_2 y_3 \cdots y_j$, and set $y = y_1^b + (y_2 y_3 \cdots y_j)^a$. Then $y \in I_h$ but y lies in none of the $\mathfrak{P}_i$'s, a contradiction. Thus $I_h \subset \mathfrak{P}_i$ for some i. Since I is homogeneous, $I \subset \mathfrak{P}_i$. Thus quo $I = 0$, contradicting the assumption that quo $I \geq 1$. Hence if *quo* $I \geq 1$, then par $I \geq 1$.

The proof now proceeds by induction on quo I. By the above paragraph, we are done if quo $I = 1$. Assume quo $I > 1$. By the above, I contains a homogeneous parameter θ. Let $S = R/(\theta)$ and $J = I/(\theta)$. By Proposition 2.3, $\dim S = \dim R - 1$. Moreover $S/J \cong R/I$, so quo $J = \dim S - \dim S/J =$

dim R − dim R/I − 1 = quo I − 1. By induction we may assume J contains a partial h.s.o.p. of cardinality quo I − 1. Lifting these parameters back to I and adjoining θ, we obtain a partial h.s.o.p. in I of cardinality quo I. Hence par $I \geq$ quo I, and the proof is complete.

Note. We will only need the equality quo I = par I of Proposition 2.5, but we have added the inequality involving $ht\ I$ for the sake of completeness. Also for the sake of completeness we include the next proposition.

PROPOSITION 2.6. *Let A be a G-algebra, and suppose that A is also an integral domain. Let I be a homogeneous ideal of A. Then $ht\ I$ =* quo I = par I.

Proof. By Proposition 2.5, it suffices to show that $ht\ I$ = quo I. Now any integral domain B which is a finitely-generated algebra over a field k has the property that all maximal chains of prime ideals have length equal to dim B (e.g., [12, Cor. 2, p. III-24]). Hence $ht\ \mathfrak{P}$ + dim $A/\mathfrak{P}$ = dim A for every prime ideal $\mathfrak{P}$ of A. Thus $ht\ I$ = inf ($ht\ \mathfrak{P}$) = inf (dim R − dim $R/\mathfrak{P}$) = dim R − sup dim $R/\mathfrak{P}$ = dim R − dim R/I = quo I, where the inf's and sup's are over all primes minimal over I. This completes the proof.

We need some information on the degrees of the elements of a system of parameters for a G-algebra A. We will prove a somewhat stronger result (Proposition 2.9) than we need for the time being, since we will require such a result in Section 5. An even stronger result can be proved, but Proposition 2.9 is adequate for our purposes. Proposition 2.9 may be regarded as an elaboration of the well-known fact (see, e.g., [1, p. 69, Ex. 16]) that if k is infinite and A is generated by A_1, then A possesses an h.s.o.p. $\theta_1, \cdots, \theta_d$ such that eachdeg $\theta_{1i} = 1$ We first require two lemmas.

LEMMA 2.7. *Let A be a G-algebra, and let $I \subset J$ be homogeneous ideals. Let $B = A/I$, and let $\bar{J}$ denote the image of J in B. Then par $\bar{J}$ = par J − par I.*

Proof. Using Proposition 2.5 and the identity $B/\bar{J} \cong A/J$, we have par $\bar{J}$ = dim B − dim $B/\bar{J}$ = (dim A − par I) − dim A/J = (dim A − dim A/J) − par I = par J − par I. This completes the proof.

LEMMA 2.8. *Let k be an infinite field, and let V be a finite-dimensional vector space over k. If $S_1, \cdots, S_m$ are subsets of V whose set-union is V, then some S_i contains a basis for V.*

Proof. Let r = dim V. We can find an infinite sequence $v_1, v_2, \cdots$ of elements of V such that any r of them form a basis for V, since choosing v_{i+1} once $v_1, v_2, \cdots, v_i$ have been chosen merely involves avoiding the zeroes of finitely many polynomials with coefficients in k. Then one of the S_i must contain r of the v_i's (in fact, infinitely many of them), so the proof is complete.

PROPOSITION 2.9. *Let A be a G-algebra such that k $(= A_0)$ is infinite. Let $I_1, \cdots, I_s$ be a sequence of homogeneous ideals of A such that each I_i is generated* *by homogeneous elements all of the same degree, say* d_i. *Furthermore assume that* d_{i-1} *divides* d_i *for* $2 \leq i \leq s$. *Let* $J_i = I_1 + I_2 + \cdots + I_i$, *and let* $p_i = \operatorname{par} J_i$. *Then* A *possesses a partial h.s.o.p.* $\theta_1, \theta_2, \cdots, \theta_{p_s}$ *such that*

$$\deg \theta_{p_{i-1}+1} = \deg \theta_{p_{i-1}+2} = \cdots = \deg \theta_{p_i} = d_i, \qquad 1 \leq i \leq s,$$

where by convention $p_0 = 0$.

Proof. The proof is by induction on p_s. The theorem is trivial for $p_s = 0$. Now assume the theorem for $p_s < p$, say, and suppose we are dealing with the situation $p_s = p > 0$. Let K_j be the ideal of A generated by the set of all x^{d_j/d_i}, where for some $i \leq j$, x is an element of I_i of degree d_i. Define $K_m = K_j + I_{j+1} + I_{j+2} + \cdots + I_m$, $j < m \leq s$.

We claim that $\operatorname{par} K_m = \operatorname{par} J_m$, $j \leq m \leq s$. Let $B = A/K_m$. Let $\bar{J}_m$ be the image of J_m in B. Since $K_m \subset J_m$, by Lemma 2.7 we have $\operatorname{par} K_m = \operatorname{par} J_m - \operatorname{par} \bar{J}_m$. But every element of $\bar{J}_m$ is nilpotent, so $\operatorname{par} \bar{J}_m = 0$. This proves the claim.

Now let j be the least integer for which $\operatorname{par} K_j > 0$, and suppose that K_j is generated by homogeneous elements $y_1, y_2, \cdots, y_q$, all of degree d_j. We now claim that some linear combination $\theta_1 = \sum \alpha_i y_i$, $\alpha_i \in k$, is a parameter. Otherwise each such θ_1 belongs to a minimal prime ideal $\mathfrak{P}$ of A satisfying $\operatorname{par} \mathfrak{P} = 0$. Since there are only finitely many minimal primes in a Noetherian ring, by Lemma 2.8 some minimal prime $\mathfrak{P}$ satisfying $\operatorname{par} \mathfrak{P} = 0$ contains a basis for the vector space spanned by the y_i's. Since $\mathfrak{P}$ is an ideal, we get $K_j \subset \mathfrak{P}$, contradicting $\operatorname{par} K_j > 0$. This proves the claim.

Let $C = A/(\theta_1)$, where θ_1 is the element constructed in the previous paragraph. Since θ_1 is homogeneous, C becomes a G-algebra by letting C_r be the image of A_r. Let $\bar{I}$ denote the image in C of an ideal I of A. Then $\bar{K}_j, \bar{I}_{j+1}, \bar{I}_{j+2}, \cdots, \bar{I}_s$ is a sequence of homogeneous ideals of C such that $\bar{K}_j$ (respectively, $\bar{I}_i$) is generated by homogeneous elements all of degree d_j (respectively, d_i). Moreover, $\bar{K}_i = \bar{K}_j + \bar{I}_{j+1} + \cdots + \bar{I}_i$, $j \leq i \leq s$. Letting $I = (\theta_1)$ and $J = \bar{K}_i$ in Lemma 2.7, we have $\operatorname{par} \bar{K}_i = \operatorname{par} K_i - 1$, $j \leq i \leq s$. By the induction hypothesis, C possesses a partial h.s.o.p. $\bar{\theta}_2, \cdots, \bar{\theta}_{p_s}$ such that $\deg \bar{\theta}_2 = \deg \bar{\theta}_3 = \cdots = \deg \bar{\theta}_{p_1} = d_1$ and $\deg \bar{\theta}_{p_{i-1}+1} = \deg \bar{\theta}_{p_{i-1}+2} = \cdots = \deg \bar{\theta}_{p_i} = d_i$, $2 \leq i \leq s$. Lifting $\bar{\theta}_2, \cdots, \bar{\theta}_{p_s}$ back to homogeneous elements $\theta_2, \cdots, \theta_{p_s}$ of A and adjoining θ_1, we obtain our desired partial h.s.o.p. $\theta_1, \theta_2, \cdots, \theta_{p_s}$. This completes the proof.

COROLLARY 2.10. *Let* A *be a* G*-algebra where* $k = A_0$ *is infinite, and let* I *be a homogeneous ideal of* A. *Let* $\operatorname{par} I = p$, *and suppose that* $y_1, y_2, \cdots, y_t$ *is a homogeneous set of generators for* I *(as an ideal of* A*). Let* $e_i = \deg y_i$, *and let* N *be the least common multiple of* $e_1, \cdots, e_t$. *Then* I *contains a partial h.s.o.p.* $\theta_1, \cdots, \theta_p$ *of cardinality* p, *such that each* θ_i *is of degree* N.

Proof. Let I_1 be the ideal of A generated by the elements y_i^{N/e_i}. Let $\bar{I} = I/I_1$ denote the image of I in A/I_1. Then every element of $\bar{I}$ is nilpotent, so par $\bar{I} = 0$. It follows from Lemma 2.7 that par I_1 = par I. The proof now follows from the case $s = 1$ of Proposition 2.9.

3. The formal subdimension of Γ.

We are now ready to resume our discussion of magic labelings.

Definition. Let Γ be a finite pseudograph, and let R^Γ be the ring defined in the previous section. Let I^Γ be the ideal of R^Γ generated by all monomials $\mathbf{x}^\alpha$, where α is a magic labeling of Γ of odd index (i.e., $\mathbf{x}^\alpha \in R_j{}^\Gamma$ for some odd integer j, where $R^\Gamma = R_0{}^\Gamma + R_1{}^\Gamma + \cdots$ is the grading defined in the previous section). The *formal subdimension* of Γ, denoted fsd Γ, is defined by

$$(2) \qquad fsd\ \Gamma = \dim \Gamma - \text{par}\ I^\Gamma.$$

THEOREM 3.1. *Let Γ be a finite pseudograph. Then $sdm\ \Gamma \leq fsd\ \Gamma$.*

Proof. Let s = par I^Γ. By Corollary 2.10, we can find a partial h.s.o.p. $\theta_1, \theta_2, \cdots, \theta_s$ of R^Γ such that each θ_i has degree equal to the least common multiple of the degrees of the generators of I^Γ. By assumption these generators all have odd degree, so each θ_i has odd degree N. Extend $\theta_1, \cdots, \theta_s$ to an h.s.o.p. $\theta_1, \cdots, \theta_d$, where $d = \dim \Gamma$. Let $e_i = \deg \theta_i$ for $s + 1 \leq i \leq d$. By Proposition 2.4, we have

$$(3) \qquad F_\Gamma(\lambda) = \sum_{n=0}^{\infty} H_\Gamma(r)\lambda^r = V_\Gamma(\lambda)/(1 - \lambda^N)^s \prod_{i=s+1}^{d} (1 - \lambda^{e_i}),$$

where $V_\Gamma(\lambda) \in \mathbf{Z}[\lambda]$. Then since N is odd, we have by (3)

$$sdm\ \Gamma \leq d - s = \dim \Gamma - \text{par}\ I^\Gamma.$$

This completes the proof.

We believe that Theorem 3.1 provides the best possible "theoretical" upper bound for $sdm\ \Gamma$ (and hence for deg $Q_\Gamma(r)$, since $1 + \deg Q_\Gamma(r) = sdm\ \Gamma$). In other words, if Γ satisfies $sdm\ \Gamma < fsd\ \Gamma$, this is because of very special properties of Γ which cannot be explained in a general way. Thus we believe that a "typical" pseudograph Γ satisfies $sdm\ \Gamma = fsd\ \Gamma$. Of course we are speaking heuristically when we use the term "typical".

Example 3.2. We give an example where $sdm\ \Gamma < fsd\ \Gamma$, and we explain why this strict inequality is due to "accidental" properties of Γ. Let Γ be the pseudograph (actually a graph) of Figure 1. Define the magic labelings

$\alpha^1 = (1, 1, 1, 0, 0, 2, 0, 0, 1, 1, 1)$, $\quad \alpha^2 = (1, 0, 0, 0, 1, 0, 1, 0, 0, 1, 0)$,

$\alpha^3 = (0, 1, 0, 1, 0, 0, 0, 1, 0, 0, 1)$, $\quad \alpha^4 = (0, 1, 0, 1, 0, 0, 1, 0, 0, 1, 0)$,

$\alpha^5 = (1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1)$.

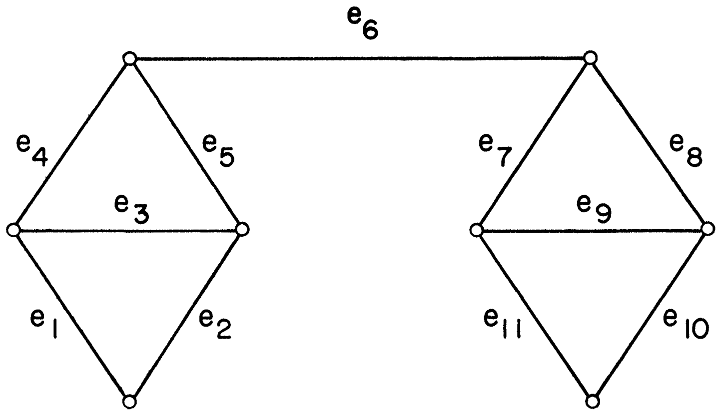

FIGURE 1

For convenience set $y_i = \mathbf{x}^{\alpha^i}$. Then a minimal set of generators for R^Γ (as an algebra over $\mathbf{C}$) consists of $y_1, y_2, \cdots, y_5$. The ideal I^Γ in the definition of *fsd* Γ is generated by y_2, y_3, y_4, y_5. It is easy to check that dim $\Gamma = 4$ and par $I^\Gamma = 3$. Hence *fsd* $\Gamma = 1$, so we would expect $F_\Gamma(\lambda)$ to have a simple pole at $\lambda = -1$. However, in fact $F_\Gamma(\lambda)$ is analytic at $\lambda = -1$. To see this, note that all relations among the generators $y_1, \cdots, y_5$ are consequences of $y_2y_3 = y_4y_5$. Hence since deg $y_1 = 2$ and deg $y_2 =$ deg $y_3 =$ deg $y_4 =$ deg $y_5 = 1$, we have

$$F_\Gamma(\lambda) = \frac{1 - \lambda^2}{(1 - \lambda)^4(1 - \lambda^2)} = \frac{1}{(1 - \lambda)^4}.$$

It is merely an "accident" that the relation between y_2, y_3, y_4, y_5, giving rise to a factor $1 - \lambda^2$ in the numerator, cancels the factor $1 - \lambda^2$ in the denominator coming from the generator y_1. There is no "theoretical" reason why y_1 should be related to y_2, y_3, y_4, y_5 in this way; indeed, y_1 is algebraically independent of y_2, y_3, y_4, y_5.

There is another way to view the above example. An h.s.o.p. for R^Γ can be taken to be $\theta_1 = y_1, \theta_2 = y_2, \theta_3 = y_3, \theta_4 = y_4 + y_5$. Now by Proposition 2.3, R^Γ is a finitely-generated module over the polynomial ring $\mathbf{C}[\theta_1, \theta_2, \theta_3, \theta_4]$. In fact, R^Γ is a *free* module with generators 1 and y_4. (For the significance of R^Γ being free, see Proposition 4.1.) Thus we get

$$F_\Gamma(\lambda) = \frac{\lambda^{\deg 1} + \lambda^{\deg y_4}}{\prod_{i=1}^{4} (1 - \lambda^{\deg \theta_i})} = \frac{1 + \lambda}{(1 - \lambda^2)(1 - \lambda)^3} = \frac{1}{(1 - \lambda)^4}.$$

Again, it is an "accident" that the factor $1 + \lambda$ in the numerator, coming from the module generators 1 and y_4, cancels the factor $1 - \lambda^2$ coming from the parameter θ_1.

Remark. The reader familiar with [14] may wish to know its relationship to the present paper. Although stated differently, Proposition 2.7 of [14] asserts essentially that if J is the ideal of R^Γ generated by all monomials $\mathbf{x}^\alpha$ where α is magic of index two, then par J = dim Γ. It then follows immediately from Proposition 2.4 of this paper that $F_\Gamma(\lambda)$ has the form (1). In [14], Proposition 2.4 of this paper has been replaced by Theorem 2.5.

Theorem 3.1 gives us a bound for *sdm* Γ, but it is not very satisfactory since it leaves open the problem of computing *fsd* Γ. We would like a purely combinatorial description of *fsd* Γ in terms of the structure of Γ. Such a description is provided by the next result.

THEOREM 3.3. *Let Γ be a finite pseudograph. Then fsd Γ =* max (dim Δ), *where Δ ranges over all positive spanning sub-pseudographs of Γ which do not possess a magic labeling of odd index.*

Note. The assumption in Theorem 3.3 that Δ is positive is clearly unnecessary, since any finite pseudograph Δ has the same dimension as its maximal spanning positive sub-pseudograph. The advantage of dealing only with positive Δ is that dim Δ (= dim R^Δ) can then be calculated by Corollary 2.2.

Proof. By Proposition 2.5 and the definition (2) of *fsd* Γ, we have

$$\text{fsd } \Gamma = \dim \Gamma - \text{quo } I^\Gamma = \dim R^\Gamma/I^\Gamma.$$

Set

$$S^\Gamma = R^\Gamma/I^\Gamma.$$

By Proposition 2.1, it follows that *fsd* Γ is the maximum number of (homogeneous) elements of S^Γ which are algebraically independent over $\mathbf{C}$. Now S^Γ is generated by monomials $\mathbf{x}^\alpha$, where α is a magic labeling of Γ. Thus *fsd* Γ is equal to the largest integer h for which there exist h magic labelings $\alpha_1, \cdots, \alpha_h$ of Γ such that $\mathbf{x}^{\alpha_1}, \cdots, \mathbf{x}^{\alpha_h}$ are algebraically independent over $\mathbf{C}$ in S^Γ. Now $\mathbf{x}^{\alpha_1}, \cdots, \mathbf{x}^{\alpha_h}$ will be algebraically independent in S if and only if the following two conditions are satisfied:

(i) If $a_1, \cdots, a_h$ are non-negative integers, the monomial $\mathbf{x}^{a_1\alpha_1+\cdots+a_h\alpha_h}$ does not lie in I^Γ. Equivalently, if α is a magic labeling of Γ, let supp α denote the set of edges of Γ on which α is positive and let $T = \bigcup_{i=1}^h \text{supp } \alpha_i$. Let Δ denote the spanning subgraph of Γ with edge set T. Then Δ has no magic labelings of odd index.

(ii) The vectors $\alpha_1, \cdots, \alpha_h$ are linearly independent over $\mathbf{Q}$.

Thus *fsd* Γ is the largest integer h obtained as follows: Δ is a positive spanning subgraph of Γ which does not possess a magic labeling of odd index, and $\alpha_1, \cdots, \alpha_h$ are magic labelings of Δ for which $\alpha_1, \cdots, \alpha_h$ are linearly independent over **Q**. But $\alpha_1, \cdots, \alpha_h$ are linearly independent over **Q** if and only if $\mathbf{x}^{\alpha_1}, \cdots, \mathbf{x}^{\alpha_h}$ are algebraically independent in R^Δ. The largest h for which $\mathbf{x}^{\alpha_1}, \cdots, \mathbf{x}^{\alpha_h}$ are algebraically independent in R^Δ is just dim Δ, so the proof follows.

4. A-sequences and Cohen-Macaulay rings. We know from Theorem 3.1 that if Γ is a finite pseudograph, then $F_\Gamma(\lambda) = W_\Gamma(\lambda)/(1 - \lambda)^d(1 + \lambda)^f$, where $d = \dim \Gamma$, $f = fsd\ \Gamma$, and $W_\Gamma(\lambda)$ is a polynomial in λ. In order to obtain more information about the generating function $F_\Gamma(\lambda)$, we need to introduce the theory of A-sequences and Cohen-Macaulay rings. If A is a G-algebra, a sequence of homogeneous elements $\theta_1, \theta_2, \cdots, \theta_r$ of A is said to be a *homogeneous A-sequence* if the following two conditions are satisfied:

(i) The ideal $(\theta_1, \theta_2, \cdots, \theta_r)$ is not all of A. Equivalently, deg $\theta_i > 0$ for $1 \leq i \leq r$.

(ii) If $1 \leq i \leq r$, then θ_i is not a zero-divisor modulo the ideal $(\theta_1, \theta_2, \cdots, \theta_{i-1})$.

Two well-known facts concerning homogeneous A-sequences are the following: Every permutation of a homogeneous A-sequence is a homogeneous A-sequence, and every homogeneous A-sequence is a partial h.s.o.p. If is not true, however, that an h.s.o.p. is a homogeneous A-sequence; and this fact leads to the next proposition.

PROPOSITION 4.1. *Let A be a G-algebra, and let $\theta_1, \cdots, \theta_d$ be an* h.s.o.p., *say with* deg $\theta_i = e_i$. *Let $B = A/(\theta_1, \cdots, \theta_d)$, endowed with the natural "quotient grading" (B_r is the image of A_r). The following four conditions are equivalent*:

(i) *$\theta_1, \cdots, \theta_d$ is an A-sequence,*

(ii) every h.s.o.p. *of A is an A-sequence,*

(iii) *A is a* free *module over the polynomial ring $k[\theta_1, \cdots, \theta_d]$ (recall from Proposition* 2.3 *that A is always a finitely-generated module over $k[\theta_1, \cdots, \theta_d]$).*

(iv) $$F_A(\lambda) = F_B(\lambda) \Big/ \prod_{i=1}^{d} (1 - \lambda^{e_i}).$$

If A satisfies any of the equivalent conditions of Proposition 4.1, then by definition A is a *Cohen-Macaulay* G-algebra. The various implications needed to prove Proposition 4.1 all can be found in the literature. The equivalence of (i) and (ii) appears, e.g., in [12, p. IV-20, Thm. 2]. Condition (iii) is mentioned in [7, p. 1036] and [13, Prop. 6.8]. Finally condition (iv) appears in [13, Cor. 6.9] and [15, Cor. 3.2].

The next result is a special case of a theorem first proved by M. Hochster [6, Thm. 1°]. Another proof appears in [10, p. 52]. Hochster's result is generalized in [8]. By using Theorem 4.2 and known properties of Cohen-Macaulay rings we could have simplified the proofs of Proposition 2.5 and Proposition 2.6 in the case $A = R^\Gamma$ (see, e.g., [11, (16.B)], but we felt it best to avoid the relatively deep Theorem 4.2 whenever possible.

THEOREM 4.2. *Let Γ be a finite pseudograph. Then R^Γ is Cohen-Macaulay.*

COROLLARY 4.3. *Let Γ be a finite pseudograph, and suppose that $\theta_1, \cdots, \theta_d$ is an* h.s.o.p. *for R^Γ with $e_i = \deg \theta_i$. Then the coefficients of the polynomial $V_\Gamma(\lambda) = F_\Gamma(\lambda) \prod_{i=1}^d (1 - \lambda^{e_i})$ are non-negative.*

Proof. Let $B = R^\Gamma/(\theta_1, \cdots, \theta_d)$. By Theorem 4.2 and Proposition 4.1 (iv), $V_\Gamma(\lambda) = F_B(\lambda) = \sum (\dim_{\mathbf{C}} B_r)\lambda^r$. This proves the corollary.

Corollary 4.3 expresses the coefficients of $V_\Gamma(\lambda)$ as dimensions of vector spaces. It would be desirable to obtain a more combinatorial interpretation of the coefficients (expressed directly in terms of Γ), but we have been unable to do so.

Corollary 4.3 raises the question of what integers $e_1, e_2, \cdots, e_d$ can be the degrees of the elements of an h.s.o.p. of R^Γ, where Γ is a pseudograph. A partial answer to this question may be deduced from Proposition 2.9 and is the subject of the next three propositions.

PROPOSITION 4.4. *Let Γ be a finiite pseudograph, and let $d = \dim \Gamma$. Then R^Γ possesses an* h.s.o.p. *$\theta_1, \theta_2, \cdots, \theta_d$ where* $\deg \theta_i = 2$ *for $1 \le i \le d$. Consequently the power series $F_\Gamma(\lambda)(1 - \lambda^2)^d$ is a polynomial with non-negative integer coefficients.*

Proof. Let I be the ideal of R^Γ generated by all monomials $\mathbf{x}^\alpha$,where α is a magic labeling of Γ of index two. It is an immediate consequence of [14, Prop. 2.7] that par $I = \dim \Gamma$. The proof now follows from Proposition 2.9 after setting $s = 1$, $I_1 = I$.

PROPOSITION 4.5. *Let Γ be a finite pseudograph with* $\dim \Gamma = d$, *and suppose that every magic labeling of Γ is a sum of magic labelings of index one. Then R^Γ possesses an* h.s.o.p. *$\theta_1, \theta_2, \cdots, \theta_d$ where* $\deg \theta_i = 1$ *for $1 \le i \le d$. Consequently the power series $F_\Gamma(\lambda)(1 - \lambda)^d$ is a polynomial with non-negative integer coefficients.*

Proof. Let J^Γ be the ideal of R^Γ generated by all monomials $\mathbf{x}^\alpha$, where α is a magic labeling of index one. By the assumption on Γ, J^Γ is the entire irrelevant ideal $R_1^\Gamma + R_2^\Gamma + \cdots$, so par $J^\Gamma = \dim \Gamma$. The proof now follows from Proposition 2.9 (or in fact directly from [1, Ex. 16, p. 69]) after setting $s = 1$, $I_1 = J^\Gamma$.

In [14, Prop. 2.9] a necessary and sufficient condition is given for Γ to satisfy the condition of Proposition 4.5. A sufficient condition is that Γ minus its loops be bipartite. Two special cases include: (a) Γ is the complete bipartite graph K_{nn}. Then $\dim \Gamma = (n - 1)^2 + 1$ and $H_\Gamma(r)$ is the number of $n \times n$ matrices of non-negative integers such that every row and column sum is equal to r. (b) Γ is K_{nn} with a loop adjoined to each vertex. Then $\dim \Gamma = n^2 + 1$ and $H_\Gamma(r)$ is the number of $n \times n$ matrices of non-negative integers such that every row and column sum is at most r.

PROPOSITION 4.6. *Let Γ be a finite pseudograph satisfying* $\dim \Gamma = d$ *and* fsd $\Gamma = f$. *Let J^Γ be the ideal of R^Γ generated by all monomials $\mathbf{x}^\alpha$, where α is a magic labeling of index one. Assume that $f = \dim \Gamma -$ par J^Γ (or equivalently,* par $J^\Gamma =$ par I^Γ, *with I^Γ as in (2)). Then R^Γ possesses an* h.s.o.p. $\theta_1, \cdots, \theta_d$

such that $\deg \theta_i = 1$ *if* $1 \leq i \leq d - f$ *and* $\deg \theta_i = 2$ *if* $d - f + 1 \leq i \leq d$. *Consequently, the power series* $F_\Gamma(\lambda)(1 - \lambda)^d(1 + \lambda)^f$ *is a polynomial with non-negative integer coefficients.* (*Of course even without the assumption* $f = \dim \Gamma - \operatorname{par} J^\Gamma$, *we know from Theorem* 3.1 *that* $F_\Gamma(\lambda)(1 - \lambda)^d(1 + \lambda)^f$ *is a polynomial with integer coefficients.*)

Proof. Let $I_1 = J^\Gamma$ and $I_2 = I$, where I is defined in the proof to Proposition 4.4. Since $\operatorname{par} I = \dim \Gamma$, the proof now follows from Proposition 2.9.

Proposition 4.6 raises the question of determining when a finite pseudograph Γ satisfies the condition $fsd\ \Gamma = \dim \Gamma - \operatorname{par} J^\Gamma$.

PROPOSITION 4.7. *Let* Γ *be a finite pseudograph, and let* J^Γ *be the ideal of* R^Γ *defined in Proposition* 4.6. *Define* $g = \max_\Delta (\dim \Delta)$, *where* Δ *ranges over all positive spanning subgraphs of* Γ *which do not possess a* 1-*factor. Then* $fsd\ \Gamma = \dim \Gamma - \operatorname{par} J^\Gamma$ *if and only if* $fsd\ \Gamma = g$.

Proof. By mimicking the proof of Theorem 3.3 we obtain $\dim \Gamma - \operatorname{par} J^\Gamma = g$. The proof now follows from Theorem 3.3.

Example 4.8. Let Γ be the pseudograph of Figure 2. Then $\dim \Gamma = 3$. By Corollary 4.5, the coefficients of $F_\Gamma(\lambda)(1 - \lambda^2)^3$ are non-negative. Indeed,

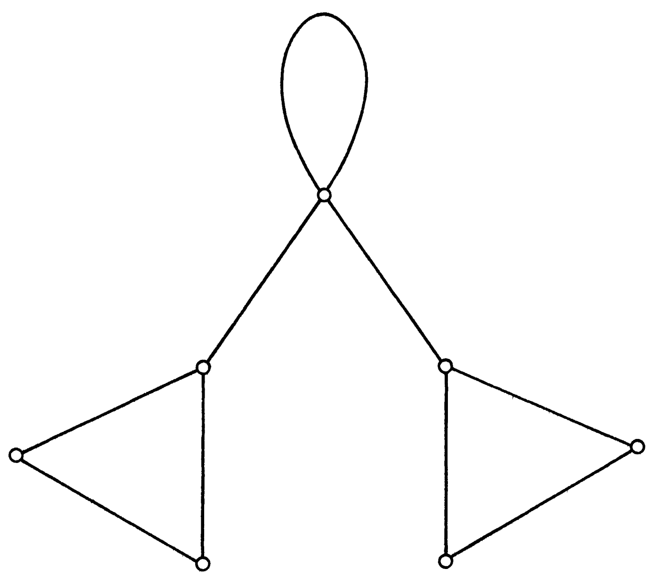

FIGURE 2

$F_\Gamma(\lambda)(1 - \lambda^2)^3 = 1 + \lambda^3$. One also can find an h.s.o.p. $\varphi_1, \varphi_2, \varphi_3$ such that $\deg \varphi_1 = \deg \varphi_2 = 2, \deg \varphi_3 = 3$. Indeed, $F_\Gamma(\lambda)(1 - \lambda^2)^2(1 - \lambda^3) = 1 + \lambda^2 + \lambda^4$, in accordance with Corollary 4.3. Moreover, *fsd* $\Gamma = 1$, so by Theorem 3.1, $F_\Gamma(\lambda)(1 - \lambda)(1 - \lambda^2)^2$ is a polynomial. In fact, this polynomial equals $1 - \lambda + \lambda^2$. Thus R^Γ does not possess an h.s.o.p. ψ_1, ψ_2, ψ_3 such that $\deg \psi_1 = 1$, $\deg \psi_2 = \deg \psi_3 = 2$. In fact, Γ has no magic labeling of index one.

In general it is difficult to tell whether a sequence $\theta_1, \cdots, \theta_r$ of homogeneous elements of R^Γ (Γ a finite pseudograph) is a partial h.s.o.p. Theorem 4.2 however, allows us answer this question when the θ_i's are monomials.

PROPOSITION 4.9. *Let Γ be a finite pseudograph, and let $\alpha_1, \alpha_2, \cdots, \alpha_s$ be magic labelings of Γ. The following two conditions are equivalent*:

(i) $\mathbf{x}^{\alpha_1}, \mathbf{x}^{\alpha_2}, \cdots, \mathbf{x}^{\alpha_s}$ *is a partial* h.s.o.p. *of* R^Γ,

(ii) *If α is a magic labeling of Γ, if $1 \leq i \leq j \leq s$, and if $\alpha - \alpha_i$ and $\alpha - \alpha_j$ are magic* (i.e., *have non-negative entries*), *then $\alpha - \alpha_i - \alpha_j$ is magic.*

Proof. (i) $\Rightarrow$ (ii). Assume (i). By Theorem 4.2, $\mathbf{x}^{\alpha_1}, \mathbf{x}^{\alpha_2}, \cdots, \mathbf{x}^{\alpha_s}$ is an R^Γ-sequence. Hence if $i \neq j$, $\mathbf{x}^{\alpha_i}, \mathbf{x}^{\alpha_j}$ is an R^Γ-sequence. By definition this means that if $\mathbf{x}^{\alpha_j}X = \mathbf{x}^{\alpha_i}Y$, where $X, Y \in R^\Gamma$, then $X = \mathbf{x}^{\alpha_i}Z$ for some $Z \in R^\Gamma$. It is easily seen that we can take X, Y, Z to be monomials. Thus the condition becomes: if $\alpha_j + \beta = \alpha_i + \gamma$ for some magic labelings β and γ, then $\beta = \alpha_i + \delta$ for some magic labeling δ. This is clearly equivalent to (ii).

(ii) $\Rightarrow$ (i) Suppose that (i) fails. For convenience write $y_i = \mathbf{x}^{\alpha_i}$. Thus for some $i \geq 2$, y_i is a zero-divisor modulo $(y_1, \cdots, y_{i-1})$. (We can assume $i \neq 1$ since R^Γ is an integral domain so each y_j is not a zero-divisor.) Thus there is a relation

$$y_iY = y_1X_1 + y_2X_2 + \cdots + y_{i-1}X_{i-1}, \tag{4}$$

where $X_1, X_2, \cdots, X_{i-1}, Y \in R^\Gamma$ and $Y \notin (y_1, \cdots, y_{i-1})$. Now Y is a linear combination of monomials, so one of these monomials $\mathbf{x}^\beta$ must appear with non-zero coefficient and satisfy $\mathbf{x}^\beta \notin (y_1, \cdots, y_{i-1})$. Since the monomials $\mathbf{x}^\alpha \in R^\Gamma$ form a basis for R^Γ, we obtain $y_i\mathbf{x}^\beta = y_j\mathbf{x}^\gamma$ for some $j < i$. Thus $\alpha_i + \beta = \alpha_j + \gamma$ but $\beta \neq \alpha_j + \delta$. Hence (ii) fails, and the proof is complete.

COROLLARY 4.10. *Let Γ be a finite pseudograph. Suppose Γ possesses s pairwise edge-disjoint spanning subgraphs $\Gamma_1, \cdots, \Gamma_s$ such that each Γ_i has a magic labeling of odd index.* (*E.g., the Γ_i's could be disjoint 1-factors of Γ.*) *Then fsd $\Gamma \leq \dim \Gamma - s$.*

Proof. Let α_i be a magic labeling of Γ_i of odd index. Since the Γ_i's are edge-disjoint, the labelings $\alpha_1, \cdots, \alpha_s$ clearly satisfy condition (ii) of Proposition 4.9. Hence $\mathbf{x}^{\alpha_1}, \mathbf{x}^{\alpha_2}, \cdots, \mathbf{x}^{\alpha_s}$ is a partial h.s.o.p. of R^Γ. Since each $\mathbf{x}^{\alpha_i} \in I^\Gamma$, we have par $I^\Gamma \geq s$. Since *fsd* $\Gamma = \dim \Gamma - \text{par } I^\Gamma$, the proof follows.

COROLLARY 4.11. *Let Γ be a finite pseudograph such that $H_\Gamma(r) \neq \delta_{0r}$. Then either $P_\Gamma(r) = Q_\Gamma(r)$ or else* $\deg Q_\Gamma(r) < \deg P_\Gamma(r)$.

Proof. If $P_\Gamma(r) \neq Q_\Gamma(r)$, then Γ has a magic labeling of odd index. Thus the hypothesis of Corollary 4.10 holds with $s = 1$, so $fsd\ \Gamma \leq \dim \Gamma - 1$. Since $\deg P_\Gamma(r) = \dim \Gamma - 1$ and $\deg Q_\Gamma(r) \leq fsd\ \Gamma - 1$, the proof follows.

5. Symmetric magic squares. Theorem 3.3 may seem like an awkward result to apply to specific graphs, but we will now give an example of its use. Throughout this section Λ_n denotes the complete graph on the vertex set $\{1, 2, \cdots, n\}$ with one loop at each vertex. Thus Λ_n has $\binom{n+1}{2}$ edges and

$$\dim \Lambda_n = \binom{n}{2} + 1.$$

The functions $H_{\Lambda_n}, P_{\Lambda_n}, Q_{\Lambda_n}, F_{\Lambda_n}$ are abbreviated S_n, P_n, Q_n, F_n respectively. As pointed out in [14, p. 610], $S_n(r)$ is equal to the number of $n \times n$ symmetric matrices of non-negative integers such that every row (and hence every column) sums to r. Such a matrix is called a *symmetric magic square.* $S_n(r)$ also has a graph-theoretic interpretation—it is the number of regular pseudographs of valency r on an n-element vertex set.

Some examples of the generating function $F_n(\lambda)$ are:

$$F_1(\lambda) = \frac{1}{1-\lambda}$$

$$F_2(\lambda) = \frac{1}{(1-\lambda)^2}$$

$$F_3(\lambda) = \frac{1+\lambda+\lambda^2}{(1-\lambda)^4(1+\lambda)}$$

$$F_4(\lambda) = \frac{1+4\lambda+10\lambda^2+4\lambda^3+\lambda^4}{(1-\lambda)^7(1+\lambda)}$$

$$F_5(\lambda) = \frac{V_5(\lambda)}{(1-\lambda)^{11}(1+\lambda)^6},$$

where

$$V_5(\lambda) = 1 + 21\lambda + 222\lambda^2 + 1082\lambda^3 + 3133\lambda^4 + 5722\lambda^5 + 7013\lambda^6 + 5722\lambda^7 + 3133\lambda^8 + 1082\lambda^9 + 222\lambda^{10} + 21\lambda^{11} + \lambda^{12}.$$

The formulas for F_3 and F_4 are due to L. Carlitz [2]. We calculated F_5 with the aid of a computer. By Theorem 5.5 below, it is only necessary to compute $S_5(r)$ for $1 \leq r \leq 6$ in order to completely determine $F_5(\lambda)$. We computed $S_5(r)$ for $1 \leq r \leq 8$, using the last two values as a check. Methods for computing $S_n(1)$ and $S_n(2)$ for any n appear in [2] and [4].

Recall that a *1-factor* of a pseudograph Γ is a spanning subgraph Γ' of Γ such that each vertex of Γ lies on exactly one edge of Γ'. Moreover, a *1-factorization* of Γ is a collection $\Gamma_1, \Gamma_2, \cdots, \Gamma_m$ of 1-factors of Γ such that each edge of Γ appears in exactly one Γ_i.

LEMMA 5.1. *For any $n \geq 1$, the graph Λ_n has a 1-factorization.*

Proof. Let K_n denote Λ_n with its loops removed. A simple result of graph theory (e.g., [5, Thm. 9.1]) states that when n is even, K_n has a 1-factorization.

Assume n is even, and suppose $\Gamma_1, \cdots, \Gamma_{n-1}$ is a 1-factorization of K_n. Let Γ_n be the spanning subgraph of Λ_n whose edges are the loops of Λ_n. Then $\Gamma_1, \cdots, \Gamma_{n-1}, \Gamma_n$ is a 1-factorization of Λ_n.

Now assume n is odd, and let v be a vertex of K_{n+1}. Choose a 1-factorization of K_{n+1}. If we remove v from K_{n+1} and replace each edge from v to any other vertex w by a loop at w, then we obtain a 1-factorization of Λ_n. This completes the proof.

I am grateful to Daniel Kleitman for providing me with the main idea for the proof of the next lemma.

LEMMA 5.2. *Let n be a positive even integer, and let Δ be a positive spanning subgraph of Λ_n which does not contain a 1-factor. Then the number $q(\Delta)$ of edges of Δ satisfies*

$$q(\Delta) \leq \binom{n-1}{2} + 1.$$

Note. The bound $\binom{n-1}{2} + 1$ is best possible. Let v be a vertex of Λ_n, and let the edges of Δ consist of the loop at v and all edges of Λ_n not adjacent to v and which are not loops. It is easily seen that Δ is positive, contains no 1-factor, and satisfies

$$q(\Delta) = \binom{n-1}{2} + 1.$$

Proof of lemma. Suppose Δ is a positive spanning subgraph of Λ_n (n even) which does not contain a 1-factor. We wish to show Δ is missing at least

$$\binom{n+1}{2} - \binom{n-1}{2} - 1 = 2n - 2$$

edges of Λ_n. Let Δ' be Δ with all loops removed. Since Δ' *a fortiori* has no 1-factor, by a theorem of Tutte [16] [5, Thm. 9.4] there is a subset S of vertices of Δ' such that the graph Ω obtained from Δ' by removing S and all edges incident to S has at least $|S| + 1$ odd components (i.e., components with an odd number of vertices). Since n is even, this means Ω must have at least $|S| + 2$ components.

Case 1. $|S| \geq 2$ and $n \geq 10$. Then Ω has at most $n - 2$ vertices and at least 4 components. Thus it must be missing at least $3(n - 5) + 3 = 3n - 12$ edges. Since $n \geq 10$, we have $3n - 12 \geq 2n - 2$, as desired.

Case 2. $|S| = 1$ and $n \geq 8$. Then Ω has $n - 1$ vertices and at least three components. It is easy to see that when $n \geq 8$, Ω will be missing at least $2n - 2$ edges unless Ω has exactly two components Ω_1 and Ω_2 with one vertex each, and one component Ω_3 with $n - 3$ vertices. There are $2(n - 3) + 1 = 2n - 5$ edges missing which would be connections among the Ω_i's. Thus if Δ is missing less than $2n - 2$ edges, there are at most two unaccounted for edges missing from Δ.

Now let θ be a subgraph of Λ_n obtained by choosing two distinct vertices v_1 and v_2, and a set V of $n - 3$ vertices disjoint from v_1 and v_2, and removing the $2(n - 3) + 1$ edges which connect each v_i to V or to v_j. We need to show that if any two edges are removed from θ so that the resulting graph Δ is positive, then Δ has a 1-factor. The condition that θ minus two edges e_1 and e_2 be positive implies that neither e_1 nor e_2 can be a loop at v_1 or v_2. Now θ restric ed to its vertices other than v_1 and v_2 is isomorphic to Λ_{n-2}. By Lemma 5.1, Λ_{n-2} has a 1-factorization. Hence if remove two edges from Λ_{n-2} (in fact, $n - 3$ edges), Λ_{n-2} retains a 1-factor. This 1-factor, together with the loops at v_1 and v_2, form a 1-factor of Δ, as was to be shown.

Case 3. $S = \varnothing$ and $n \geq 8$. Thus Δ $(= \Omega)$ has at least two odd components. If it has more than two components, then it will immediately be missing at least $2n - 2$ edges unless exactly two components have one vertex and one component has the remaining $n - 2$ vertices. In this case, $2n - 3$ edges are missing which would connect the three components. Hence no other edges can be missing, but in this case the loops form a 1-factor.

Hence assume Δ has exactly two components. Then these components must be odd, from which it follows immediately that Δ will be missing at least $2n - 2$ edges unless one component consists of a single vertex v. In this case, there are $n - 1$ edges missing which would connect v to the remaining component. Let θ consist of Λ_n with all edges incident to v removed except for the loop at v. We wish to show that if $n - 1$ edges are removed from θ so that the resulting graph Δ is positive, then Δ has a 1-factor. Clearly the positivity of Δ implies that we cannot remove the loop at v. The subgraph of θ obtained by removing v is isomorphic to Λ_{n-1}, which by Lemma 5.1 has a 1-factorization. Hence if any $n - 1$ edges are removed from Λ_{n-1}, a 1-factor remains. This 1-factor, together with the loop at v, yields the desired 1-factor of Δ.

Case 4. *Small values of n not covered by the preceding cases.* Simple modifications of the above arguments, or independent *ad hoc* arguments, will eliminate the remaining possibilities. We leave the details to the reader, so the proof of the lemma is complete.

THEOREM 5.3. *We have*

$$\operatorname{fsd} \Lambda_n = \begin{cases} \binom{n-1}{2}, & n \text{ odd}, \\ \binom{n-2}{2}, & n \text{ even}. \end{cases}$$

Proof. First assume n is odd. By Lemma 5.1, Λ_n has a 1-factorization. Thus by Corollary 4.10,

$$\text{fsd } \Lambda_n \le \dim \Lambda_n - n = \binom{n-1}{2}.$$

On the other hand, let Δ be the subgraph obtained from Λ_n by removing all loops, so $\Delta \cong K_n$. Clearly Δ is positive and since n is odd, possesses no magic labelings of odd index. By Theorem 3.3,

$$\text{fsd } \Lambda_n \ge \dim \Delta = \binom{n-1}{2}.$$

Thus

$$\text{fsd } \Lambda_n = \binom{n-1}{2}.$$

Now assume n is even. Let Δ be as in the note following the statement of Lemma 5.2. Then again by Theorem 3.3,

$$\text{fsd } \Lambda_n \ge \dim \Delta = \binom{n-2}{2}.$$

Now let Δ be any positive spanning subgraph of Λ_n (n even) which does not have a magic labeling of odd index, so *a fortiori* Δ does not have a 1-factor. By Theorem 3.3, it suffices to show that

$$\dim \Delta \le \binom{n-2}{2}.$$

Let b be the number of bipartite components of Δ.

Case 1. $b = 0$. Now by Lemma 5.2, the number $q(\Delta)$ of edges of Δ satisfies

$$q(\Delta) \le \binom{n-1}{2} + 1.$$

Thus by Corollary 2.2,

$$\dim \Delta \le q(\Delta) - n + 1 \le \binom{n-2}{2},$$

as desired.

Case 2. $b \ge 1$. If any of the bipartite components of Δ consists of a single vertex, then $\dim \Delta = 0$. Thus we may assume each bipartite component of Δ has at least two vertices, so $b \le n/2$. Now Δ can be written uniquely as a disjoint union $\Delta_1 + \Delta_2$, where Δ_1 is bipartite and Δ_2 has no bipartite components. Let p_i (respectively q_i) denote the number of vertices (respectively edges) of Δ_i, for $i = 1$ or 2. Thus $p_1 + p_2 = n$. Now any positive bipartite pseudograph with at least one edge has a 1-factor, since every magic labeling of a bipartite graph is the sum of magic labelings of index one (see [14, Prop. 2.9]). Thus Δ_2 has no 1-factor since Δ has no 1-factor. Since Λ_{p_2} has a 1-factorization, we obtain $q_2 \leq \binom{p_2}{2}$. Since Δ_1 is bipartite with no multiple edges, $q_1 \leq p_1^2/4$. Since $b \geq 1$, we have $p_1 \geq 2$. It follows from the conditions

$$p_1 \geq 2, \quad p_2 \geq 0, \quad p_1 + p_2 = n, \quad q_1 \leq p_1^2/4, \quad q_2 \leq \binom{p_2}{2}$$

that $q_1 + q_2 \leq 1 + \binom{n-2}{2}$.

Hence

$$\dim \Delta = q_1 + q_2 - n + b + 1 \leq 1 + \binom{n-2}{2} - n + \frac{n}{2} + 1 \leq \binom{n-2}{2}, \qquad n > 2.$$

Since the case $n = 2$ is trivial, the proof is complete.

Remark. It should be noted that our proof of Theorem 5.3 did not use the fact that R^Γ is a Cohen-Macaulay ring (Theorem 4.2). Although the proof did use Corollary 4.10 (and therefore Proposition 4.9), we only used the implication (ii) $\Rightarrow$ (i) of Proposition 4.9. This implication requires only the relatively easy fact that a homogeneous R^Γ-sequence is an h.s.o.p. It is the implication (i) $\Rightarrow$ (ii) that requires the fact that R^Γ is Cohen-Macaulay.

Note that for $1 \leq n \leq 5$, $fsd\ \Lambda_n = sdm\ \Lambda_n$. It seems plausible that $fsd\ \Lambda_n = sdm\ \Lambda_n$ for all n, but we have no idea how to prove this fact.

Let $f = fsd\ \Lambda_n$ as given by Theorem 5.3, let

$$d = \dim \Lambda_n = 1 + \binom{n}{2},$$

and let

$$V_n(\lambda) = \left(\sum_{r=0}^{\infty} S_n(r)\lambda^r\right)(1 - \lambda)^d(1 + \lambda)^f.$$

We know that $V_n(\lambda)$ is a polynomial with integer coefficients, we would like to show that these coefficients are non-negative. In view of Propositions 4.6 and 4.7, it suffices to show that $fsd\ \Lambda_n = \max_\Delta (\dim \Delta)$, where Δ ranges over all positive spanning subgraphs of Λ_n which do not contain a 1-factor. However, this result was actually shown in the proof of Theorem 5.3. The point is that in Lemma 5.2, Δ is merely assumed not to contain a 1-factor, rather than the stronger fact of having no magic labeling of odd index. Thus we have shown:

PROPOSITION 5.4. *Let* $d = \dim \Lambda_n$, $f = fsd\ \Lambda_n$. *Then* R^{Λ_n} *possesses an* h.s.o.p. $\theta_1, \theta_2, \cdots, \theta_d$ *such that* $\deg \theta_i = 1$ if $1 \leq i \leq d - f$ *and* $\deg \theta_i = 2$ *if* $d - f + 1 \leq i \leq d$. *Consequently,* $V_n(\lambda)$ *has non-negative coefficients.*

In conclusion, we collect together all our results which pertain to the function $S_n(r)$, in particular Corollary 2.2, Theorem 5.3, Proposition 5.4, [34, Cor. 1.4], and [14, Lemma 4.2], to obtain the following result.

THEOREM 5.5. *Let $n \geq 1$, and let $S_n(r)$ be the number of $n \times n$ symmetric matrices of non-negative integers such that every row (and hence every column) sums to r. Let*

$$d = \binom{n}{2} + 1$$

and

$$f = \begin{cases} \binom{n-1}{2}, & n \text{ odd} \\ \binom{n-2}{2}, & n \text{ even}. \end{cases}$$

Let $V_n(\lambda) = (\sum_{r=0}^{\infty} S_n(r)\lambda^r)(1 - \lambda)^d(1 + \lambda)^f$. Then $V_n(\lambda)$ is a polynomial with integer coefficients satisfying the following additional properties:

(i) $\deg V_n(\lambda) = d + f - n$.

(ii) $\lambda^{d+f-n}V_n(1/\lambda) = V_n(\lambda)$.

(iii) $V_n(0) = 1$, so by (ii) *$V_n(\lambda)$ is monic.*

(iv) *the coefficients of $V_n(\lambda)$ are non-negative.*

We remark that property (iv) can be improved by examining the structure of the ring R^{Λ_n} in more detail. For instance, it follows from [15, Thm. 5.15] that R^{Λ_n} is a Gorenstein ring. (Property (ii) is a consequence of this fact, but actually (ii) was used to prove that R^{Λ_n} is Gorenstein.) From this one can deduce that if $0 \leq i \leq d + f - n$, then the coefficient of λ^i in $V_n(\lambda)$ is positive. It is possible to obtain better information about the coefficients (see [15] for some relevant techniques), but we do not pursue this here.

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