COHEN-MACAULAY COMPLEXES*

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1. SIMPLICIAL COMPLEXES

Let Δ be a finite simplicial complex (or complex for short) on the vertex set $V = \{x_1,\ldots,x_n\}$. Thus, Δ is a collection of subsets of V satisfying the two conditions: (i) $\{x_i\} \in \Delta$ for all $x_i \in V$, and (ii) if $F \in \Delta$ and $G \subset F$, then $G \in \Delta$. There is a certain commutative ring A_Δ which is closely associated with the combinatorial and topological properties of Δ. We will discuss this association in the special case when A_Δ is a Cohen-Macaulay ring. Lack of space prevents us from giving most of the proofs and from commenting on a number of interesting sidelights. However, a greatly expanded version of this paper is being planned.

Let Δ be a complex (= finite simplicial complex). If $F \in \Delta$, we call F a face of Δ. If F has $i + 1$ elements (denoted card $F = i + 1$), we say $\dim F = i$. Let $d = \delta + 1 = \max\{\text{card } F \mid F \in \Delta\}$. We write $\dim \Delta = \delta = d - 1$. If every maximal face of Δ has dimension δ, then Δ is called pure (or homogeneous by topologists). Let f_i be the number of i-dimensional faces of Δ. Thus $f_0 = n$. The vector $f = (f_0, f_1, \ldots, f_\delta)$ is called the f-vector of Δ. Now define a function on the non-negative integers by

$$H(\Delta,m) = \begin{cases} 1, & m = 0 \\ \sum_{i=0}^{\delta} f_i \binom{m-1}{i}, & m > 0. \end{cases} \tag{1}$$

Define integers h_i by

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M. Aigner (ed.), Higher Combinatorics, 51-62.

$$(1-\lambda)^d \sum_{m=0}^{\infty} H(\Delta,m)\lambda^m = \sum_{i=0}^{\infty} h_i\lambda^i .$$

It is easily seen that $h_i = 0$ if $i > d$. The vector $h=(h_0,h_1,\ldots,h_d)$ is called the h-vector of Δ. Knowing the f-vector of Δ is equivalent to knowing its h-vector.

Let $|\Delta|$ denote the underlying topological space of Δ, as defined in topology. The notation $\Delta = \langle abc, acd, bcd, bde\rangle$ means that the maximal faces of Δ are $\{a,b,c\}$, $\{a,c,d\}$, $\{b,c,d\}$, and $\{b,d,e\}$. For this Δ, the f-vector is (5,8,4), the h-vector is (1,2,1,0), and $|\Delta|$ is a 2-cell.

Let k be a field, fixed once and for all. All homology groups appearing in this paper are taken over the coefficient field k. Let $A = k[x_1,\ldots,x_n]$ be the polynomial ring over k whose variables are the vertices of Δ. Let I_Δ be the ideal of A generated by all squarefree monomials $x_{i_1}x_{i_2}\ldots x_{i_j}$ such that $\{x_{i_1},\ldots,x_{i_j}\} \notin \Delta$. For instance, if $\Delta = \langle abc, acd, bcd, bde\rangle$, then $I_\Delta = (ae,ce,abd)$. (We only need to include the minimal "non-faces" of Δ as generators of I_Δ, since I_Δ is an ideal.) Let $A_\Delta = A/I_\Delta$. This ring was first considered by M. Hochster (who suggested it to his student G. Reisner [10] for further study) and independently by this writer [14] [15]. In order to study the algebraic properties of A_Δ, we will require some concepts from commutative algebra. We will survey these concepts in the context of "standard k-algebras," although much of what we say can be generalized considerably.

2. BETTI NUMBERS OF RINGS

Let $A = k[x_1,\ldots,x_n]$ as above, and let I be any homogeneous ideal of A (i.e., I is generated by homogeneous polynomials). Set R=A/I. We call such a ring R a standard k-algebra. In particular, R is graded as $R = R_0+R_1+\cdots$, where R_i is the k-vector space of homogeneous polynomials of degree i contained in R. Thus $R_0 = k$, R_1 generates R as a k-algebra, $R_iR_j \subset R_{i+j}$, and $\dim_k R_i < \infty$. The Hilbert function $H(R,m)$ of R is defined by $H(R,m) = \dim_k R_m$. It was first shown by Hilbert that $H(R,m)$ is a polynomial for m sufficiently large, the Hilbert polynomial of R. The Krull dimension of R, denoted dim R, can be defined to be one more than the degree of the Hilbert polynomial of R.

If $R = A/I$ is a standard k-algebra, then a finite free resolution of R (as an A-module) is an exact sequence $0 \to M_j \to M_{j-1} \to \cdots \to M_0 \to R \to 0$ of A-modules, where each M_i is a free A-module of finite rank. A theorem of Hilbert implies that a finite free resolution of R always exists. There is a unique such resolution which minimizes the rank of each M_i; this resolution is called minimal. Define the i-th Betti number $\beta_i=\beta_i(R)$ of R to be the rank of the A-module M_i appearing in the minimal free resolution of R. In particular, $\beta_0=1$ and β_1 is the minimal number of generators of I. In the language of homological algebra, $\beta_i = \dim_k \operatorname{Tor}_i^A(R,k)$. For further information, see [12].

Example. Let $\Delta = \langle ab,bc,ac,cd \rangle$, so $I_\Delta = (ad,bd,abc)$. Then the minimal free resolution of A_Δ has the form $0 \to M_2 \to M_1 \to M_0 \to A_\Delta \to 0$, where rank $M_0 = 1$, rank $M_1 = 3$, rank $M_2 = 2$. With an appropriate choice of bases $\{X\}$ for M_0, $\{Y_1,Y_2,Y_3\}$ for M_1, and $\{Z_1,Z_2\}$ for M_2, the maps are given by $X \mapsto 1$, $Y_1 \mapsto adX$, $Y_2 \mapsto bdX$, $Y_3 \mapsto abcX$, $Z_1 \mapsto bY_1 - aY_2$, $Z_2 \mapsto bcY_1 - dY_3$. We have $\beta_0 = 1$, $\beta_1 = 3$, $\beta_2 = 2$, and $\beta_i = 0$ if $i \geq 3$.

If R is a standard k-algebra, let h be the largest integer i for which $\beta_i(R) \neq 0$. It is known that $n - d \leq h \leq n$, where $d = \dim R$ and n is the number of variables in A. The integer h is the homological dimension of R (as an A-module), denoted $\operatorname{hd}_A R$ or just hd R. If hd R = n − d then R is said to be a Cohen-Macaulay ring. The integer β_{n-d} is then called the type of R, denoted type R. If R is a Cohen-Macaulay ring of type one, then R is said to be Gorenstein. In this case, one can show $\beta_i = \beta_{h-i}$, where h = hd R. If R is Cohen-Macaulay, it is known that $\operatorname{Ext}_A^i(R,A) = 0$ unless i = hd R. Letting $\Omega(R) = \operatorname{Ext}_A^h(R,A)$, where h = hd R, this means that if we "dualize" the minimal free resolution of R by applying the functor $\operatorname{Hom}_A(\cdot, A)$, then we obtain a minimal free resolution for $\Omega(R)$, regarded as an A-module. $\Omega(R)$ is called the canonical module of R. Given that R is Cohen-Macaulay, one has that R is Gorenstein if and only if $\Omega(R) \cong R$. Thus the minimal free resolution of a Gorenstein standard k-algebra is "self-dual", a much stronger result than $\beta_i = \beta_{h-i}$.

3. CHARACTERIZING HILBERT FUNCTIONS

We now consider the relationship between the structure of R and its Hilbert function H(R,m). A non-void set M of monomials $x^\alpha y^\beta \cdots$ is called an order ideal of monomials if whenever $u \in M$ and v divides u, then $v \in M$. A finite or infinite sequence $h = (h_0,h_1,\ldots)$ of integers is called an O-sequence if there exists an order ideal M of monomials containing exactly h_i monomials of degree i. For instance, (1,3,2,2) is an O-sequence, since we can take $M = \{1, x, y, z, x^2, xy, x^3, x^2y\}$. A finite order ideal M of monomials is said to be pure if the maximal elements of M (ordered by divisibility) all have the same degree. We define a pure O-sequence in the obvious way. For instance, (1,3,1) is an O-sequence but not a pure O-sequence. Clearly, if $(h_0,h_1,\ldots)$ is an O-sequence, then $h_0 = 1$ and

$$0 \leq h_i \leq \binom{h_1 + i - 1}{i}, \qquad (3)$$

since the corresponding order ideal M has h_1 variables actually appearing, and there are $\binom{h_1+i-1}{i}$ monomials of degree i in h_1 variables. An explicit numerical condition for a sequence $(h_0,h_1,\ldots)$ to be an O-sequence appears in [15], though the crux of this result was first proved by Macaulay. No similar characterization of pure O-sequences is known.

The next result characterizes the Hilbert function of a Cohen-Macaulay standard k-algebra. This result is due to Macaulay, but is first stated in "modern" terminology in [14]. See [16, Cor. 3.10] for a proof.

Theorem 1. Let H be a function on the non-negative integers, and let k be any field. Then H is the Hilbert function of a Cohen-Macaulay standard k-algebra of Krull dimension d if and only if the sequence $(h_0,h_1,\ldots)$ defined by

$$(1-\lambda)^d \sum_{m=0}^{\infty} H(m)\lambda^m = \sum_{i=0}^{\infty} h_i\lambda^i \tag{4}$$

is an O-sequence with finitely many non-zero terms.

If R is a Cohen-Macaulay standard k-algebra of Krull dimension d and Hilbert function H, then we call the sequence $h = (h_0,h_1,\ldots)$ defined by (4) the h-vector of R. If $h_i = 0$ for $i > s$, we also write $h = (h_0,h_1,\ldots,h_s)$ for this h-vector.

We are now in a position to define a concept intermediate between Cohen-Macaulay and Gorenstein which will be of interest to us. Suppose that R is a Cohen-Macaulay standard k-algebra with h-vector $(h_0,h_1,\ldots,h_s)$, $h_s \neq 0$. It is easy to see that $h_s \leq$ type R. If $h_s =$ type R, then we say that R is a level ring, and we call $(h_0,h_1,\ldots,h_s)$ a level sequence. A level ring with $h_s = 1$ is just a Gorenstein ring, and in this case we call $(h_0,h_1,\ldots,h_s)$ a Gorenstein sequence. Clearly, every level sequence is an O-sequence. Unlike the Cohen-Macaulay case, no characterization of level sequences, or even of Gorenstein sequences, is known. The next result gives some information about level sequences, though undoubtedly stronger restrictions can be obtained.

Theorem 2. Let $h = (h_0,h_1,\ldots,h_s)$ be a level sequence with $h_s \neq 0$.

(i) If i and j are non-negative integers with $i + j \leq s$, then $h_i \leq h_j h_{i+j}$. In particular, if h is a Gorenstein sequence then $h_i = h_{s-i}$.

(ii) The vector $(h_s,h_{s-1},\ldots,h_0)$ is a sum of h_s O-sequences.

(iii) If $0 \leq t \leq s$, then $(h_0,h_1,\ldots,h_t)$ is a level sequence.

For instance, (1,4,10,2) is an O-sequence but by Theorem 2(i) is not a level sequence. Similarly, (1,4,2,2) is an O-sequence but by Theorem 2(ii) is not a level sequence. On the other hand, (1,3,5,4,5,3,1) is an O-sequence but not a Gorenstein sequence, although this example is not covered by Theorem 2. A characterization of Gorenstein sequences with $h_1 \leq 3$ appears in [16, Thm. 4.2]. Finally, we remark that it is easily seen that every pure O-sequence is a level sequence but not conversely, e.g., (1,3,1).

4. APPLICATIONS TO SIMPLICIAL COMPLEXES

We are now in a position to apply the above results on standard k-algebras to those of the form A_Δ. We begin with a simple result whose proof appears in [15].

Theorem 3. Let Δ be a complex with $d = 1 + \dim \Delta$. Then $d = \dim A_\Delta$, and the Hilbert function $H(A_\Delta, m)$ is the function $H(\Delta, m)$ of (1).

Corollary. Suppose A_Δ is Cohen-Macaulay. Then the h-vector of Δ is equal to the h-vector of A_Δ. Consequently, the h-vector of Δ is an O-sequence.

The above corollary raises the question of determining for which Δ is A_Δ Cohen-Macaulay, or more generally of computing hd A_Δ. The answer to this question follows from the following unpublished result of M. Hochster. First we require some notation. Let V be the set of vertices of Δ, and let $W \subset V$. Let Δ_W denote the restriction of Δ to W, i.e., $\Delta_W = \{F \in \Delta \mid F \subset W\}$. Throughout this paper, the notation H (respectively, $\tilde{H}$) denotes homology (respectively, reduced homology), either simplicial or singular (whichever is appropriate), over the coefficient field k, with the conventions $\tilde{H}_{-1}(\Gamma) = 0$ unless $\Gamma = \phi$, $\tilde{H}_i(\phi) = 0$ if $i \geq 0$, $\tilde{H}_{-1}(\phi) = k$, $\tilde{H}_i(\Gamma) = 0$ if $i < -1$.

Theorem 4. The Betti numbers of A_Δ are given by

$$\beta_i(A_\Delta) = \sum \dim_k \tilde{H}_{j-i-1}(\Delta_W) \ ,$$

where the sum is over all subsets W of the set V of vertices of Δ, and where card W = j.

Theorem 4 yields a topological criterion for computing hd A_Δ and therefore determining whether or not A_Δ is Cohen-Macaulay, but this criterion is quite cumbersome to use. A simpler condition for A_Δ to be Cohen-Macaulay was given by G. Reisner [10] prior to the discovery of Theorem 4. The equivalence of (i) and (ii) below is Reisner's result, while the equivalence of (ii) and (iii) is a simple exercise in topology. First recall that if $F \in \Delta$, then the link of F is defined by

$$\mathit{lk}\ F = \{G \in \Delta \mid F \cap G = \phi \text{ and } F \cup G \in \Delta\}.$$

In particular, $\mathit{lk}\ \phi = \Delta$.

Theorem 5. The following three conditions are equivalent.

(i) A_Δ is Cohen-Macaulay.

(ii) For all $F \in \Delta$ (including $F = \phi$), $\tilde{H}_i(\mathit{lk}\ F) = 0$ if $i \neq \dim \mathit{lk}\ F$.

(iii) If $X = |\Delta|$, then $\tilde{H}_i(X) = H_i(X,X-p) = 0$ for all $p \in X$ and $i \neq \dim X$.

When A_Δ is Cohen-Macaulay, we call Δ a Cohen-Macaulay complex. The property of being a Cohen-Macaulay complex depends on k. It follows, however, from the Universal Coefficient Theorem that Δ is Cohen-Macaulay over some k if and only if it is Cohen-Macaulay over the rational numbers.

Note that Theorem 5 implies that the question of whether or not Δ is Cohen-Macaulay depends only on $|\Delta|$ (and the coefficient field k). Recently J. Munkres has shown, using Theorem 4, that for any Δ the integer n-hd A_Δ depends only on $|\Delta|$ (and k).

If $h = (h_0, h_1, \ldots, h_d)$ is the h-vector of a Cohen-Macaulay complex Δ, then by the corollary to Theorem 3 and (3), $h_i \leq \binom{n-d+i-1}{i}$. When $|\Delta|$ is a sphere, this condition is equivalent to a condition on the f-vector of Δ of the form $f_i \leq c_i(n,d)$, where $c_i(n,d)$ is a certain explicit number depending on H on i,n, and d. A complex satisfying the above condition on h_i is said to satisfy the Upper Bound Conjecture (UBC). If $X = |\Delta|$ is a topological manifold with or without boundary, then $H_i(X, X-p) = 0$ for all $p \in X$ and $i < \dim X$. Hence by Theorem 5, Δ is Cohen-Macaulay if and only if $\tilde{H}_i(X) = 0$ for $i < \dim X$. In particular, Δ is Cohen-Macaulay if Δ is a sphere or cell. Thus the UBC holds for spheres and cells. For further details, see [15]. An example of a complex which fails to satisfy the UBC is <abcd,ae,be,ce>, whose h-vector is (1,1,0,-3,2). No example is known of a complex Δ which fails to satisfy the UBC for which $|\Delta|$ is a manifold with or without boundary.

5. CONSTRUCTIBILITY AND SHELLABILITY

We now give a result which shows that the corollary to Theorem 3 completely characterizes the h-vector of a Cohen-Macaulay complex. We say that a complex Δ is constructible (a concept due to M. Hochster) if it can be obtained by the following recursive procedure: (i) any simplex is constructible, and (ii) if Δ' and Δ'' are constructible of the same dimension δ, and if $\Delta' \cap \Delta''$ is constructible of dimension $\delta-1$, then $\Delta' \cup \Delta''$ is constructible. A straightforward Mayer-Vietoris argument, combined with Theorem 5, shows that Δ is Cohen-Macaulay if it is constructible. (A simple direct algebraic proof can also be given; see [14].)

Suppose that in building up a constructible polytope, one can always take Δ'' to be a simplex. Equivalently, Δ is pure and its maximal faces can be ordered $F_1, F_2, \ldots, F_\mu$ so that for $i = 2,3,\ldots,\mu$, we have that $(F_1 \cup F_2 \cup \cdots \cup F_{i-1}) \cap F_i$ is a non-void union of faces F of F_i satisfying $\dim F_i - \dim F = 1$. Then Δ is said to be shellable. This differs somewhat from other definitions of "shellable" in the literature, in that we place no restrictions on when $(F_1 \cup \cdots \cup F_{i-1}) \cap F_i$ can consist of all faces F of F_i of codimension one. See [5] for an interesting account of shellable complexes.

Theorem 6. Let $h = (h_0, h_1, \ldots, h_d)$ be a sequence of integers. The following four conditions are equivalent.

(i) h is an O-sequence,
(ii) h is the h-vector of a Cohen-Macaulay complex Δ,
(iii) h is the h-vector of a constructible complex Δ,
(iv) h is the h-vector of a shellable complex Δ.

The most important examples of shellable complexes are the boundary complexes of simplicial convex polytopes. We do not know an example of a constructible complex which is not shellable. However, in Section 7 are examples of constructible complexes for which it is unclear whether they are shellable; and it seems quite likely that a constructible complex need not be shellable.

6. GORENSTEIN COMPLEXES

If A_Δ is Gorenstein then Δ is called a Gorenstein complex. (As usual, this depends on k). We now give a characterization of Gorenstein complexes which can be deduced from Theorem 4 by using either topological or combinatorial arguments. Recall that if Γ and Δ are complexes on disjoint vertex sets V and W, then their join $\Gamma * \Delta$ is a complex on $V \cup W$ defined by $\Gamma * \Delta = \{F \cup G \mid F \in \Gamma \text{ and } G \in \Delta\}$.

Theorem 7. Δ is a Gorenstein complex if and only if it is a join $\sigma * \Gamma$, where σ is a simplex and where

$$\tilde{H}_i(X) = 0 \text{ for } i < \delta; \ \dim_k \tilde{H}_\delta(X) = 1$$

$$H_i(X, X-p) = 0 \text{ for } i < \delta; \ \dim_k H_\delta(X, X-p) = 1 \text{ for all } p \in X, \quad (5)$$

where $X = |\Gamma|$ and $\delta = \dim X$.

Since (5) automatically holds when X is a manifold, we see in particular that Δ is Gorenstein if X is a sphere, a result first proved by M. Hochster (unpublished). We also remark that it is possible for Δ to be Gorenstein (over any field k) without X being a topological manifold, e.g., when X is the suspension of Kneser's "dodecahedral space."

Suppose $(h_0, h_1, \ldots, h_d)$ is the h-vector of a Gorenstein complex Δ. We may assume that $h_d \neq 0$, since if $h_s \neq 0$ and $h_{s+1} = 0$, then $(h_0, \ldots, h_s)$ is the h-vector of the complex Γ of Theorem 7. By Theorem 2(ii), we then have $h_i = h_{d-i}$. This relation is equivalent to a condition on the f-vector of Δ known as the Dehn-Sommerville equations.

An outstanding open problem in the theory of convex polytopes is to characterize the h-vector $(h_0, \ldots, h_d)$ of the boundary complex of a simplicial convex d-polytope. McMullen's still open "g-conjecture" [9] states that the desired characterization is given by the following two conditions:

$$h_i = h_{d-i} \text{ for all } i, \tag{6}$$
$$(h_0, h_1-h_0, h_2-h_1, \ldots, h_e-h_{e-1}) \text{ is an O-sequence, where } e = [d/2].$$

We ask whether (6) also holds when $|\Delta|$ is a sphere or even more generally when Δ is Gorenstein. There are special cases for which it is possible to verify (6). For instance, in [9] it is shown by geometric means (Gale diagrams) that if $|\Delta|$ is a sphere satisfying $n \leq 4 + \dim \Delta$, then (6) holds. This is also an immediate consequence of Theorem 7 and [16, Thm. 4.2], and in fact one needs only to assume that Δ is a Gorenstein complex satisfying $n \leq 4 + \dim \Delta$. The next theorem gives another such result. It is an easy consequence of Theorem 4 and the corollary to Theorem 3. First we require a definition. If $|\Delta|$ is a δ-dimensional manifold with boundary, then the _boundary complex_ $\partial\Delta$ of Δ is the complex whose maximal faces are the $(\delta-1)$-dimensional faces of Δ which lie on only one δ-dimensional face. Thus $|\partial\Delta| = \partial|\Delta|$, so if $|\Delta|$ is a δ-dimensional cell, then $|\partial\Delta|$ is a $(\delta-1)$-dimensional sphere.

Theorem 8. Suppose that $|\Delta|$ is a d-dimensional manifold with boundary such that Δ is Cohen-Macaulay and $\partial\Delta$ is Gorenstein (e.g., $|\Delta|$ is a cell), and such that any face $F \in \Delta-\partial\Delta$ satisfies $\dim F \geq \frac{1}{2}(d-1)$. Then the h-vector $(h_0, h_1, \ldots, h_d)$ of $\partial\Delta$ satisfies (6).

A result of Klee [8] implies that if Δ is Gorenstein with h-vector $(h_0, \ldots, h_d)$, $h_d \neq 0$, then $h_1 + h_2 + \cdots + h_{d-1} \geq (d-1)h_1$. In [16] it is shown that (1,13,12,13,1) is a Gorenstein sequence. It follows that a Gorenstein sequence need not be the h-vector of a Gorenstein complex, in contrast to the Cohen-Macaulay case.

As a generalization of Theorem 7, we can ask for a description of the canonical module $\Omega(A_\Delta)$ when A_Δ is Cohen-Macaulay. If $|\Delta|$ is a manifold with boundary, there is overwhelming evidence (but not yet a proof) that $\Omega(A_\Delta)$ is isomorphic to the ideal of A_Δ generated by the squarefree monomials $x_{i_1}x_{i_2}\cdots x_{i_j}$ for which $\{x_{i_1}, \ldots, x_{i_j}\} \in \Delta - \partial\Delta$.

7. INDEPENDENT SETS AND BROKEN CIRCUITS

We now discuss some applications of Cohen-Macaulay complexes to the theory of pregeometries (or "matroids") in the sense of Crapo-Rota [4]. A _finite pregeometry_ Γ consists of a finite set V of vertices (or "points"), and a collection Δ of subsets of V, called _independent sets_, such that (i) Δ is a complex, and (ii) for every subset W of V, the induced complex Δ_W is pure. To avoid trivialities, we will also assume $\{v\} \in \Delta$ for all $v \in V$, so we may identify Γ with Δ. We call Δ a _G-complex_. For example, if V is a finite set of points in a vector space and Δ is the collection of linearly independent subsets of V, then Δ is a G-complex. If V is the set of edges of a finite graph G and Δ is the collection of subsets of V containing no cycle, then Δ is a G-complex. For further examples, see [4]. We also refer the reader to [4] for any unexplained terminology in this section.

It is easy to see, using the so-called "Tutte-Grothendieck decomposition" [2], that a G-complex Δ is constructible and is therefore Cohen-Macaulay. Using Theorem 4, one can obtain a simple expression for the Betti numbers $\beta_i(A_\Delta)$. We need to compute $N = \dim_k \tilde{H}_{j-i-1}(\Delta_W)$ where $W \subset V$ and card $W = j$. Let $\delta = \dim \Delta_W$. Now Δ_W is a Cohen-Macaulay complex, so $N = 0$ unless $j-i-1 = \delta$. If $j-i-1 = \delta$, then $N = (-1)^\delta(\chi(\Delta_W)-1)$, where χ is the Euler characteristic. It is known that $\chi(\Delta_W) - 1 = 0$ unless V-W is a flat (closed set) of the dual pregeometry $\tilde{\Delta}$. When V-W is a flat, then $N = |\tilde{\mu}(V-W,V)|$, where $\tilde{\mu}$ is the Möbius function (in the sense of [11]) of the lattice $L(\tilde{\Delta})$ of flats of $\tilde{\Delta}$. Moreover, (card W)$-\delta-1$ is just the corank of V-W in $\tilde{\Delta}$, i.e., the length of the longest chain between V-W and V in $L(\tilde{\Delta})$. Hence we obtain:

Theorem 9. Let Δ be a G-complex on a vertex set V. Then $\beta_i(A_\Delta) = \Sigma|\tilde{\mu}(X,V)|$, where X ranges over all flats of $\tilde{\Delta}$ of corank i.

Compare this with the so-called "Whitney number of the first kind" $\Sigma|\tilde{\mu}(\phi,X)|$, where X ranges over all flats of $\tilde{\Delta}$ of rank i. Theorem 9 implies that when Δ is a G-complex, the type of A_Δ is $|\tilde{\mu}(\phi,V)|$. On the other hand, if $(h_0,h_1,\ldots,h_d)$ is the h-vector of any complex Δ satisfying $\dim \Delta = d-1$, then an easy computation reveals $h_d = (-1)^{d-1}(\chi(\Delta)-1)$. Hence if Δ is a G-complex then $h_d = |\tilde{\mu}(\phi,V)| = \text{type } A_\Delta$. There follows:

Corollary. If Δ is a G-complex, then A_Δ is a level ring of type $|\tilde{\mu}(\phi,V)|$. Hence the h-vector of Δ is a level sequence.

Not every level sequence is the h-vector of some G-complex, e.g., (1,3,1), and it would be of considerable interest to characterize such h-vectors. In this direction, we have:

Conjecture. If Δ is a G-complex, then the h-vector of Δ is a pure O-sequence (as defined in Section 5).

Closely related to G-complexes are the "broken circuit complexes." Let $x_1,x_2,\ldots,x_n$ be an ordering of the vertices of a pregeometry Δ. A broken circuit is obtained by deleting the highest labeled element from any circuit (= minimal dependent set) of Δ. The broken circuit complex (or BC-complex) of Δ with respect to the ordering $x_1,\ldots,x_n$ is the complex whose faces are the subsets of V which do not contain a broken circuit. Let Λ denote the broken circuit complex of Δ (with respect to the ordering $x_1,\ldots,x_n$). If $(f_0,f_1,\ldots,f_\delta)$ is the f-vector of Λ, where $\delta = \dim \Lambda$, then $p_\Delta(\lambda) = \lambda^{\delta+1} - f_0\lambda^\delta + \cdots - (-1)^\delta f_\delta$ is the characteristic polynomial of Δ and is thus independent of the ordering chosen for the vertices (see [11, §7]). If Δ consists of the acyclic sets of edges of a graph G, then $\lambda^c \cdot p_\Delta(\lambda)$ is the chromatic polynomial of G, where c is the number of components of G. Hence the theory of Cohen-Macaulay complexes is applicable to chromatic polynomials.

It is easy to see that a BC-complex Λ is constructible and therefore Cohen-Macaulay. Hence the h-vector of Λ is an O-sequence. This improves an observation of Wilf [17, Thm. 2], who was the first person to study broken circuit complexes _qua_ complexes. For additional information on BC-complexes, see [3].

The fact that the h-vector of a BC-complex Λ is an O-sequence by no means characterizes such h-vectors, and it would be extremely interesting to obtain additional restrictions by a more detailed analysis of A_Λ. It has been conjectured that the f-vector $(f_0,\ldots,f_\delta)$ of a BC-complex is _unimodal_, i.e., for some i we have $f_0 \le f_1 \le \cdots \le f_i$, $f_i \ge f_{i+1} \ge \cdots \ge f_\delta$. Unfortunately, this fact does not follow simply from the h-vector being an O-sequence. For instance, the vector h = (1, 500, 55, 220, 715, 2002) is an O-sequence, and the corresponding f-vector (with d = 5) is f = (1, 505, 2065, 3395, 3325, 3493). Hence, by Theorem 6, f is the f-vector of some Cohen-Macaulay (or even shellable) complex Δ. We remark that without Theorem 6 it is difficult to find an example even of a pure complex whose f-vector is not unimodal.

8. COHEN-MACAULAY POSETS

Let P be a finite poset (= partially ordered set). Let $\Delta(P)$ denote the complex whose vertices are the elements of P and whose faces are the chains (totally ordered subsets) of P. If $\Delta(P)$ is a Cohen-Macaulay complex, then we call P a _Cohen-Macaulay poset_. (As usual, this depends on k.) There are two main classes of such posets known. (i) A finite semimodular lattice is a Cohen-Macaulay poset. This follows from Theorem 5 and work of Folkman [7] and Farmer [6]. More generally, we conjecture that a finite _admissible lattice_ in the sense of [13] is Cohen-Macaulay. (ii) Let Σ be a finite regular cell complex, e.g., a finite simplicial complex. (Certain more general structures can be allowed.) Suppose that the underlying topological space of Σ satisfies condition (iii) of Theorem 5. If P is the set of faces of Σ, ordered by inclusion, then P is Cohen-Macaulay. Indeed, $\Delta(P)$ is just the first barycentric subdivison of Σ.

Cohen-Macaulay posets were first considered explicitly by Baclawski [1]. His Theorems 6.1 and 6.2 are special cases of the next result, which was conjectured by this writer and proved (unpublished) by J. Munkres. First, let us define the _rank_ $\rho(x)$ of an element x of a finite poset P to be the length of the longest chain of P whose top element is x. Thus $\rho(x) = 0$ if and only if x is a minimal element of P. If $x \le y$ in P, set $\rho(x,y) = \rho(y) - \rho(x)$. If $\Delta(P)$ is pure, then $\rho(x,y)$ is the length of _any_ unrefinable chain between x and y, and $\rho(x,y) = \dim \Delta(P)$ if and only if x is a minimal element and y is a maximal element of P.

Theorem 10. Let P be a Cohen-Macaulay poset with rank function ρ, and let i be a non-negative integer. Let P_i be the set of all $x \in P$ satisfying $\rho(x) \neq i$. Give P_i the ordering induced from P. Then P_i is a Cohen-Macaulay poset.

If P is a Cohen-Macaulay poset with Möbius function μ, and if $x \leq y$ in P, then the open interval (x,y) is a Cohen-Macaulay poset and $(-1)^{\ell}\mu(x,y) = \dim_k \tilde{H}_{\ell}(\Delta((x,y)))$, where $\ell = \rho(x,y)$. Hence, $(-1)^{\ell}\mu(x,y) \geq 0$, i.e., the Möbius function of P alternates in sign. Theorem 10 implies that if we remove any set of "levels" from P, the Möbius function of the resulting poset continues to alternate in sign.

If P is a finite poset for which $\Delta(P)$ is pure, and if $\mu(x,y) = (-1)^{\rho(x,y)}$ for every $x \leq y$ in P, then P is called an Eulerian poset. It is not hard to see that a Cohen-Macaulay poset P is Gorenstein if and only if when we remove from P all elements x which are related to every element of P, and then adjoin a unique maximal and unique minimal element, the resulting poset is Eulerian.

The above considerations suggest that the Cohen-Macaulay posets are a natural class of posets whose Möbius functions merit further study.

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Late note. The work of M. Hochster to which we have referred (especially our Theorem 4) appears in Hochster's paper "Cohen-Macaulay rings, combinatorics, and simplicial complexes", based on a talk presented at the Oklahoma Ring Theory Conference, March 11-13, 1976. This paper contains many other interesting results on the structure of the ring A_Δ.

Later note. Regarding the conjecture in Section 6 concerning $\Omega(A_\Delta)$ when $|\Delta|$ is a manifold with boundary, Hochster has proved the following result. Suppose Δ is Cohen-Macaulay and $|\Delta|$ is a manifold with boundary. Let I be the ideal of A_Δ generated by all square free monomials $x_{i_1}x_{i_2}\cdots x_{i_j}$ for which $\{x_{i_1},\ldots,x_{i_j}\} \in \Delta - \partial\Delta$. Then I is isomorphic to $\Omega(A_\Delta)$ if and only if $\partial\Delta$ is Gorenstein (e.g., if $|\Delta|$ is orientable).