SIAM J. ALG. DISC. METH.
Vol. 1, No. 2, June 1980

0196-5212/80/0102-0005 $01.00/0

WEYL GROUPS, THE HARD LEFSCHETZ THEOREM, AND THE SPERNER PROPERTY*

RICHARD P. STANLEY†

Abstract. Techniques from algebraic geometry, in particular the hard Lefschetz theorem, are used to show that certain finite partially ordered sets Q^X derived from a class of algebraic varieties X have the k-Sperner property for all k. This in effect means that there is a simple description of the cardinality of the largest subset of Q^X containing no $(k+1)$-element chain. We analyze, in some detail, the case when $X = G/P$, where G is a complex semisimple algebraic group and P is a parabolic subgroup. In this case, Q^X is defined in terms of the "Bruhat order" of the Weyl group of G. In particular, taking P to be a certain maximal parabolic subgroup of $G = SO(2n+1)$, we deduce the following conjecture of Erdös and Moser: Let S be a set of $2\ell+1$ distinct real numbers, and let $T_1, \cdots, T_k$ be subsets of S whose element sums are all equal. Then k does not exceed the middle coefficient of the polynomial $2(1+q)^2(1+q^2)^2 \cdots (1+q^\ell)^2$, and this bound is best possible.

1. The Sperner property. Let P be a finite partially ordered set (or *poset*, for short), and assume that every maximal chain of P has length n. We say that P is *graded of rank* n. Thus P has a unique *rank function* $\rho: P \to \{0, 1, \cdots, n\}$ satisfying $\rho(x) = 0$ if x is a minimal element of P, and $\rho(y) = \rho(x) + 1$ if y covers x in P (i.e., if $y > x$ and no $z \in P$ satisfies $y > z > x$). If $\rho(x) = i$, then we say that x has *rank* i. Define $P_i = \{x \in P: \rho(x) = i\}$ and set $p_i = p_i(P) = \text{card } P_i$. The polynomial $F(P, q) = p_0 + p_1 q + \cdots + p_n q^n$ is called the *rank-generating function* of P. We say that P is *rank-symmetric* if $p_i = p_{n-i}$ for all i, and that P is *rank-unimodal* if $p_0 \leqq p_1 \leqq \cdots \leqq p_i \geqq p_{i+1} \geqq \cdots \geqq p_n$ for some i.

An antichain (also called a *Sperner family* or *clutter*) is a subset A of P, such that no two distinct elements of A are comparable. The poset P is said to have the *Sperner property* (or *property* S_1) if the largest size of an antichain is equal to $\max\{p_i: 0 \leqq i \leqq n\}$. More generally, if k is a positive integer then P is said to have the *k-Sperner property* (or property S_k) if the largest subset of P containing no $(k+1)$-element chain has cardinality $\max\{p_{i_1} + \cdots + p_{i_k}: 0 \leqq i_1 < \cdots < i_k \leqq n\}$. If P has property S_k for all $k \leqq n$, then following [21] we say that P has *property* S. For further information concerning the Sperner property and related concepts, see for instance [15], [16], [17].

Using some results from algebraic geometry, we will give several new classes of graded posets which have property S. These posets will all be rank-symmetric and rank-unimodal. First we must consider a property of posets related to property S. Suppose P is graded of rank n and is rank-symmetric. Again following [21], we say that P has *property* T if for all $0 \leqq i \leqq [n/2]$, there exist p_i pairwise disjoint saturated chains $x_i < x_{i+1} < \cdots < x_{n-i}$ where $x_j \in P_j$. It is clear that P is then rank-unimodal.

LEMMA 1.1. *Let P be a finite graded rank-symmetric poset of rank n. The following three conditions are equivalent*:

(i) *P is rank-unimodal and has property* S.

(ii) *P has property* T.

(iii) *Let V_i be the complex vector space with basis P_i. Then for $0 \leqq i < n$, there exist linear transformations φ_i: $V_i \to V_{i+1}$ satisfying the following two properties*:

(a) *If $0 \leqq i \leqq [n/2]$, then the composite transformation $\varphi_{n-i-1}\varphi_{n-i-2} \cdots \varphi_{i+1}\varphi_i$: $V_i \to V_{n-i}$ is invertible.*

(b) *Let $x \in P_i$ and $\varphi_i(x) = \sum_{y \in P_{i+1}} c_y y$. Then $c_y = 0$ unless $x < y$.*

Proof. (i)⇔(ii). This is a special case of [21, Thms. 2 and 3].

* Received by the editors June 1, 1979.

† Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139. The research was supported in part by the National Science Foundation under Grant MCS 77-01947.

(iii)$\Rightarrow$(ii). (I am grateful to Joseph Kung for supplying the following argument, which is considerably simpler than my original proof.) Assume (iii). Identify φ_i with its matrix with respect to the bases P_i and P_{i+1}. If φ is a matrix whose rows are indexed by a set S and whose columns are indexed by T, and if $S' \subset S$ and $T' \subset T$, then let $\varphi[S', T']$ denote the submatrix of φ with rows indexed by S' and columns by T'. By the Binet-Cauchy theorem (e.g., [1, § 36]) we have

$$\det(\varphi_{n-i-1} \cdots \varphi_i) = \sum (\det \varphi_i[Q_i, Q_{i+1}]) \cdot (\det \varphi_{i+1}[Q_{i+1}, Q_{i+2}]) \cdots (\det \varphi_{n-i-1}[Q_{n-i-1}[Q_{n-i}]),$$

where the sum is over all sequences of subsets $Q_i = P_i$, $Q_{i+1} \subset P_{i+1}$, $Q_{i+2} \subset P_{i+2}, \cdots, Q_{n-i-1} \subset P_{n-i-1}$, $Q_{n-i} = P_{n-i}$ such that $|Q_{i+1}| = |Q_{i+2}| \cdots = |Q_{n-i-1}| = p_i$. By (a), some term in the above sum is nonzero. Hence, the expansion of each factor $\det \varphi_k[Q_k, Q_{k+1}]$ in this term contains a nonzero term. By (b), this nonzero term defines a map $\sigma: Q_k \to Q_{k+1}$ such that $x < \sigma(x)$ for all $x \in Q_k$. Piecing together these two-element chains over all k yields (ii).

(ii)$\Rightarrow$(iii). The steps of the above argument can be reversed, provided we pick the φ_i's as generically as possible, i.e., all the entries of the matrices $\varphi_0, \varphi_1, \cdots, \varphi_{n-1}$ should be chosen to be algebraically independent over $\mathbb{Q}$, except for entries forced to equal 0 by condition (b). This completes the proof. □

2. Varieties with cellular decompositions. We now are in a position to invoke algebraic geometry. Let X be a complex projective variety of complex dimension n. Suppose that there are finitely many pairwise-disjoint subsets C_i of X, each isomorphic as an algebraic variety to complex affine space of some dimension n_i, such that (i) the union of the C_i's is X, and (ii) $\bar{C}_i - C_i$ is a union of some of the C_j's. (Here $\bar{C}_i$ denotes the closure of C_i either in the Hausdorff or Zariski topology—under the present circumstances the two closures coincide.) Following [4, p. 500], we then say that the C_i's form a *cellular decomposition* of X. The simplest and most familiar example is complex projective space $\mathbb{P}^n$ itself. Recall that $\mathbb{P}^n$ may be regarded as the set of nonzero $(n+1)$-tuples $x = (x_0, x_1, \cdots, x_n) \in \mathbb{C}^{n+1}$, modulo the equivalence relation $x \sim \lambda x$ $(\lambda \in \mathbb{C}^*)$. The set of elements of $\mathbb{P}^n$ of the form $(0, \cdots, 0, 1, x_{n-i+1}, \cdots, x_n)$ forms a subvariety isomorphic to $\mathbb{C}^i$. Hence we have the cellular decomposition $\mathbb{P}^n = \mathbb{C}^n \cup \mathbb{C}^{n-1} \cup \cdots \cup \mathbb{C}^0$.

If X is any complex projective variety and Y is a closed subvariety, then e.g., by [4] or [18, Chap. 5, § 4], Y represents an element (cocycle) $[Y]$ of the cohomology group $H^*(X, \mathbb{C})$. If X is irreducible of (complex) dimension n, and Y is irreducible of dimension m, then in fact $[Y] \in H^{2(n-m)}(X, \mathbb{C})$. If X is irreducible of dimension n and has a cellular decomposition $\{C_i\}$, it follows that the closures $\bar{C}_i$ represent cohomology classes $[\bar{C}_i] \in H^{2(n-m)}(X, \mathbb{C})$ where $C_i \cong \mathbb{C}^m$. (For this fact, we don't need condition (ii) in our definition of cellular decomposition.) The following fundamental result concerning varieties with a cellular decomposition appears in [4, p. 501], [22, § 6] in the case when X is nonsingular. The extension to singular varieties follows from [14]. (Again, condition (ii) is not actually necessary.)

THEOREM 2.1. *Let X be a complex projective variety of complex dimension n, and suppose that X has a cellular deomposition $\{C_i\}$. Then the cohomology classes $[\bar{C}_i]$ form a basis (over $\mathbb{C}$) for $H^*(X, \mathbb{C})$. In particular, $H^{2m+1}(X, \mathbb{C}) = 0$ for all $m \in \mathbb{Z}$, while if X is irreducible then $H^{2(n-m)}(X, \mathbb{C})$ has a basis consisting of those classes $[\bar{C}_i]$ for which $C_i \cong \mathbb{C}^m$.* □

Now given a cellular decomposition $\{C_i\}$ of X, define a partial ordering $Q^X = Q^X(C_1, C_2, \cdots)$ on the C_i's by setting $C_i \geqq C_j$ in Q^X if $C_i \subset \bar{C}_j$. If X is irreducible of dimension n, then it can be shown, using standard techniques from algebraic geometry, that Q^X is graded of rank n, with the rank function given by $\rho(C)=n\text{-dim } C$. If, moreover, X is nonsingular, then Poincaré duality implies that Q^X is rank-symmetric. Theorem 2.1 then implies that we may identify the vector space V_i of Lemma 1.1 (iii) with $H^{2i}(X, \mathbb{C})$ by identifying $C \in Q_i^X$ with $[\bar{C}] \in H^{2i}(X, \mathbb{C})$.

We now wish to define linear transformations $\varphi_i: V_i \to V_{i+1}$ (or equivalently, $\varphi_i: H^{2i}(X, \mathbb{C}) \to H^{2(i+1)}(X, \mathbb{C})$) satisfying conditions (a) and (b) of Lemma 1.1 (iii). This will enable us to conclude that Q^X has property S. Let Y be a hyperplane section of X, i.e., the intersection of X (regarded as being imbedded in some projective space $\mathbb{P}^N$) with a hyperplane of $\mathbb{P}^N$. If X is irreducible, then Y is a closed subvariety of X of dimension $n-1$ which represents a cohomology class $[Y] \in H^2(X, \mathbb{C})$. The cup product operation on cohomology then yields a linear transformation $\varphi_i: H^{2i}(X, \mathbb{C}) \to H^{2(i+1)}(X, \mathbb{C})$ defined as multiplication by $[Y]$. In other words, $\varphi_i(K)=[Y]\cdot K$. We now verify that when X is nonsingular and irreducible (so Q^X is graded and rank-symmetric), then these linear transformations φ_i satisfy conditions (a) and (b) of Lemma 1.1 (iii). First we dispose of condition (b). I am grateful to Steve Kleiman for providing a proof of this result.

LEMMA 2.2. *Let X be a complex projective variety with a cellular decomposition $\{C_i\}$, and let Y be a hyperplane section (or in fact any closed subvariety) of X. If $[Y]\cdot[\bar{C}_i]=\sum \alpha_j[\bar{C}_j]$* in $H^*(X, \mathbb{C})$, then $\alpha_j = 0$ unless $C_j \subset \bar{C}_i$.

Proof. Let $A(W)$ denote the Chow group of the variety W, i.e., the group of cycles modulo rational equivalence. If W is nonsingular and has a cellular decomposition $\{D_i\}$, then it is mentioned in [22, § 6] that the cycles $\bar{D}_i$ form a basis for $A(W)$, and that the corresponding map $A(W) \to H^*(W, \mathbb{Z})$ is an isomorphism of groups. It follows from [14] that this result continues to hold when W is singular. Now returning to our hypotheses, the C_j's contained in $\bar{C}_i$ form a cellular decomposition of $\bar{C}_i$. Hence a hyperplane section of $\bar{C}_i$ is rationally equivalent to a linear combination of the $\bar{C}_j$ that are contained in $\bar{C}_i$. A priori, the rational equivalence is on $\bar{C}_i$, but it may be considered as a rational equivalence on X. Hence $\alpha_j = 0$ unless $C_j \subset \bar{C}_i$ because the $[\bar{C}_i]$ are linearly independent in $H^*(X, \mathbb{C})$. □

Lemma 2.2 shows that condition (b) of Lemma 1.1 (iii) holds for Q^X (assuming X is nonsingular and irreducible, so we know Q^X is graded and rank-symmetric). Condition (a) is implied by the following basic result, known as the "hard Lefschetz theorem" (although the first rigorous proof was given by Hodge). See [34] for a brief history and survey of this theorem, and for its extension to characteristic p. Other references include [24, p. 187], [29], [10, Corollary, p. 75], [30, p. 44], [19, Chap. 0, § 7].

LEMMA 2.3 (the hard Lefschetz theorem). *Let X be a nonsingular irreducible complex projective variety of complex dimension n. Let Y be a hyperplane section of X. If $0 \leqq i \leqq n$, then the linear transformation $H^i(X, \mathbb{C}) \to H^{2n-i}(X, \mathbb{C})$ given by multiplication by $[Y]^{n-i}$ is an isomorphism.*

Putting Lemmas 1.1, 2.2, and 2.3 together, we obtain the main result of this paper.

THEOREM 2.4. *Let X be a nonsingular irreducible complex projective variety of complex dimension n with a cellular decomposition $\{C_i\}$. Then Q^X is graded of rank n, rank-symmetric, rank-unimodal, and has property* S.

For future use, we record the following simple result. The proof is evident.

PROPOSITION 2.5. *Let X and Y be complex projective varieties, with cellular decompositions $\{C_i\}$ and $\{D_j\}$ respectively. Then the product variety $X \times Y$ has a cellular decomposition with cells $C_i \times D_j$,* and $Q^{X\times Y} \cong Q^X \times Q^Y$.

It follows from Theorem 2.4 and Proposition 2.5 that if $P = Q^X$ and $P' = Q^Y$ for nonsingular irreducible complex projective varieties X and Y, each having a cellular decomposition, then $P \times P'$ has property S. More generally, Canfield [7] and independently Proctor, Saks, and Sturtevant [36] have shown that the product $P \times P'$ of any two graded, rank-symmetric, rank-unimodal posets P and P', each with property S, also has property S. (An even more general result has subsequently been proved by Saks [37].) For our purposes, however, it suffices to consider only Proposition 2.5.

3. Weyl groups. It remains to find interesting examples of varieties X with cellular decompositions and to describe the resulting posets Q^X. The best known examples of such varieties are the following. Let G be a complex semisimple algebraic group, and let P be a *parabolic subgroup* of G (i.e., a closed subgroup which contains a maximal solvable subgroup B of G. B is known as a *Borel subgroup.*) Then the coset space G/P has the structure of a non-singular irreducible complex projective variety, and the Bruhat decomposition of G affords a cellular decomposition $\{C_i\}$ of G/P. The cells C_i are known as *generalized Schubert cells.* See [5, § 3] for further details.

When $X = G/P$, a description of the poset Q^X can be given in terms of the Weyl groups W of G, and W_J of P [5, § 3], [11] as follows. Every Weyl group W is a finite Coxeter group, i.e., W is a finite group with a finite set $S = \{s_1, \ldots, s_m\}$ of generators such that for all $1 \leqq k \leqq m$, $1 \leqq i < j \leqq m$ and certain integers $n_{ij} \geqq 2$, W is defined by the relations $s_k^2 = 1$ and $(s_i s_j)^{n_{ij}} = 1$. The pair (W, S) is called a *Coxeter system.*

A *parabolic subgroup* of W (with respect to S) is any subgroup W_J generated by a subset J of S. Thus $W_\phi = \{1\}$ and $W_S = W$. The *length* $\ell(w)$ of an element $w \in W$ is the smallest integer $q \geqq 0$ for which w is a product of q elements of S. Define a partial order, called the *Bruhat order*, on W as follows. We say $w \leqq w'$ if there exist conjugates $t_1, \cdots, t_j$ of the elements of S such that $w' = wt_1t_2 \cdots t_j$ and $\ell(wt_1t_2 \cdots t_{i+1}) > \ell(wt_1t_2 \cdots t_i)$ for all $0 \leqq i < j$. The following properties (among others) of the Bruhat order of a finite Coxeter group W are known:

1. The Bruhat order makes W into a graded poset (which we still call W).

2. The function ℓ is the rank function of W, and the rank-generating function of W is given by

$$F(W, q) = \prod_{i=1}^{m} (1 + q + q^2 + \cdots + q^{e_i}) \tag{1}$$

for certain positive integers e_i known as the *exponents* of W. One may regard (1) as the definition of the exponents. For other equivalent definitions, see, e.g., [6, Chap. 5, § 6.2] or [8, Chap. 10]. Note that (1) implies the well-known fact that $|W| = \prod (e_i + 1)$, and that W has rank $e_1 + \cdots + e_m$.

3. If $J \subset S$, then each coset wW_J of W_J in W contains a unique element w_J of minimal length. For any $v \in W_J$ we have $\ell(w_J v) = \ell(w_J) + \ell(v)$.

4. Let W^J be the set of minimal length coset representatives w_J. Then W^J is a graded subposet of W such that the rank function of W^J is the restriction of the rank function of W.

5. (W_J, J) is itself a finite Coxeter system, say with exponents $f_1, \cdots, f_t$. Then W^J has the rank-generating function

$$F(W^J, q) = \frac{F(W, q)}{F(W_J, q)} = \frac{\prod_{i=1}^{m} (1 + q + q^2 + \cdots + q^{e_i})}{\prod_{j=1}^{t} (1 + q + q^2 + \cdots + q^{f_j})}. \tag{2}$$

For proofs of these results and further information on Coxeter groups, see e.g., [6], [8], [11]. For a connection between the posets W^J and combinatorics, different from the one given here, see [23].

Now we return to the varieties $X = G/P$, where G is a complex semisimple algebraic group and P a parabolic subgroup of G. It is known [6, p. 29], [5, § 3] that the parabolic subgroups of G containing a given Borel subgroup B are in one-to-one correspondence with the parabolic subgroups W_J of the Weyl group W of G (with respect to a fixed set S of Coxeter generators of W). Moreover, the poset Q^X corresponding to the cellular decomposition of $X = G/P$ obtained from the Bruhat decomposition of G is isomorphic to the partial order on W^J defined above. Hence from Theorem 2.4 we conclude:

THEOREM 3.1. *Let* (W, S) *be a Coxeter system for which* W *is a Weyl group. Let* $J \subset S$ *and let* W^J *be the poset defined above. Then* W^J *is rank-symmetric, rank-unimodal, and has property* S.

A Coxeter system (W, S) is *irreducible* if one cannot write S as a nontrivial disjoint union $T \cup T'$ such that $W = W_T \times W_{T'}$. If (W, S) is reducible, say $W = W_T \times W_{T'}$, then we also have $W = W_T \times W_{T'}$ as posets, and similarly for W^J. Thus by Proposition 2.5 nothing is lost by considering only irreducible Coxeter systems. Now all finite irreducible Coxeter systems are known (e.g., [6, p. 193]). There are the infinite families of type $A_n (n \geqq 1)$, $B_n (n \geqq 2)$, and $D_n (n \geqq 4)$, together with seven "exceptional" systems E_6, E_7, E_8, F_4, G_2, H_3, H_4 and the dihedral groups $I_2(p)$ of order $2p$ for $p = 5$ or $p \geqq 7$. ($I_2(3)$ coincides with A_2, $I_2(4)$ with B_2, and $I_2(6)$ with G_2.) For all of these systems (W, S), W is a Weyl group except for H_3, H_4, $I_2(p)$, $p = 5$ or $p \geqq 7$. It is easy to check that Theorem 3.1 remains valid for the dihedral groups $I_2(p)$, and for H_3. Presumably the remaining case H_4 can also be checked directly, so in fact one could determine those finite Coxeter systems (probably all of them) for which Theorem 3.1 remains valid.

4. Type A_n. We now want to describe the posets W^J in greater detail, for the types A_n, B_n, D_n. First consider A_{n-1}. Then W is the symmetric group $\mathfrak{S}_n$ of all permutations of $\{1, 2, \cdots, n\}$. The exponents are $1, 2, \cdots, n-1$, and as Coxeter generators we may take the "adjacent transpositions" $s_i = (i, i+1)$, $1 \leqq i \leqq n-1$. Regard a permutation $\pi \in \mathfrak{S}_n$ as a linear array $a_1 a_2 \cdots a_n$, where $\pi(i) = a_i$. Then a direct translation of the definition of the Bruhat order yields the following: $\pi \leqq \sigma$ in W if σ can be obtained from π by a sequence of operations which interchange i and j in a permutation $a_1 a_2 \cdots a_n$ provided i appears to the left of j and $i < j$. We abbreviate this operation as

$$i < j \longrightarrow j > i. \tag{3}$$

Thus the notation "$i < j$" in (3) means that i and j appear in the given order (i.e., i to the left of j) and $i < j$. For instance, $213 \leqq 312$ (obtained by $2 < 3 \longrightarrow 3 > 2$) and $24153 \leqq 35241$ (obtained, e.g., by $2 < 3 \to 3 < 2$, $1 < 2 \to 2 > 1$, $4 < 5 \to 5 > 4$). The rank $\ell(\pi)$ of $\pi = a_1 a_2 \cdots a_n \in W$ is equal to the number $i(\pi)$ of *inversions* of π, i.e., the number of pairs (i, j) for which $i < j$ and $a_i > a_j$. Thus $12 \cdots n$ is the unique permutation of rank 0 and $n \cdots 21$ is the unique permutation of highest rank $\binom{n}{2}$. It is well-known (e.g., [9, § 6.4]) that

$$\sum_{\pi \in \mathfrak{S}_n} q^{i(\pi)} = (1+q)(1+q+q^2) \cdots (1+q+\cdots+q^{n-1}),$$

which of course agrees with (1). Figure 1 depicts the Bruhat order of $\mathfrak{S}_3$.

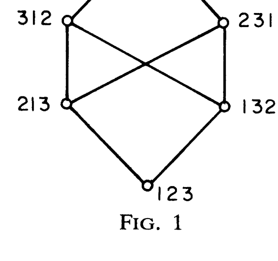

FIG. 1

Now let $J \subset S = \{s_1, \cdots, s_{n-1}\}$ where $s_i = (i, i+1)$. If we let $\mathfrak{S}(a, b)$ denote the group of all permutations of $\{a, a+1, \cdots, b\}$, then it is clear that $W_J = \mathfrak{S}(1, c_1) \times \mathfrak{S}(c_1+1, c_2) \times \cdots \times \mathfrak{S}(c_{j-1}+1, n)$ for some integers $1 \leqq c_1 < c_2 < \cdots < c_{j-1} < n$, where $j = n - |J|$. If $\pi = a_1 a_2 \cdots a_n \in W$, then the coset πW_J consists of all $c_1!(c_2 - c_1)! \cdots (n - c_{j-1})!$ permutations obtained from π by permuting among themselves the elements within the sets $N_1 = \{1, 2, \cdots, c_1\}$, $N_2 = \{c_1+1, \cdots, c_2\}, \cdots, N_j = \{c_{j-1}+1, \cdots, n\}$. The coset representative $\pi_J \in \pi W_J$ with the least number of inversions is that element of πW_J for which the elements of the above sets N_i appear in their natural order. Hence W^J consists of those $n!/c_1!(c_2 - c_1)! \cdots (n - c_{j-1})!$ permutations for which the elements of each of the sets N_i appear in their natural order; or, as it is sometimes called, the set of *shuffles* of $N_1, \cdots, N_j$. The rank-generating function of W^J is given by

$$F(W^J, q) = \frac{(\mathbf{n})!}{(\mathbf{c}_1)!(\mathbf{c}_2 - \mathbf{c}_1)! \cdots (\mathbf{n} - \mathbf{c}_{j-1})!} \tag{4}$$

where $(\mathbf{k})! = (1-q)(1-q^2) \cdots (1-q^k)$. The right-hand side of (4) is known as a *q-multinomial coefficient* and is commonly denoted $\begin{bmatrix} n \\ c_1, c_2 - c_1, \cdots, n - c_{j-1} \end{bmatrix}$. Figure 2 illustrates the poset W^J in the case $n = 4$, $J = \{(12)\}$.

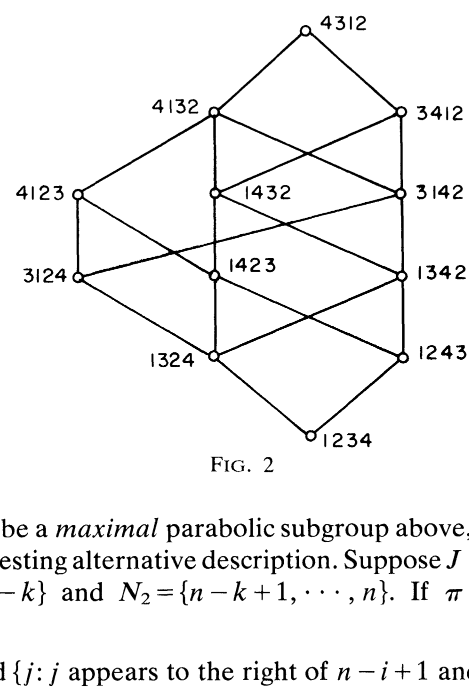

FIG. 2

If we take W_J to be a *maximal* parabolic subgroup above, i.e., $|J| = n - 2$, then the poset W^J has an interesting alternative description. Suppose $J = S - \{(n-k, n-k+1)\}$, so $N_1 = \{1, 2, \cdots, n-k\}$ and $N_2 = \{n-k+1, \cdots, n\}$. If $\pi = a_1 a_2 \cdots a_n \in W^J$ and $1 \leqq i \leqq k$, then set

$$\ell_i(\pi) = \text{card}\,\{j\colon j \text{ appears to the right of } n-i+1 \text{ and } j < n-i+1\}. \tag{5}$$

Clearly $\ell(\pi) = \sum_{i=1}^{k} \ell_i(\pi)$. The mapping $\pi \mapsto (\ell_1(\pi), \cdots, \ell_k(\pi))$ is a bijection between W^J and all integer sequences $0 \leqq \ell_1 \leqq \cdots \leqq \ell_k \leqq n-k$. Moreover, $\pi \leqq \pi'$ in W^J if and only if $\ell_i \leqq \ell_i'$ for $1 \leqq i \leqq k$. Hence, W^J is isomorphic to the poset of all partitions of integers into at most k parts, with largest part at most $n-k$, i.e., a partition whose Ferrers diagram (e.g., [9, § 2.4]) fits into a $k \times (n-k)$ rectangle. These partitions are ordered by inclusion of their Ferrers diagrams. Since the union and intersection of Ferrers diagrams is again a Ferrers diagram, it follows that the poset W^J is actually a distributive lattice, which we will denote by $L(k, n-k)$. Figure 3 depicts $L(2, 3)$.

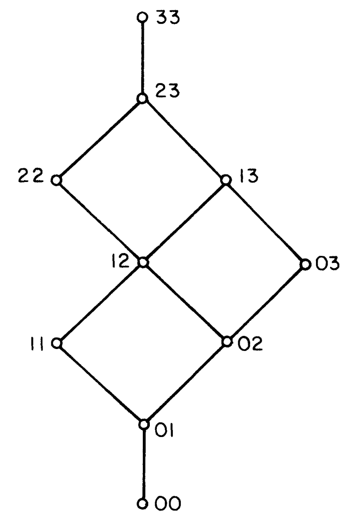

FIG. 3

In terms of the characterization [3, Thm. 3, p. 46] of a finite distributive lattice L as the lattice $\mathbf{2}^P$ of semi-ideals (also called "order ideals" or "decreasing subsets") of a poset P, we have $L(k, n-k) = \mathbf{2}^{\mathbf{k}\times(\mathbf{n}-\mathbf{k})}$, where $\mathbf{i}$ denotes an i-element chain. The rank-generating function of this lattice is the q-binomial coefficient $\left[{n \atop k}\right] = (\mathbf{n})!/(\mathbf{k})!(\mathbf{n}-\mathbf{k})!$. It is by no means a priori obvious that W^J is rank-unimodal; this was first shown essentially by Sylvester in 1878 (see [40] for historical details) and no combinatorial proof is known. I am grateful to Tony Iarrobino for originally calling to my attention that the hard Lefschetz theorem implies the unimodality of the coefficients of $\left[{n \atop k}\right]$. It was my attempt to understand this fact which eventually led to the present paper.

By applying Theorem 3.1 to the lattice $L(k, m)$, we can deduce a "multiset analogue" to a conjecture of Erdös and Moser [13, (12)]. (Regarding their actual conjecture, see Corollary 5.3 below.) I am grateful to Ranee Gupta for her comments on this result.

COROLLARY 4.1. *Fix positive integers k, m, and j. Let $A = \{a_0, a_1, \cdots, a_m\}$ be a set of $m+1$ distinct real numbers. Let $B_1, \cdots, B_r$ be subsets of A with exactly k elements* ***with repeated elements allowed.*** *(One may think of B_s as being an $m+1$-tuple $(\alpha_0, \cdots, \alpha_m)$ of nonnegative integers such that $\sum \alpha_i = k$, where α_i is the number of repetitions of a_i.) Let $\sum B_s$ denote the sum of the elements of B_s, i.e., $\sum B_s = \sum \alpha_i a_i$. Suppose that there are at most j distinct numbers among $\sum B_1, \cdots, \sum B_r$. Then r is less than or equal to the sum of the j middle coefficients of the polynomial $\begin{bmatrix} m+k \\ k \end{bmatrix}$. Moreover, this value of r is achieved by taking $A = \{0, 1, \cdots, m\}$ and $B_1, \cdots, B_r$ to have element sums consisting of the j middle elements of the set $\{0, 1, \cdots, km\}$. (If $km - j$ is even, then there are two equivalent choices of the "j middle coefficients" and "j middle elements.")*

Proof. Regarding $B_s = (\alpha_0, \cdots, \alpha_m)$ associate with B_s the sequence $\lambda_s = (\ell_1, \cdots, \ell_k) \in L(k, m)$ defined by setting exactly α_i of the ℓ_k's equal to i. It is easy to see that the subset $\{\lambda_1, \cdots, \lambda_r\}$ of $L(k, m)$ contains no $(j+1)$-element chain provided there are only j distinct numbers among $\sum B_1, \cdots, \sum B_r$. The proof now follows from Theorem 3.1 and the fact that the rank-generating function of $L(k, m)$ is $\begin{bmatrix} k+m \\ k \end{bmatrix}$. □

As a variation of the preceding corollary, we have

COROLLARY 4.2. *Fix positive integers k, m, and j. Let $A'=\{a_1,\cdots,a_m\}$ be a set of m distinct nonzero real numbers. Let $B_1,\cdots,B_r$ be subsets of A' with* ***at most*** *k elements with repeated elements allowed. Suppose that there are at most j distinct numbers among $\sum B_1,\cdots,\sum B_r$. Then r is less than or equal to the sum of the j middle coefficients of the polynomial $\begin{bmatrix} m+k \\ k \end{bmatrix}$. Moroever, this value of r is achieved by taking $A'=\{1,\cdots,m\}$ and $B_1,\cdots,B_r$ to have element sums consisting of the j middle elements of the set $\{0,1,\cdots,km\}$.*

Proof. Apply Corollary 4.1 to the set $A=A'\cup\{0\}$. □

Remark. The cellular decomposition of G/P in the case $W(G)=\mathfrak{S}_n$ and $W(P)=\mathfrak{S}_k\times\mathfrak{S}_{n-k}$ can be described quite concretely. The group G is given by $SL(n,\mathbb{C})$, which acts linearly on n-dimensional complex projective space $\mathbb{P}^{n-1}$. Let V be a $(k-1)$-dimensional subspace (or $(k-1)$-plane) of $\mathbb{P}^{n-1}$, and let P be the subgroup of G leaving V invariant. (Then P is a maximal parabolic subgroup of G.) The coset ϕP transforms V into the subspace ϕV, and this sets up a one-to-one correspondence between $X=G/P$ and the $(k-1)$-planes in $\mathbb{P}^{n-1}$. Hence X is the *Grassmann manifold* $G(k-1,n-1)$ of all $(k-1)$-planes in $\mathbb{P}^{n-1}$. Regard the elements of $\mathbb{P}^{n-1}$ as (equivalence classes of) n-tuples $(x_1,\cdots,x_n)\in\mathbb{C}^n-\{0\}$. A $(k-1)$-plane V in $\mathbb{P}^{n-1}$ has a unique ordered basis $w_1,\cdots,w_k$ for which the matrix $\begin{bmatrix} w_1 \\ \vdots \\ w_k \end{bmatrix}$ is in row-reduced echelon form. Choose integers $0\leqq a_1\leqq a_2\leqq\cdots\leqq a_k\leqq n-k$, and suppose we specify that for each i, the first 1 in w_i occurs in coordinate a_i+i. The set of all such V forms a subset $C(a_1,\cdots,a_k)$ of $G(k-1,n-1)$ isomorphic to $\mathbb{C}^{k(n-k)-a_1-\cdots-a_k}$; indeed, there are $n-k-a_i$ coordinates in w_i which can be specified arbitrarily, and the remaining coordinates are predetermined. By considering all sequences $0\leqq a_1\leqq\cdots\leqq a_k\leqq n-k$, we obtain a cellular decomposition of $G(k-1,n-1)$. Thus the cells $C(a_1,\cdots,a_k)$ are in one-to-one correspondence with the elements $(a_1,\cdots,a_k)$ of $L(k,n-k)$. For instance, when $k=2$ and $n=4$ the cells correspond to the following row-reduced echelon matrices:

$$\underset{C(0,0)}{\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \end{bmatrix}},\ \underset{C(0,1)}{\begin{bmatrix} 1 & * & 0 & * \\ 0 & 0 & 1 & * \end{bmatrix}},\ \underset{C(0,2)}{\begin{bmatrix} 1 & * & * & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}},$$

$$\underset{C(1,1)}{\begin{bmatrix} 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{bmatrix}},\ \underset{C(1,2)}{\begin{bmatrix} 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}},\ \underset{C(2,2)}{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}.$$

A little thought shows that $\overline{C(a_1,\cdots,a_k)}\supset C(b_1,\cdots,b_k)$ if and only if $a_i\leqq b_i$ for $1\leqq i\leqq k$. Thus we see directly that $Q^X\cong L(k,n-k)$. The closure of the cell $C(a_1,\cdots,a_k)$ is called a *Schubert variety*, and its cohomology class is called a *Schubert cycle*, which we shall denote by $\Omega(a_1,\cdots,a_k)$. (A more common notation is $\Omega(a'_1,\cdots,a'_k)$ where $a'_i=n-k+i-1-a_{k-i+1}$.) The Schubert cycle $\omega=\Omega(0,0,\cdots,0,1)\in H^2(X,\mathbb{C})$ turns out to be the class of a hyperplane section. According to a special case of Pieri's formula in the Schubert calculus, the product of $\Omega(a_1,\cdots,a_k)$ with ω in $H^*(X,\mathbb{C})$ is equal to the sum of all $\Omega(b_1,\cdots,b_k)$ such that $b_i\geqq a_i$ and $\sum b_i=1+\sum a_i$. In other words, $\omega\cdot\Omega(a_1,\cdots,a_k)=\sum\Omega(b_1,\cdots,b_k)$, where the sum is over all sequences $(b_1,\cdots,b_k)$ covering $(a_1,\cdots,a_k)$ in $L(k,n-k)$. Thus we have a direct verification of Lemma 2.2. For further information on these matters, see, for example, [26], [27], [41].

5. Type B_n. We next turn our attention to type B_n. In this case W is the group of all $n \times n$ signed permutation matrices (i.e., matrices with entries 0, ± 1 with one nonzero entry in every row and column). W has order $2^n n!$ and exponents $1, 3, 5, \cdots, 2n-1$. Identify the matrix $(m_{ij}) \in W$ with the ordered pair (π, ε), where $\pi \in \mathfrak{S}_n$ is given by $m_{i,\pi(i)} = \pm 1$ and $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_n) \in \{\pm 1\}^n$ by $\varepsilon_i = m_{i,\pi(i)}$. We then have the multiplication rule $(\pi, \varepsilon)(\pi', \varepsilon') = (\pi\pi', \delta)$, where $\delta_i = \varepsilon_{\pi'(i)}\varepsilon'_i$. We sometimes will abbreviate a group element such as (24513, $(-1, 1, -1, -1, 1)$) by $\bar{2}\ 4\ \bar{5}\ \bar{1}\ 3$, and thus regard W as consisting of all "barred permutations" of $\{1, 2, \cdots, n\}$. For the Coxeter generators of W we take the set $S = \{s_1, \cdots, s_n\}$, where s_i is the adjacent transposition $(i, i+1)$, $1 \leqq i \leqq n-1$, and $s_n = \bar{1}\ 2\ 3 \cdots n$. A little thought shows that $\pi \leqq \sigma$ in W if σ can be obtained from π by a sequence of the following seven types of operations on barred permutations:

a) $i \longrightarrow \bar{i}$,
b) $i < j \longrightarrow j > i$,
c) $\bar{i} < j \longrightarrow j > \bar{i}$,
d) $\bar{i} < j \longrightarrow \bar{j} > i$,
e) $\bar{i} > j \longrightarrow j < \bar{i}$,
f) $i > \bar{j} \longrightarrow j < \bar{i}$,
g) $\bar{i} > \bar{j} \longrightarrow \bar{j} < \bar{i}$.

For instance, Fig. 4 illustrates W when $n = 2$.

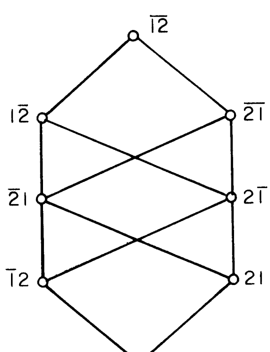

FIG. 4

If $(\pi, \varepsilon) \in W$, then one can check that

$$\ell(\pi) = i(\pi) + \sum_j (2d_j + 1), \tag{6}$$

where $i(\pi)$ is the number of inversions of π, j ranges over all integers for which $\varepsilon_j = -1$, and d_j is the number of k's appearing in $\pi = a_1 a_2 \cdots a_n$ to the left of a_j for which $k < a_j$. For instance, $\ell(\bar{3}\ 1\ 5\ \bar{4}\ 2) = 11$, since $i(\pi) = 5$, $d_1 = 0$, $d_4 = 2$. It is easy to give a direct combinatorial proof that

$$\sum_{\pi \in W} q^{\ell(\pi)} = \prod_{i=1}^{n} (1 + q + q^2 + \cdots + q^{2i-1}),$$

agreeing with (1).

Now let $J \subset S$. Let $\bar{\mathfrak{S}}(a, b)$ denote the group of all signed permutations of $\{a, a+1, \cdots, b\}$. Then W_J has the form

$$W_J = \bar{\mathfrak{S}}(1, c_1) \times \mathfrak{S}(c_1+1, c_2) \times \mathfrak{S}(c_2+1, c_3) \times \cdots \times \mathfrak{S}(c_{j-1}+1, n), \tag{7}$$

where $0 \leqq c_1 < c_2 < \cdots < c_{j-1} < n$. The case $c_1 = 0$ corresponds to $s_n \notin J$. If $c_1 = 0$ then $j = n - |J|$; otherwise $j = n - |J| + 1$. Set $N_1 = \{1, 2, \cdots, c_1\}$, $N_2 = \{c_1+1, \cdots, c_2\}, \cdots, N_j = \{c_{j-1}+1, \cdots, n\}$. One can check that W^J consists of all $(a_1 a_2 \cdots a_n, \varepsilon) \in W$ satisfying:

(i) $\varepsilon_i = 1$ if $a_i \in N_1$.

(ii) If $a_r, a_s \in N_i$ with $r < s$ and $\varepsilon_r = \varepsilon_s = 1$, then $a_r < a_s$.

(iii) If $a_r, a_s \in N_i$ with $r < s$ and $\varepsilon_r = \varepsilon_s = -1$, then $a_r > a_s$.

(iv) If $a_r, a_s \in N_i$ and $\varepsilon_r = 1$, $\varepsilon_s = -1$, then $a_r > a_s$.

For instance, if $W_J = \bar{\mathfrak{S}}(1, 2) \times \mathfrak{S}(3, 7) \times \mathfrak{S}(8, 9)$, then a typical element of W^J is $5\ \bar{4}\ 1\ \bar{8}\ 6\ 2\ 7\ 9\ \bar{3}$. The letters 1, 2 are unbarred and appear in increasing order. Similarly 3, 4 are barred and decrease, 5, 6, 7 are unbarred and increase, 8 is barred and "decreases," and 9 is unbarred and "increases."

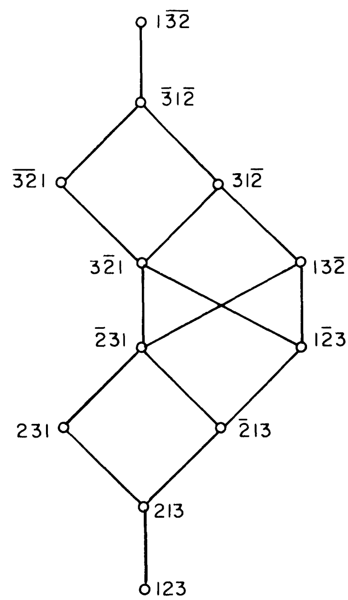

FIG. 5

Figure 5 illustrates W^J when $n = 3$ and $J = \{s_1, s_3\}$. We see that, unlike the situation for A_n, W^J need not be a distributive lattice (or even just a lattice) when J is a maximal subset of S. There is one case, however, in which W^J is a distributive lattice, viz., $J = \{s_1, s_2, \cdots, s_{n-1}\}$, so $W_J = \mathfrak{S}(1, n)$. In this case we will denote W^J by $M(n)$. To see that $M(n)$ is indeed a distributive lattice, observe that for every sequence $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_n) \in \{\pm 1\}^n$, there is a unique $\pi \in \mathfrak{S}_n$ for which $(\pi, \varepsilon) \in M(n)$. Identify ε with the subset of $\{1, 2, \cdots, n\}$ consisting of those integers t for which $\varepsilon_t = -1$. Then the partial order on $M(n)$ is given by $\{a_1, \cdots, a_j\} \leqq \{b_1, \cdots, b_k\}$ if $a_1 < \cdots < a_j$, $b_1 < \cdots < b_k$, $j \leqq k$, and $a_{j-i} \leqq b_{k-i}$ for $0 \leqq i \leqq j-1$. It is then easily seen that $M(n)$ is a distributive lattice. The poset P for which $M(n) = \mathbf{2}^P$ is given by $P = \mathbf{2}^{\mathbf{2} \times (\mathbf{n}-\mathbf{1})}$. Figure 6 illustrates $M(4)$.

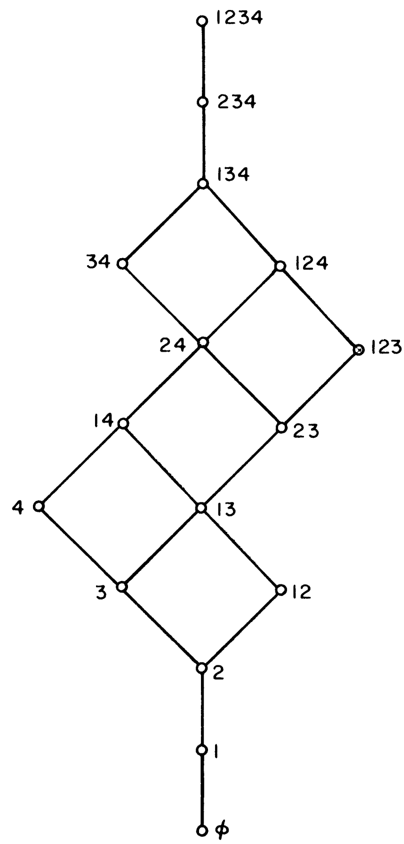

FIG. 6

Lindström [30] conjectured that $M(n)$ has property S_1, while in fact we now know that $M(n)$ has property S and is rank-unimodal. (I am grateful to Larry Harper for calling my attention to Lindström's conjecture.) The rank-generating function of $M(n)$ is $(1+q)(1+q^2)\cdots(1+q^n)$. The unimodality of the coefficients of this polynomial was first explicitly proved by Hughes [25], based on a result of Dynkin (see [40] for further information). Presumably, however, this result could also be proved analytically using the methods of [12]. Lindström [30], [31] shows that the structure of $M(n)$ is related to a conjecture [13, (12)] of Erdös and Moser (see also [12], [38], [42]). In fact, Corollary 5.3 below provides a more general result. I am grateful to Ranee Gupta for pointing out an error in my original treatment of the Erdös–Moser conjecture.

COROLLARY 5.1. *Let A be a set of distinct real numbers. Assume that ν elements of A are negative, ζ are equal to* 0 *(so $\zeta = 0$ or* 1*), and π are positive. Let $B_1, \cdots, B_r$ be subsets of A whose element sums take on at most k distinct values. Then r does not exceed the sum of the k middle coefficients of the polynomial*

$$G_{\nu\zeta\pi}(q) = 2^\zeta(1+q)(1+q^2)\cdots(1+q^\nu)\cdot(1+q)(1+q^2)\cdots(1+q^\pi)$$

(there being two equivalent choices of the "k middle coefficients" when $\binom{\nu+1}{2}+\binom{\pi+1}{2}-k$ is even). Moreover, this value of r is achieved by taking $A = \{-1, -2, \cdots, -\nu\} \cup \{1, 2, \cdots, \pi\} \cup Z$, where $Z = \phi$ or $\{0\}$ depending on whether $\zeta = 0$ or 1.

Proof. Since 0 can be adjoined to a set without affecting its element sum we may assume $\zeta = 0$. Let $M(\nu)^*$ denote the order-dual of $M(\nu)$.(The elements of $M(\nu)$ and $M(\nu)^*$ coincide, but $C \leqq C'$ in $M(\nu)^*$ if and only if $C \geqq C'$ in $M(\nu)$.) Regard elements of the product $M(\nu)^* \times M(\pi)$ as consisting of pairs (C, D), where C is a subset of $\{1, 2, \cdots, \nu\}$, and D is a subset of $\{1, 2, \cdots, \pi\}$. Suppose that the elements of A are $\alpha_\nu < \cdots < \alpha_1 < 0 < \beta_1 < \cdots < \beta_\pi$ and that $B_s = \{\alpha_{i_1}, \cdots, \alpha_{i_h}, \beta_{j_1}, \cdots, \beta_{j_m}\}$. Associate with B_s the set $(C_s, D_s) = (\{i_1, \cdots, i_h\}, \{j_1, \cdots, j_m\}) \in M(\nu)^* \times M(\pi)$. It is easy to see that the subset $\{(C_1, D_1), \cdots, (C_r, D_r)\}$ of $M(\nu)^* \times M(\pi)$ contains no $(k+1)$-element chain provided there are most k distinct element sums of $B_1, \cdots, B_r$. Now it is not difficult to see that $M(\nu)^* \cong M(\nu)$. (For instance, given the set $T = \{i_1, \cdots, i_h\} \in M(\nu)$ with $1 \leqq i_1 < \cdots < i_h \leqq \nu$, define T^* to be the set of nonzero parts of the partition λ which is conjugate (in the sense of [9, p. 100]) to the partition whose parts are $\nu - i_h, \nu - 1 - i_{h-1}, \cdots, \nu - h + 1 - i_1, \nu - h, \nu - h - 1, \cdots, 1$. Then the mapping $T \to T^*$ is an isomorphism $M(\nu) \to M(\nu)^*$. See also § 7 for a more general result.) The proof now follows from Theorem 3.1 and Proposition 2.5 (or from Theorem 3.1 alone applied to the appropriate *reducible* Weyl group) and the fact that the rank-generating function of $M(\nu)^* \times M(\pi)$ is $G_{\nu 0 \pi}(q)$. □

We now want to consider the situation where $\nu + \zeta + \pi$ is fixed, but ν, ζ, and π can vary. First we need:

LEMMA 5.2. *Let $G(q)$ be a polynomial of degree d with symmetric unimodal coefficients. Fix positive integers j and k. Then the sum of the middle k coefficients of $G(q)(1+q^{j+1})$ does not exceed the sum of the middle k coefficients of $G(q)(1+q^j)$.*

Proof. Let $G(q) = \alpha(0) + \alpha(1)q + \cdots + \alpha(d)q^d$. For simplicity of notation we assume $d = 2d'$, $j = 2j'$, $k = 2k'$. The other cases are done similarly. The middle k coefficients of $G(q)(1+q^j)$ are

$$\alpha(d'+j'-k'+i) + \alpha(d'-j'-k'+i), \qquad 0 \leqq i \leqq k-1.$$

The middle k coefficients of $G(q)(1+q^{j+1})$ are

$$\alpha(d'+j'-k'+i+1) + \alpha(d'-j'-k'+i), \qquad 0 \leqq i \leqq k-1.$$

(Here we set $\alpha(t) = 0$ if $t < 0$.) If Ω applied to a polynomial denotes the sum of its middle k coefficients, then

$$\Omega G(q)(1+q^j) - \Omega G(q)(1+q^{j+1}) = \alpha(d'+j'-k') - \alpha(d'+j'+k').$$

Since $\alpha(i) = \alpha(d-i)$ and $\alpha(0) \leqq \alpha(1) \leqq \cdots \leqq \alpha(d')$, it follows that $\alpha(d'+j'-k') \geqq \alpha(d'+j'+k')$, completing the proof. □

COROLLARY 5.3. *Let A be a set of n distinct real numbers, and let $B_1, \cdots, B_r$ be subsets of A whose element sums take on at most k distinct values. Let $\nu = [(n-1)/2]$ and $\pi = [n/2]$. Then r does not exceed the sum of the k middle coefficients of the polynomial*

$$2(1+q)(1+q^2)\cdots(1+q^\nu)\cdot(1+q)(1+q^2)\cdots(1+q^\pi).$$

Moreover, this value of r is achieved by choosing $A = \{-\nu, -\nu+1, \cdots, \pi\}$.

Proof. For fixed $n = \nu + \zeta + \pi$, it follows from Lemma 5.2 that the sum of the middle k coefficients of $G_{\nu\zeta\pi}(q)$ is maximized by choosing $\zeta = 1$, $\nu = [(n-1)/2]$, $\pi = [n/2]$. The proof follows from Corollary 5.1. □

The actual conjecture [13, (12)] of Erdös and Moser is equivalent to the case $k = 1$, and n odd, of Corollary 5.3. A purely combinatorial derivation of the Erdös–Moser conjecture from the fact that $M(n)$ has property S appears in [35].

6. Type D_n. If (W, S) is a Coxeter system of type D_n, then W is the subgroup of the group W' of type B_n consisting of all (π, ε) such that $\prod_{i=1}^{n} \varepsilon_i = +1$. W has order $2^{n-1}n!$ and exponents $1, 3, 5, \cdots, 2n-5, 2n-3, n-1$. We may take $S = \{s_1, \cdots, s_n\}$ where $s_i = (i, i+1)$ if $1 \leqq i \leqq n-1$ (as in type B_n) and $s_n = \bar{2}\ \bar{1}\ 3\ 4 \cdots n$. We then have the following seven transformation rules for obtaining w' from w when $w \leqq w'$ in W:

a) $i < j \longrightarrow \bar{j} > \bar{i}$,
b) $i < j \longrightarrow j > i$,
c) $i < j \longrightarrow j > \bar{i}$,
d) $\bar{i} < j \longrightarrow \bar{j} > i$,
e) $\bar{i} > j \longrightarrow j < \bar{i}$,
f) $i > \bar{j} \longrightarrow j < \bar{i}$,
g) $\bar{i} > \bar{j} \longrightarrow \bar{j} < \bar{i}$.

Note that rules b–g coincide with those for B_n, and that rule a for D_n is obtained by applying rule b and rule a twice for B_n. It follows that if $\pi \leqq \sigma$ in W then $\pi \leqq \sigma$ in W'. The converse, however, is false. For instance, $21 < \overline{21}$ in W' but 21 and $\overline{21}$ are incomparable in W. Figure 7 depicts W when $n = 2$.

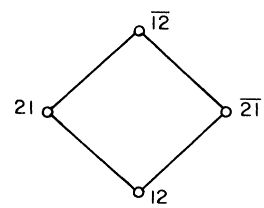

FIG. 7

If $(\pi, \varepsilon) \in W$, then

$$\ell(\pi) = i(\pi) + 2 \sum_j d_j,$$

where $i(\pi)$ and d_j have the same meaning as in (6). For instance, $\ell(\bar{3}\ 1\ 5\ \bar{4}\ 2) = 9$ for D_5, while $\ell(\bar{3}\ 1\ 5\ \bar{4}\ 2) = 11$ for B_5.

Now let $J \subset S$. In so far as describing the poset W^J is concerned, we may assume that if $s_n = \overline{21}34 \cdots n \in J$ then also $s_1 = 213 \cdots n \in J$, since interchanging s_1 and s_n induces an automorphism of the Coxeter system (W, S). Thus if we let $\mathfrak{S}(a, b)$ denote the group of all signed permutations of $\{a, a+1, \cdots, b\}$ with an even number of -1's, then W_J has the form

$$W_J = \hat{\mathfrak{S}}(1, c_1) \times \mathfrak{S}(c_1+1, c_2) \times \cdots \times \mathfrak{S}(c_{j-1}+1, n),$$

where $0 \leqq c_1 < c_2 < \cdots < c_{j-1} < n$ and $c_1 \neq 1$. The case $c_1 = 0$ corresponds to $s_n \notin J$. Defining $N_1 = \{1, 2, \cdots, c_1\}$, $N_2 = \{c_1+1, \cdots, c_2\}, \cdots, N_j = \{c_{j-1}+1, \cdots, n\}$, one can check that W^J consists of all $(a_1 a_2 \cdots a_n, \varepsilon) \in W$ satisfying:

(i) $\varepsilon_1 = 1$ if $a_i \in N_1$ and $a_i > 1$.
(ii)–(iv) Same as for type B_n.
(v) 1 precedes every other element of N_1 (even if 1 is barred).

For instance, Fig. 8 depicts W^J when $n = 3$ and $J = \{12\}$, i.e., $W_J = \hat{\mathfrak{S}}(1, 2) \times \mathfrak{S}(3, 3)$, so $N_1 = \phi$, $N_2 = \{1, 2\}$, $N_3 = \{3\}$. Note that this poset is isomorphic to that of Fig. 2; this is no accident since Coxeter systems of types A_3 and D_3 are isomorphic. (Recall that to obtain nonisomorphic systems, one may take A_n for $n \geqq 1$, B_n for $n \geqq 2$, and D_n for $n \geqq 4$.)

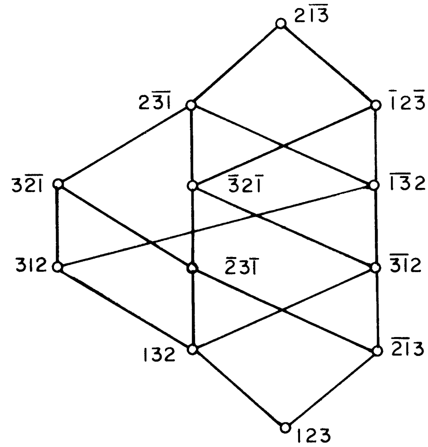

FIG. 8

As in the case of B_n, W^J need not be a distributive lattice when J is maximal. For instance, take $n = 4$ and $J = \{s_1, s_3, s_4\} = S - \{(23)\}$, so $W_J = \hat{\mathfrak{S}}(1, 2) \times \mathfrak{S}(3, 4)$. Then the rank-generating function of W^J is given by

$$F(W^J, q) = 1 + q + 3q^2 + 3q^3 + 4q^4 + 4q^5 + 3q^6 + 3q^7 + q^8 + q^9,$$

and it is easy to check that there does not exist a distributive lattice with this rank-generating function. As in the situation for B_n, there is one special case for which W^J *is* a distributive lattice. Take $J = \{s_1, s_2, \cdots, s_{n-1}\}$, so $W_J = \mathfrak{S}(1, n)$. If we regard $M(n)$ (as defined in the previous section) as consisting of all subsets of $\{1, 2, \cdots, n\}$, then W^J turns out to be the subposet of $M(n)$ consisting of all sets of even cardinality. But it is easily seen that this subposet is isomorphic to $M(n-1)$, so nothing new is obtained.

7. Final comments. In view of the examples $L(m, n)$ and $M(n)$, it is natural to ask under what circumstances is W^J a distributive lattice. I am grateful to Robert Proctor for supplying the following answer to this question. The Coxeter generators S of an irreducible Weyl group W correspond to the *fundamental representations* $\lambda_i (1 \leqq i \leqq n)$ of a certain complex simple Lie algebra $\mathfrak{g}$. By direct computation facilitated by representation theory, Proctor has shown that (except for the representations λ_1 and λ_2 of G_2) W^J is distributive if and only if the irreducible representation of $\mathfrak{g}$ with highest weight $\sum_{i \notin J} \lambda_i$ is *miniscule*, as defined in [6, p. 226]. These representations have special significance in other contexts; see [39] and more generally [28]. It turns out that for all the distributive W^J's except $L(m, n)$ and $M(n)$, it is easy to check Property S directly.

Proctor has also shown that if W is a Weyl group with largest element v (in the Bruhat order) and if W^J (for any $J \subset S$) has largest element y, then the bijection from W^J to W^J given by $w \to vwy^{-1}v^{-1}$ is an anti-automorphism of W^J. Thus W^J is self-dual whenever W is a Weyl group. We do not know whether the more general posets Q^X of Theorem 2.4 need always be self-dual.

We conclude with an open problem. Let P be a finite graded rank-symmetric poset of rank n, with rank function ρ. P is called a *symmetric chain order* (e.g., [17, §3], [20], [21]) if it can be partitioned into pairwise disjoint saturated chains $x_i < x_{i+1} < \cdots < x_{n-i}$ such that $\rho(x_j) = j$. It is easy to see that a symmetric chain order satisfies Property T and hence is rank-unimodal. Easy examples show that a rank-symmetric poset satisfying Property T need not be a symmetric chain order.

Our open problem is the following: Are all the posets Q^X of Theorem 2.4 (or at least the special cases W^J of Theorem 3.1) symmetric chain orders? Since any poset Q^X given by Theorem 2.4 has property T, there are pairwise disjoint chains connecting all of Q_i^X to Q_{i+1}^X when $i < n/2$, and all of Q_i^X to Q_{i-1}^X when $i > n/2$. Piecing together these chains yields a partition of Q^X into saturated chains all of which pass through the middle rank (when n is even) or middle two ranks (when n is odd). However, it is by no means clear whether these chains may be chosen to be symmetric about the middle.

Emden Gansner has pointed out to me that for type A_n, there is a rank-preserving, order-preserving bijection $\mathbf{1}\times\mathbf{2}\times\cdots\times\mathbf{n}\overset{\varphi}{\to} W=\mathfrak{S}_n$, where $\mathbf{1}\times\mathbf{2}\times\cdots\times\mathbf{n}=\{(b_1,\cdots,b_n)\colon 0\leqq b_i<i\}$. Namely, $\varphi(b_1,\cdots,b_n)$ is that permutation $\pi=a_1a_2\cdots a_n\in\mathfrak{S}_n$ such that b_i is the number of elements j appearing in π to the right of i and satisfying $j<i$. Since any product of chains is a symmetric chain order (e.g., [17, pp. 30–31]), it follows that $\mathfrak{S}_n$ (with the Bruhat order) is also a symmetric chain order. A similar argument for types B_n and D_n produces rank-preserving order-preserving bijections $\mathbf{2}\times\mathbf{4}\times\cdots\times\mathbf{2n}\to\bar{\mathfrak{S}}_n$ and $\mathbf{2}\times\mathbf{4}\times\cdots\times\mathbf{2(n-1)}\times\mathbf{n}\to\hat{\mathfrak{S}}_n$. Hence $\bar{\mathfrak{S}}_n$ and $\hat{\mathfrak{S}}_n$ are also symmetric chain orders. However, we do not know for instance whether $L(m,n)$ and $M(n)$ are always symmetric chain orders. Lindström [32] has shown that $L(3,n)$ is a symmetric chain order, and D. West [44] has shown that $L(4,n)$ is a symmetric chain order. Littlewood [33, pp. 193–203] claims to prove that $L(m,n)$ is indeed a symmetric chain order for all m and n. However, his proof is invalid. Specifically, it relies on the "method of chains" of Aitken [45], and this method is not correct as stated by Aitken. For the reader's benefit we will discuss the nature of Aitken's error in more detail. Let $P=\{x_1,\cdots,x_n\}$ be a finite poset, and let $\Phi=(a_{ij})$ be the $n\times n$ matrix defined by $a_{ij}=0$ unless $x_i<x_j$ in P; otherwise the a_{ij}'s are independent indeterminates over $\mathbb{Q}$. Remove a chain C_1 of maximum cardinality c_1 from P, then remove a chain C_2 of maximum cardinality c_2 from $P-C_1$, etc. Aitken essentially claims first that the numbers $c_1, c_2,\cdots$, are independent of the choice of chains $C_1, C_2,\cdots$, and second that the numbers $c_1, c_2,\cdots$ are the sizes of the Jordan blocks of Φ. The first claim is clearly false. However, Littlewood's proof would still be valid if there were *some* way of choosing $C_1, C_2,\cdots$ so that the second claim is true. Even this weaker result is false. Let P be the poset of Fig. 9. We have no choice but to take $c_1=4$, $c_2=1$, $c_3=1$. However, the Jordan block sizes of Φ are 4 and 2. A corrected version of Aitken's result appears in [37]. If this corrected result is used in conjunction with Littlewood's method, it yields the result that $L(m,n)$ has property T. Thus we have an alternative proof, avoiding the hard Lefschetz theorem (though actually Littlewood's method essentially proves the hard Lefschetz theorem for the Grassmann variety), that $L(m,n)$ has property T.

A further property of posets which implies the Sperner property is the LYM *property* [17, § 4]. However, Griggs has observed that $L(4,3)$ fails to satisfy the LYM property.

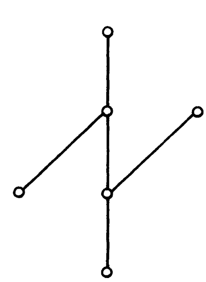

FIG. 9

Note added in proof. A proof that $L(3, m)$ and $L(4, m)$ have symmetric chain decompositions was first given by W. Riess, *Zwei Optimierungsprobleme auf Ordnungen*, Arbeitsberichte des Institute für Mathematische Maschinen und Datenverarbeitung (Informatik) 11, Number 5, Erlangen, April 1978.

REFERENCES

[1] A. C. AITKEN, *Determinants and Matrices*, 3rd ed., Oliver and Boyd, London, 1944.

[2] G. E. ANDREWS, *A theorem on reciprocal polynomials with applications to permutations and compositions*, Amer. Math. Monthly, 82 (1975), pp. 830–833.

[3] G. BIRKHOFF, *Lattice Theory*, third ed., American Mathematical Society, Providence, RI, 1967.

[4] A. BOREL AND A. HAEFLIGER, *La classe d'homologie fondamentale d'un espace analytique*, Bull. Soc. Math. France, 89 (1961), pp. 461–513.

[5] A. BOREL AND J. TITS, *Compléments à l'article: "Groupes réductifs,"* Inst. Hautes Études Sci. Publ. Math., No. 41 (1972), pp. 253–276.

[6] N. BOURBAKI, *Groupes et algèbres de Lie*, Éléments de Mathématique, Fasc. XXXIV, Hermann, Paris, 1968.

[7] E. R. CANFIELD, *A Sperner property preserved by product*, Linear and Multilinear Algebra, to appear.

[8] R. W. CARTER, *Simple Groups of Lie Type*, Wiley, New York, 1972.

[9] L. COMTET, *Advanced Combinatorics*, Reidel, Boston, 1974.

[10] M. CORNALBA AND P. A. GRIFFITHS, *Some transcendental aspects of algebraic geometry*, Algebraic Geometry, Arcata 1974, R. Hartshorne, ed., American Mathematical Society, Providence, RI, 1975, pp. 3-110.

[11] V. V. DEODHAR, *Some characterizations of Bruhat ordering on a Coxeter group and determination of the relative Möbius function*, Invent. Math., 39 (1977), pp. 187–198.

[12] R. C. ENTRINGER, *Representation of m as* $\sum_{k=-n}^{n} \varepsilon_k k$, Canad. Math. Bull., 11 (1968), pp. 289–293.

[13] P. ERDÖS, *Extremal problems in number theory*, Theory of Numbers, A. L. Whiteman, ed., American Mathematical Society, Providence, RI, 1965, pp. 181–189.

[14] W. FULTON, *Rational equivalence on singular varieties*, Inst. Hautes Etudes Sci. Publ. Math., no. 45 (1975), pp. 147–167.

[15] C. GREENE, *Some partitions associated with a partially ordered set*, J. Combinatorial Theory, 20 (1976), pp. 69–79.

[16] C. GREENE AND D. J. KLEITMAN, *The structure of Sperner k-families*, J. Combinatorial Theory, 20 (1976), pp. 41–68.

[17] C. GREENE AND D. J. KLEITMAN, *Proof techniques in the theory of finite sets*, Studies in Combinatorics, G.-C. Rota, ed., Mathematical Association of America, Washington, DC, 1978, pp. 22–79.

[18] P. GRIFFITHS AND J. ADAMS, *Topics in Algebraic and Analytic Geometry*, Princeton University Press, Princeton, NJ, 1974.

[19] P. GRIFFITHS AND J. HARRIS, *Principles of Algebraic Geometry*, Wiley, New York, 1978.

[20] J. R. GRIGGS, *Sufficient conditions for a symmetric chain order*, SIAM J. Appl. Math., 32 (1977), pp. 807–809.

[21] J. R. GRIGGS, *On chains and Sperner k-families in ranked posets*, J. Combinatorial Theory, to appear.

[22] A. GROTHENDIECK, *Sur quelques propriétés fondamentales en théorie des intersections*, Séminaire C. Chevalley, Ecole Normale Supérieure, Paris, 2 (1958), Chap. 4.

[23] L. H. HARPER, *Stabilization and the edgesum problem*, Ars Combinatoria, 4 (1977), pp. 225–270.

[24] W. V. D. HODGE, *The Theory and Applications of Harmonic Integrals*, 2nd ed., Cambridge University Press, London, 1952.

[25] J. W. B. HUGHES, *Lie algebraic proofs of some theorems on partitions*, Number Theory and Algebra, H. Zassenhaus, ed., Academic Press, New York, 1977, pp. 135–155.

[26] S. L. KLEIMAN, *Problem 15. Rigorous foundation of Schubert's enumerative calculus, Mathematical Developments Arising from Hilbert Problems*, F. E. Browder, ed., American Mathematical Society, Providence, RI, 1976, pp. 445–482.

[27] S. L. KLEIMAN AND D. LAKSOV, *Schubert calculus*, Amer. Math. Monthly, 79 (1972), pp. 1061–1082.

[28] V. LAKSHMIBAI, C. MUSILI AND C. S. SESHADRI, *Geometry of G/P*, Bull. Amer. Math. Soc., 1 (1979), pp. 432–435.

[29] S. LEFSCHETZ, *L'Analysis situs et la Géométrie Algébrique*, Gauthier-Villars, Paris, 1924; reprinted in Selected Papers, Chelsea, New York, 1971.

[30] B. LINDSTRÖM, *Conjecture on a theorem similar to Sperner's*, Combinatorial Structures and their Applications, R. Guy, H. Hanani, N. Sauer, and J. Schonheim, eds., Gordon and Breach, New York, 1970, p. 241.

[31] B. LINDSTRÖM, *Om en elementär sats i kombinatoriken och några tillämpningar*, Nordisk Mat. Tidskr., 17 (1969), pp. 61–70.

[32] ———, *A partition of $L(3, n)$ into saturated symmetric chains*, European J. Combinatorics, to appear.

[33] D. E. LITTLEWOOD, *The Theory of Group Characters and Matrix Representations of Groups*, 2nd ed., Oxford University Press, London, 1950.

[34] W. MESSING, *Short sketch of Deligne's proof of the hard Lefschetz theorem*, Algebraic Geometry, Arcata 1974, R. Hartshorne, ed., American Mathematical Society, Providence, RI, 1975, pp. 563–580.

[35] G. W. PECK, *Erdös' conjecture on sums of distinct numbers*, Studies in Applied Math., to appear.

[36] R. A. PROCTOR, M. E. SAKS AND D. G. STURTEVANT, *Product partial orders with the Sperner property*, Discrete Math., to appear.

[37] M. E. SAKS, *Dilworth numbers, incidence maps and product partial orders*, this Journal, this issue pp. 211–215.

[38] A. SÁRKÖZI AND E. SZEMERÉDI, *Über ein Problem von Erdös und Moser*, Acta Arith., 11 (1966), pp. 205–208.

[39] C. S. SESHADRI. *Geometry of G/P-I (Standard monomial theory for a miniscule P)*, C. P. Ramanujan: A Tribute, Springer-Verlag, 1978, published for the Tata Institute of Fundamental Research, Bombay.

[40] R. STANLEY, *Unimodal sequences arising from Lie algebras*, Proc. Young Day, to appear.

[41] ———, *Some combinatorial aspects of the Schubert calculus*, Combinatoire et Représentation du Groupe Symétrique, D. Foata, ed., Lecture Notes in Math. No. 579, Springer-Verlag, New York, 1977, pp. 217–251.

[42] J. H. VAN LINT, *Representation of 0 as $\sum_{k=-N}^{N} \varepsilon_k k$*, Proc. Amer. Math. Soc., 19 (1967), pp. 182–184.

[43] A. WEIL, *Introduction à l'étude des variétés Kähleriennes*, Hermann, Paris, 1958.

[44] D. B. WEST, *A symmetric chain decomposition of $L(4, n)$*, preprint.

[45] A. C. AITKEN, *The normal form of compound and induced matrices*, Proc. London Math. Soc., (2) 38 (1934), pp. 354–376.