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Differentiably Finite Power Series

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A formal power series $\sum f(n)x^n$ is said to be differentiably finite if it satisfies a linear differential equation with polynomial coefficients. Such power series arise in a wide variety of problems in enumerative combinatorics. The basic properties of such series of significance to combinatorics are surveyed. Some reciprocity theorems are proved which link two such series together. A number of examples, applications and open problems are discussed.

1. INTRODUCTION

Recently there has been interest [2], [3], [16] in the problem of computing quickly the coefficients of a power series $F(x)=\sum_{n\geq 0} f(n)x^n$, where say $F(x)$ is defined by a functional equation or as a function of other power series. If the coefficients $f(n)$ have a combinatorial meaning, then a fast algorithm for computing $f(n)$ would also be of combinatorial interest. Here we consider a class of power series, which we call *differentiably finite* (or D-finite, for short), whose coefficients can be quickly computed in a simple way. We consider various operations on power series which preserve the property of being D-finite, and give examples of operations which don't preserve this property. We mention some classes of power series for which it seems quite difficult to decide whether they are D-finite. Everything we say can be extended routinely from power series to Laurent series having finitely many terms with negative exponents, though for simplicity we will restrict ourselves to power series. Moreover, we will consider only complex coefficients, though virtually all of what we do is valid over any field of characteristic zero (and much is valid over any field).

The class of D-finite power series has been subject to extensive investigation, particularly within the theory of differential equations. However, a systematic exposition of their properties from a combinatorial point of view seems not to have been given before. Many of our results can therefore be found scattered throughout the literature, so this paper should be regarded as about 75% expository. To simplify and unify the concepts and proofs we have used the terminology and elementary theory of linear algebra, though all explicit dependence on linear algebra could be avoided without great difficulty.

Let us now turn to the basic definition of this paper. First note that the field $\mathbb{C}((x))$ of all formal Laurent series over $\mathbb{C}$ of the form $\sum_{n\geq n_0} f(n)x^n$ for some $n_0\in\mathbb{Z}$ contains the field $\mathbb{C}(x)$ of rational functions of x, and $\mathbb{C}((x))$ has the structure of a vector space over $\mathbb{C}(x)$.

DEFINITION 1.1. A formal power series $y\in\mathbb{C}[[x]]$ is said to be *differentiably finite* (or D-finite) if y together with all its derivatives $y^{(n)}=\mathrm{d}^n y/\mathrm{d}x^n$, $n\geq 1$, span a finite-dimensional subspace of $\mathbb{C}((x))$, regarded as a vector space over the field $\mathbb{C}(x)$.

THEOREM 1.2. *The following three conditions on a formal power series $y\in\mathbb{C}[[x]]$ are equivalent.*

(i) *y is D-finite.*

(ii) *There exist finitely many polynomials $q_0(x),\ldots,q_k(x)$, not all 0, and a polynomial $q(x)$, such that*

$$q_k(x)y^{(k)}+\cdots+q_1(x)y'+q_0(x)y=q(x). \tag{1}$$

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(iii) *There exist finitely many polynomials* $p_0(x), \ldots, p_m(x)$, *not all* 0, *such that*

$$p_m(x)y^{(m)}+\cdots+p_1(x)y'+p_0(x)y=0. \tag{2}$$

PROOF

(i)$\Rightarrow$(ii). Suppose y is D-finite. Let the dimension of the vector space over $\mathbb{C}(x)$ spanned by $y, y', y'', \ldots$ be k. Then with $q(x)=0$, (1) is just the relation of linear dependence (with denominators cleared) which exists among the $k+1$ series $y, y', \ldots, y^{(k)}$.

(ii)$\Rightarrow$(iii). Suppose (1) holds with $\deg q(x)=d$. Differentiating (1) $d+1$ times (with respect to x) yields (2).

(iii)$\Rightarrow$(i) Suppose (2) holds with $p_m(x)\neq 0$. Dividing by $p_m(x)$ shows that $y^{(m)}\in \langle y, y', \ldots, y^{(m-1)}\rangle$, where $\langle\cdots\rangle$ denotes span over $\mathbb{C}(x)$. Differentiating (2) with respect to x yields $y^{(m+1)}\in\langle y, y', \ldots, y^{(m)}\rangle=\langle y, y', \ldots, y^{(m-1)}\rangle$. Continued differentiation yields $y^{(m+i)}\in\langle y, y', \ldots, y^{(m-1)}\rangle$ for all $i\geq 0$, so y is D-finite.

We now consider the question of characterizing the *coefficients* of a D-finite power series.

DEFINITION 1.3. Let $\mathbb{N}$ denote the set of non-negative integers. A function $f:\mathbb{N}\to\mathbb{C}$ is said to be *polynomially recursive* (or P-recursive) if there exist finitely many polynomials $P_0(n), \ldots, P_d(n)$, with $P_d(n)$ not identically zero, such that for all $n\in\mathbb{N}$,

$$P_d(n)f(n+d)+P_{d-1}(n)f(n+d-1)+\cdots+P_0(n)f(n)=0. \tag{3}$$

Equation (3) defines $f(n+d)$ in terms of $f(n), f(n+1), \ldots, f(n+d-1)$, provided $P_d(n)\neq 0$. Hence if n_0 is large enough so that $P_d(n)\neq 0$ for all $n\geq n_0$, then (3) can be used to compute rapidly (and with relatively little storage space) the sequence of values $f(n+d)$ for $n\geq n_0$.

We next show that the property of being P-recursive depends only on the behaviour of $f(n)$ for n large. Equivalently, altering finitely many values of $f(n)$ does not affect whether or not $f(n)$ is P-recursive. Although this result is very easy to prove directly, it is convenient for what follows to formulate it using the concept of germs. Define two functions $f, g:\mathbb{N}\to\mathbb{C}$ to be *equivalent* if $f(n)=g(n)$ for all n sufficiently large. This clearly defines an equivalence relation; equivalence classes are called *germs* (more properly, "germs at ∞ of functions $f:\mathbb{N}\to\mathbb{C}$"). The germ containing f is called the *germ of f* and will be denoted $[f]$. Clearly addition and (pointwise) multiplication of functions is compatible with the above equivalence relation, so we can speak of the sum and product of germs.

THEOREM 1.4. *If $f, g:\mathbb{N}\to\mathbb{C}$ have the same germ, then f is P-recursive if and only if g is P-recursive. Thus it makes sense to speak of a P-recursive germ.*

PROOF. Suppose $f(n)=g(n)$ for all $n>n_0$. If f satisfies (3) for all $n\geq 0$, then g satisfies

$$Q(n)[P_d(n)g(n+d)+\cdots+P_0(n)g(n)]=0$$

for all $n\geq 0$, where $Q(n)=n(n-1)\cdots(n-n_0)$. Symmetrically, if g is P-recursive, then so is f.

We now come to the connection between P-recursive functions and D-finite power series. This result is alluded to in [13, p. 299].

THEOREM 1.5 *The formal power series $y=\sum_{n\geq 0} f(n)x^n\in\mathbb{C}[[x]]$ is D-finite if and only if $f(n)$ is P-recursive.*

PROOF. Suppose y is D-finite and that (2) holds. Since

$$x^i y^{(i)} = \sum_{n\geq 0} (n+i-j)(n+i-j-1)\cdots(n-j+1)x^n,$$

when we equate coefficients of x^n on both sides of (2) we will obtain a recurrence of the form (3) for $f(n)$.

Conversely, suppose $f(n)$ is P-recursive and satisfies (3). The polynomial $P_i(n)$ can be written (uniquely) as a linear combination of the polynomials $(n+i)_j = (n+i)(n+i-1)\dots(n+i-j+1), j\geq 0$. Hence the series $\sum P_i(n)f(n+i)x^n$ is a linear combination of those of the form $\sum_{n\geq 0}(n+i)_j f(n+i)x^n$. It is easy to see that for some polynomial $R_i(x)$,

$$\sum_{n\geq 0} (n+i)_j f(n+i)x^n = R_i(x) + x^{j-i}y^{(j)}.$$

Hence if we multiply (3) by x^n and sum on n, then after multiplying by a sufficiently high power of x we obtain a (non-zero) equation of the form (1), so y is D-finite.

We conclude this section with an analogue of Definition 1.1 for P-recursive functions. We cannot regard the space of all functions $f:\mathbb{N}\to\mathbb{C}$ as a vector space over the field $\mathbb{C}(n)$ of rational functions $R:\mathbb{N}\to\mathbb{C}$ because the rational function $R=P/Q$, where P and Q are polynomials, is not defined when $Q(n)=0$. However, the space of germs of functions $f:\mathbb{N}\to\mathbb{C}$ has an obvious structure of a vector space over the field $\mathbb{C}(n)$, viz., if $R\in\mathbb{C}(n)$ then define $R\cdot[f]$ to be the germ of any function g agreeing with $R(n)f(n)$ for all $n\in\mathbb{N}$ for which $R(n)$ is defined.

THEOREM 1.6. *A function $f:\mathbb{N}\to\mathbb{C}$ is P-recursive if and only if the span of the germs $[f(n)], [f(n+1)], [f(n+2)], \dots$ is a finite-dimensional subspace of the space of all germs of functions $f:\mathbb{N}\to\mathbb{C}$, regarded as a vector space over the field $\mathbb{C}(n)$.*

PROOF. Suppose (3) holds. Then

$$[f(n+d)] = -\sum_{i=0}^{d-1}\frac{P_i(n)}{P_d(n)}[f(n+i)],$$

so $[f(n+d)]\in\langle[f(n)],[f(n+1)],\dots,[f(n+d-1)]\rangle$, where $\langle\cdots\rangle$ denotes span over $\mathbb{C}(n)$. Substituting successively $n+1, n+2, \dots$ for n in (3) yields $[f(n+e)]\in\langle[f(n)],[f(n+1)],\dots,[f(n+d-1)]\rangle$ for all $e\geq d$.

Conversely, if the germs $[f(n)], [f(n+1)], \dots$ span a finite-dimensional subspace, then some non-trivial linear relationship $\sum_{i=0}^{d} R_i(n)[f(n+i)]=0$ holds, where each $R_i(n)\in\mathbb{C}(n)$. Clear the denominators of the $R_i(n)$'s and use Theorem 1.4 to conclude that $f(n)$ is P-recursive.

2. ALGEBRAIC PROPERTIES OF D-FINITE POWER SERIES

We now consider what kinds of operations can be performed on D-finite power series which again produce D-finite power series. This will yield a large class of examples of D-finite series. Note that Theorem 1.2 already shows that many familiar power series are D-finite, for example $y=e^x$ (since $y'-y=0$), $y=\sin x$ and $y=\cos x$ (since $y''+y=0$), $y=\log(1+x)$ (since $(1+x)y'=1$), etc. Theorem 1.5 also yields many quick examples, such as $y=\sum_{n\geq 0} n!x^n$, since $f(n)=n!$ satisfies $f(n+1)-(n+1)f(n)=0$. However, it is not evident at this point which (if any) of the power series $\sec x$, $e^{\sqrt{1-x}}$, $\sqrt{1-x}\,e^x$, $e^x+\sum_{n\geq 0} n!x^n$, $e^{e^{x}-1}$, and $\sqrt{1+\log(1-x^2)}$ are D-finite.

We first discuss an important class of D-finite power series. Recall that a power series $y \in \mathbb{C}[[x]]$ is said to be *algebraic* if $1, y, y^2, y^3, \ldots$ span a finite-dimensional vector space over $\mathbb{C}(x)$. Equivalently, there exist finitely many polynomials $Q_0(x), \ldots, Q_d(x)$, not all 0, such that $Q_d(x)y^d + \cdots + Q_1(x)y + Q_0(x) = 0$.

THEOREM 2.1. *If $y \in \mathbb{C}[[x]]$ is algebraic, then y is D-finite.*

REMARK. This simple result was well-known to early workers on algebraic functions, though I don't know where the first explicit statement appears. A proof may be found in [7] or [22, Theorem 5.1].

Theorems 1.5 and 2.1 show that the coefficients of an algebraic function can be quickly computed, though considerable conditioning may first be necessary to find the recurrence (3). A different method for rapidly computing the coefficients of an algebraic function appears in [16].

Note that the converse of Theorem 2.1 is certainly false. For instance, the power series e^x and $\Sigma n!x^n$ are D-finite but not algebraic.

EXAMPLE 2.2. Let $R(x)$ be a rational power series such that $R(0) \neq 0$. Thus there exists $y \in \mathbb{C}[[x]]$ satisfying $y^k = R(x)$. Then y satisfies the differential equation $kR(x)y' - R'(x)y = 0$.

Recall that the set $\mathcal{A}$ of all algebraic power series forms a subalgebra of $\mathbb{C}[[x]]$. Moreover, if $y \in \mathcal{A}$ and if the reciprocal $y^{-1} \in \mathbb{C}[[x]]$ (i.e., if $y(0) \neq 0$), then $y^{-1} \in \mathcal{A}$. If $u, v \in \mathcal{A}$ and $v(0) = 0$ (so that the formal composition $u(v(x))$ is defined), then $u(v(x)) \in \mathcal{A}$. (For if $\Sigma Q_i(x)u(x)^i = 0$, then $\Sigma Q_i(v(x))u(v(x))^i = 0$. Hence $u(v(x))$ is algebraic over $\mathbb{C}(v(x))$. Since $v(x)$ is algebraic over $\mathbb{C}(x)$, $u(v(x))$ is therefore algebraic over $\mathbb{C}(x)$.) If $y \in \mathcal{A}$ and $y = a_1x + \cdots$, $a_1 \neq 0$ (so the formal compositional inverse $y^{\langle -1 \rangle}$ exists, with the defining property $y^{\langle -1 \rangle}(y(x)) = y(y^{\langle -1 \rangle}(x)) = x$), then $y^{\langle -1 \rangle} \in \mathcal{A}$ (for if $\Sigma Q_i(x)y^i = 0$, then $\Sigma Q_i(y^{\langle -1 \rangle})x^i = 0$). We now consider to what extent these properties carry over to D-finite power series.

THEOREM 2.3. *The set $\mathcal{D}$ of D-finite power series forms a subalgebra of* $\mathbb{C}[[x]]$.

PROOF. If $y \in \mathbb{C}[[x]]$, let V_y denote the vector space over $\mathbb{C}(x)$ spanned by $y, y', y'', \ldots$. Now let $u, v \in \mathcal{D}$ and $\alpha, \beta \in \mathbb{C}$. Set $y = \alpha u + \beta v$. Then $y, y', y'', \ldots \in V_u + V_v$. Thus (taking dimensions over $\mathbb{C}(x)$),

$$\dim V_y \leq \dim(V_u + V_v) \leq \dim V_u + \dim V_v < \infty.$$

Hence by definition, y is D-finite.

It remains to show $uv \in \mathcal{D}$. Let $V = \mathbb{C}((x))$, regarded as a vector space over $\mathbb{C}(x)$. There is a unique linear transformation $\phi: V_u \otimes_{\mathbb{C}(x)} V_v \to V$ satisfying $\phi(u^{(i)} \otimes v^{(j)}) = u^{(i)}v^{(j)}$. By Leibnitz' rule for differentiating a product, the image of ϕ contains V_{uv}. Hence

$$\dim V_{uv} \leq \dim(V_u \otimes V_v) = (\dim V_u)(\dim V_v) < \infty,$$

so $uv \in \mathcal{D}$.

EXAMPLE 2.4. Let r be a positive integer, and set

$$B_r(n) = \sum_{k=0}^{n} \binom{n}{k}^r.$$

Define $F_r(x)=\Sigma_{n\geq 0} x^n/n!^r$. Clearly $F_r(x)$ is D-finite, since the coefficients $f(n)=1/n!^r$ satisfy $f(n+1)-(n+1)^r f(n)=0$. Hence by Theorem 2.3, $F_r(x)^2$ is D-finite. But

$$F_r(x)^2=\sum_{n\geq 0}\frac{B_r(n)x^n}{n!^r}.$$

In general, if $g(n)$ is P-recursive and satisfies $\sum_{i=0}^{d} P_i(n)g(n+i)=0$, then $h(n)=g(n)n!^r$ is also P-recursive since $\sum_{i=0}^{d} P_i(n)(n+d)^r(n+d-1)^r\ldots(n+i+1)^r h(n+i)=0$. (Theorem 2.10 below gives a more general result.) Hence $B_r(n)$ is P-recursive. Franel [10] in fact conjectured that $B_r(n)$ satisfies a recurrence of the form

$$(2n)^{r-1}B_r(n)=f_0(2n-1)B_r(n-1)+f_1(2n-2)B_r(n-2)$$
$$+\cdots+f_q(2n-1-q)B_r(n-q-1),$$

where $q=[(r-1)/2]$, and where $f_0,\ldots,f_q$ are polynomials with integer coefficients, each of degree $r-1$, satisfying $f_i(-m)=(-1)^{r-1}f_i(m)$. Apparently this conjecture is still open, although it probably should succumb to methods from the theory of hypergeometric functions.

The next example shows that unlike the situation for algebraic power series, the reciprocal of a D-finite power series need not be D-finite.

EXAMPLE 2.5. Let $y=\sec x$. Clearly $z=y^{-1}$ is D-finite, since $z''+z=0$. Carlitz [4, Theorem 4] has shown that y is not D-finite. Here we give a simpler proof. (For yet another proof, see Section 4(a).) Suppose to the contrary y satisfies (2). Now $y'=y\sqrt{y^2-1}$, $y''=y^3+y^2-y$, and in general by induction it is easily seen that $y^{(2i+1)}=L_i(y)\sqrt{y^2-1}$ and $y^{(2i)}=M_i(y)$, where L_i and M_i are polynomials (with complex coefficients), both of degree $2i+1$. Making these substitutions into (2) yields a *non-zero* polynomial equation in x, y, and $\sqrt{y^2-1}$ satisfied by y. Hence y is algebraic, contrary to a well-known and easily proved result. (For example, if y were algebraic, so would be $y^{-1}+\sqrt{y^{-2}-1}=e^x$. But e^x can be seen to be non-algebraic in several ways, such as by Eisenstein's Theorem [17, Part 8, Chapter 3, Section 2], or by differentiating the alleged polynomial equation of smallest degree satisfied by e^x, or by observing that the coefficients of e^x are rational but $e=e^1$ is transcendental.)

We now consider the effect of functional composition on D-finite power series: first some negative results.

EXAMPLE 2.6. Let $y=(\log(1+x^2))^{\frac{1}{2}}$. Note that:

(i) $z=y^2\in\mathcal{D}$, since $z'(1+x^2)-2x=0$.

(ii) Set $F(x)=\sqrt{1+x}$ and $G(x)=\log(1+x^2)-1$, so $F\in\mathcal{A}$ and $G\in\mathcal{D}$. Then $y=F(G(x))$.

We now show $y\notin\mathcal{D}$. It follows from (i)–(ii) above that square roots of D-finite power series need not be D-finite, and that algebraic power series composed with D-finite power series need not be D-finite. We have

$$y'=\frac{1}{y}\frac{x}{1+x^2},$$

$$y''=\frac{1}{y}\cdot\frac{1-x^2}{(1+x^2)^2}-\frac{y'}{y^2}\cdot\frac{x}{1+x^2}$$

$$=\frac{1}{y}\cdot\frac{1-x^2}{(1+x^2)^2}-\frac{1}{y^3}\cdot\frac{x^2}{(1+x^2)^2}$$

and in general by induction,

$$y^{(i)} = \frac{1}{y} R_{1i}(x) + \frac{1}{y^3} R_{2i}(x) + \cdots + \frac{1}{y^{2i-1}} R_{ii}(x),$$

where each $R_{ji}(x) \in \mathbb{C}(x)$ and $R_{ii}(x) \neq 0$. If y satisfied (2), then making this substitution into (2) would yield a *non-zero* polynomial equation satisfied by y. However y cannot be algebraic, since, for example, it would easily follow that e^x is algebraic.

A simple example (brought to my attention by Ira Gessel) of a D-finite power y whose inverse $y^{\langle -1 \rangle}$ is not D-finite is given by $y = \tan^{-1} x$. One shows that $\tan x$ is not D-finite by an argument similar to that of Example 2.5, or by the techniques of [4].

In view of Example 2.6, the next result comes as somewhat of a surprise.

THEOREM 2.7. *If $F(x) \in \mathscr{D}$ and $G(x) \in \mathscr{A}$ with $G(0) = 0$, then $F(G(x)) \in \mathscr{D}$.*

PROOF. Let $y = F(G(x))$. We can write $y^{(i)}$ as a linear combination of $F(G(x))$, $F'(G(x))$, $F''(G(x)), \ldots$, with coefficients in $\mathbb{C}[G, G', G'', \ldots]$, the ring of polynomials in $G, G', G'', \ldots$ with complex coefficients. Since G is algebraic, it follows from differentiating the polynomial equation satisfied by G that each $G^{(i)}$ is a rational function of x and G. Hence $\mathbb{C}[G, G', G'', \ldots] \subset \mathbb{C}(x, G)$. Let V be the vector space of all $\mathbb{C}(x, G)$-linear combinations of $F(G(x))$, $F'(G(x)), \ldots$. Since $F, F', F'', \ldots$ span a finite-dimensional vector space over $\mathbb{C}(x)$ (because F is D-finite), it follows that $F(G(x)), F'(G(x)), \ldots$ span a finite-dimensional vector space over $\mathbb{C}(G)$ and hence over $\mathbb{C}(x, G)$. Since V is finite over $\mathbb{C}(x, G)$ and $\mathbb{C}(x, G)$ is finite over $\mathbb{C}(x)$, V is finite over $\mathbb{C}(x)$. Since each $y^{(i)} \in V$, it follows that $y \in \mathscr{D}$.

EXAMPLE 2.8. Let $M(n)$ be the number of extreme points (vertices) of the convex polytope of all $n \times n$ *symmetric* doubly-stochastic matrices. Katz [14] obtains a formula for $M(n)$ and uses it to compute $M(n)$ for $1 \leq n \leq 6$. It can be shown (either from Katz' formula or by a more direct argument) that

$$y = \sum_{n \geq 0} \frac{M(n) x^n}{n!} = \left(\frac{1+x}{1-x}\right)^{\frac{1}{4}} \exp\left(\frac{x}{2} + \frac{x^2}{2}\right). \tag{4}$$

It follows immediately from Theorems 2.3 and 2.7 that y is D-finite. (This is also easy to see directly, since $y'/y \in \mathbb{C}(x)$.) We easily compute from (4) that

$$M(n+4) = M(n+3) + (n+2)^2 M(n+2) - \binom{n+3}{2} M(n+1) - (n+3)(n+2)(n+1) M(n).$$

This yields a much faster method for computing $M(n)$ than Katz' formula.

Similarly, if $M^*(n)$ denotes the number of extreme points of the convex polytope of all $n \times n$ symmetric doubly *substochastic* matrices [15] (i.e., symmetric matrices of non-negative real numbers, with every row sum at most one), then one can show that

$$\sum_{n \geq 0} \frac{M^*(n) x^n}{n!} = \left(\frac{1+x}{1-x}\right)^{\frac{1}{4}} \exp\left(x + \frac{1}{2} x^2 + \frac{1}{2} \frac{x}{1-x^2}\right).$$

Again it is easy to compute the recurrence (3) satisfied by $M^*(n)$.

EXAMPLE 2.9. If $f: \mathbb{N} \to \mathbb{C}$, then the *nth difference of f at* 0 is defined by

$$\Delta^n f(0) = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} f(i),$$

or equivalently,

$$f(n)=\sum_{i=0}^{n}\binom{n}{i}\Delta^i f(0).$$

If $F(x)=\sum_{n\geq 0} f(n)x^n$, then let $F^*(x)=\sum_{n\geq 0}(\Delta^n f(0))x^n$. It is easy to see that

$$F^*(x)=\frac{1}{1+x}F\left(\frac{x}{1+x}\right) \quad \text{and} \quad F(x)=\frac{1}{1-x}F^*\left(\frac{x}{1-x}\right).$$

It follows from Theorems 2.3 and 2.7 that $F(x)$ is D-finite if and only if $F^*(x)$ is D-finite. (Since the composition of algebraic functions is algebraic, we also have that $F(x)$ is algebraic if and only if $F^*(x)$ is algebraic.)

The final operation on power series which we consider is the *Hadamard product.* If $y=\sum f(n)x^n$ and $z=\sum g(n)x^n$, then by definition the Hadamard product $y*z$ is the power series

$$y*z=\sum f(n)g(n)x^n.$$

It is well-known that the Hadamard product $y*z$ of algebraic power series need not be algebraic. The standard example [13, p. 298], [17, Part 8, no. 148], [6, p. 25], [11, pp. 271–272] is given by

$$y=z=(1-4x)^{-\frac{1}{2}}=\sum_{n\geq 0}\binom{2n}{n}x^n.$$

(It is known, however, that the Hadamard product of an algebraic power series and a rational power series is algebraic [13, Theorem 8]). In contrast to the algebraic case, we have the following result.

THEOREM 2.10. *Let y and z be D-finite power series. Then $y*z$ is D-finite. Equivalently, if $f(n)$ and $g(n)$ are P-recursive, then $f(n)g(n)$ is P-recursive.*

PROOF. By Theorem 1.4, it suffices to show that the germ $[f(n)g(n)]$ is P-recursive. In view of Theorem 1.5, we can simply mimic the proof that the product of D-finite power series is D-finite (Theorem 2.3). Let V be the vector space (over the field $\mathbb{C}(n)$) of all germs of functions $f:\mathbb{N}\to\mathbb{C}$. Given $f:\mathbb{N}\to\mathbb{C}$, let V_f be the subspace of V spanned by $[f(n)]$, $[f(n+1)]$, $[f(n+2)], \ldots$. There is a unique linear transformation $\phi: V_f\otimes_{\mathbb{C}(n)} V_g\to V$ satisfying $[f(n+i)]\otimes[g(n+j)]\overset{\phi}{\mapsto}[f(n+i)g(n+j)]$. Clearly the image of ϕ contains V_{fg}, so

$$\dim V_{fg}\leq \dim(V_f\otimes V_g)=(\dim V_f)(\dim V_g)<\infty.$$

Hence fg is P-recursive.

A proof of Theorem 2.10 avoiding tensor products appears in [13, p. 299] and is attributed to Hurwitz. This proof as stated is slightly inaccurate, though it is easy to fix by working with n sufficiently large (or equivalently, with germs).

3. RECIPROCITY.

If $f:\mathbb{N}\to\mathbb{C}$ satisfies a linear homogeneous recurrence with *constant* coefficients, then the recurrence can be "run backwards" to define f for all $n\in\mathbb{Z}$. Define the generating functions $F(x)=\sum_{n\geq 0} f(n)x^n$ and $\bar{F}(x)=\sum_{n\geq 1} f(-n)x^n$. Then $F(x)$ and $\bar{F}(x)$ are rational functions, and Popoviciu's theorem [18], [22, Theorem 4.4] states that $\bar{F}(x)=-F(1/x)($ as rational functions). We want to extend this result to P-recursive functions. One difficulty is that it may not be possible to run the recurrence (3) backwards because of the possibility that $P_0(n)=0$ for some $n<0$. Thus for the time being we assume that $f(n)$ is defined *a priori* for all $n \in \mathbb{Z}$.

THEOREM 3.1. *Let $D = \mathrm{d}/\mathrm{d}x$, and let $\Omega = \sum_{i=0}^{m} p_i(x)D^i$ be a linear differential operator with polynomial coefficients $p_i(x)$. Thus for any $z = \sum_{n\geq 0} g(n)x^n \in \mathbb{C}[[x]]$ we have*

$$\Omega z = \sum_{n\geq 0} [P_d(n)g(n+d)+\cdots+P_0(n)g(n)]x^{n+j} + q_z(x) \tag{5}$$

for certain polynomials $P_0, \ldots, P_d$ independent of z, a polynomial q_z depending on z, and a non-negative integer j independent of z. Let $f:\mathbb{Z}\to\mathbb{C}$ satisfy (3) for all $n\in\mathbb{Z}$. Define $y=\sum_{n\geq 0} f(n)x^n \in \mathbb{C}[[x]]$ and $\bar{y} = \sum_{n<0} f(n)x^n \in \mathbb{C}[[1/x]]$. Then

$$\sum_{i=0}^{m} p_i(x)y^{(i)} = q_y(x) \tag{6}$$

$$\sum_{i=0}^{m} p_i(x)\bar{y}^{(i)} = -q_y(x). \tag{7}$$

PROOF. Equation (6) is clear. Now Ω is clearly a linear operator on the space of all Laurent series $\sum_{n=-\infty}^{\infty} g(n)x^n$. Since f satisfies (3) for all $n\in\mathbb{Z}$, we have $\Omega(y+\bar{y})=0$, so (7) follows.

In the special case $m=0$, Equations (6) and (7) take the form $p(x)y=q(x)$ and $p(x)\bar{y}=-q(x)$ for certain polynomials p and q with $\deg q<\deg p$. If we substitute $1/x$ for x in the second equation, then we obtain Popoviciu's theorem.

Theorem 3.1 has the defect that given $f(n)$, we must choose a *particular* differential operator Ω in order to deduce the reciprocal formulas (6) and (7). Ideally, given the D-finite power series $y=\sum_{n\geq 0} f(n)x^n$, we would like to take *any* differential equation (6) and deduce (7). In general this is not possible. For instance, although $y=\sum_{n\geq 0} n!x^n$ is D-finite, there is *no* function $f:\mathbb{Z}\to\mathbb{C}$ satisfying $f(n)=n!$ for $n\geq 0$ and satisfying a non-trivial recurrence (3) for all $n\in\mathbb{Z}$. Suppose then we simply assume we are given $f:\mathbb{Z}\to\mathbb{C}$ satisfying (3) for all $n\in\mathbb{Z}$. There are still difficulties. For example, consider the equation

$$(n+1)f(n)=0. \tag{8}$$

One solution $f:\mathbb{Z}\to\mathbb{C}$ to (8) is given by

$$f(n)=\begin{cases}0, & n\neq -1\\ 1, & n=-1.\end{cases}$$

Thus $y=\sum_{n\geq 0} f(n)x^n$ satisfies $y=0$, but $\bar{y}=\sum_{n\geq 1} f(-n)x^{-n}$ does not satisfy the reciprocal equation $\bar{y}=0$. The reciprocity theorem fails because, in a sense, the "correct" equation satisfied by $f(n)$ for $n\geq 0$ is $f(n)=0$, not $(n+1)f(n)=0$. The factor of $n+1$ introduced a "spurious" degree of freedom into the behavior of $f(n)$ for $n<0$. In order for a reciprocity theorem to hold for *any* differential equation satisfied by $y=\sum_{n\geq 0} f(n)x^n$, we need an hypothesis which guarantees that the values of $f(n)$ for $n\geq 0$ and for $n<0$ are correctly coupled. First we need the following lemma.

LEMMA 3.2. *Suppose $f:\mathbb{N}\to\mathbb{C}$ is P-recursive. Let d be the least non-negative integer such that f satisfies a recurrence of the form (3). Then there are unique polynomials $P_0(n),\ldots,P_d(n)$ such that* (i) *$P_d(n)$ is monic,* (ii) *f satisfies* (3), *and* (iii) *if*

$Q_0(n), \ldots, Q_d(n)$ are polynomials such that

$$Q_d(n)f(n+d)+\cdots+Q_0(n)f(n)=0, \tag{9}$$

then for some polynomial $R(n)$, we have $Q_i(n)=R(n)P_i(n)$ for $0 \leq i \leq d$.

PROOF. Given d, choose $P_0(n), \ldots, P_d(n)$ so that $P_d(n)$ is monic, f satisfies (3), and $\deg P_d(n)$ is minimal. Now suppose f satisfies (9). Let $K(n)$ be the greatest monic common divisor of $P_d(n)$ and $Q_d(n)$, and choose polynomials $A(n)$, $B(n)$ so that $A(n)P_d(n)+B(n)Q_d(n)=K(n)$. Then taking $A(n)$ times (3) plus $B(n)$ times (9), we see from the definition of $P_d(n)$ that $P_d(n)=K(n)$, so $P_d(n)R(n)=Q_d(n)$ for some polynomial $R(n)$. Now taking $R(n)$ times (3) minus (9), we get from the minimality of d that $P_i(n)R(n)=Q_i(n)$ for $0 \leq i \leq d$.

LEMMA 3.3. *Let $f: \mathbb{N} \to \mathbb{C}$ be p-recursive, and let (3) be the unique equation of least degree satisfying the conditions of Lemma 3.2. Suppose that $P_d(n)$ has no integral zeros. Fix any extension (also called f) of f to $\mathbb{Z}$ satisfying (3) for all $n \in \mathbb{Z}$. (If no such extension exists, then this lemma is inapplicable.) Suppose that f also satisfies*

$$Q_e(n)f(n+e)+\cdots+Q_0(n)f(n)=0 \tag{10}$$

for all n sufficiently large (where the Q_i are polynomials in n). Then (10) continues to hold for all $n \in \mathbb{Z}$.

PROOF. Suppose (10) holds for all $n \geq k \geq 0$. First note $e \geq d$, since

$$\left[\prod_{i=0}^{k-1}(n-i)\right](Q_e(n)f(n+e)+\cdots+Q_0(n)f(n))=0 \tag{11}$$

holds for all $n \in \mathbb{Z}$. We now use induction on e. If $e=d$, then by Lemma 3.2 (11) is a multiple of (3). Since $P_d(n) \neq 0$ for $n \in \mathbb{Z}$, it follows that (10) holds for all $n \in \mathbb{Z}$.

Now assume $e>d$ and that the lemma holds for all $e'<e$. Substitute $n+e-d$ for n in (3), multiply by $Q_e(n)$, and subtract Equation (10) multiplied by $P_d(n+e-d)$. We get an equation of degree less than e satisfied by $f(n)$ for all n sufficiently large. By the induction hypothesis, this equation holds for all $n \in \mathbb{Z}$. Since equation (3) holds for all $n \in \mathbb{Z}$, it follows that equation (10) times $P_d(n+e-d)$ holds for all $n \in \mathbb{Z}$. Since $P_d(n+e-d) \neq 0$ for $n \in \mathbb{Z}$, the result follows.

THEOREM 3.4. *Let $f: \mathbb{N} \to \mathbb{C}$ be P-recursive, and let (3) be the unique equation of least degree satisfying the conditions of Lemma 3.2. Suppose that $P_d(n)$ has no integral zeros. Fix any extension of f to $\mathbb{Z}$ satisfying (3) for all $n \in \mathbb{Z}$. (If no such extension exists, then this theorem is inapplicable.) Define*

$$y=\sum_{n \geq 0} f(n)x^n, \qquad \bar{y}=\sum_{n<0} f(n)x^n.$$

Suppose that

$$\sum_{i=0}^{m} p_i(x)y^{(i)}=q(x)$$

is any linear differential equation with polynomial coefficients $p_i(x)$ and $q(x)$ satisfied by y. Then

$$\sum_{i=0}^{m} p_i(x)\bar{y}^{(i)}=-q(x).$$

PROOF. Let $\Omega = \sum_{i=0}^{m} p_i(x)D^i$. Then for any $z = \sum_{n\geq 0} g(n)x^n \in \mathbb{C}[[x]]$ there are polynomials $P_0(n), \ldots, P_d(n)$ independent of z, a polynomial $q_z(x)$ depending on z, and a non-negative integer j independent of z, for which (5) holds. It follows that $P_d(n)f(n+d) + \cdots + P_0(n)f(n) = 0$ for all n sufficiently large. By Lemma 3.3, this equation holds for all $n \in \mathbb{Z}$. The proof now follows from Theorem 3.1.

We now have a satisfactory generalization of Popoviciu's theorem to D-finite power series. It would also be desirable to have a generalization to algebraic power series. Of course every algebraic series is D-finite so Theorems 3.1 and 3.4 apply, but ideally we want a reciprocity theorem which only refers to the polynomial equation satisfied by the algebraic series y, not to the differential equation which y satisfies. We will give one such result here, based on an idea of James Shearer. However, a more general result might be true—see the remarks after the proof of the following proposition.

PROPOSITION 3.5. *Let $y = \sum_{n\geq 0} f(n)x^n \in \mathbb{C}[[x]]$ be D-finite and analytic at $x = 0$, and suppose that f satisfies* (3) *for all $n \in \mathbb{N}$. Assume that y satisfies the following three additional conditions.*

(i) *The polynomial $P_0(n)$ of* (3) *has no zeros at negative integers. Thus we can uniquely define $f(n)$ for all $n \in \mathbb{Z}$ so that* (3) *holds.*

(ii) *Let $\Omega = \sum_{i=0}^{m} p_i(x)D^i$ be the linear differential operator satisfying* (5). *Then* $\deg p_i(x) \leq i$.

(iii) *The analytic function y has a branch y^* at ∞ satisfying*

$$y^* = \sum_{n<0} f^*(n)x^n$$

for some complex numbers $f^*(n)$.

Then $f^*(n) = -f(-n)$. Thus $-\bar{y} = -\sum_{n<0} f(n)x^n$ satisfies any functional equation (defined in terms of analytic operations) satisfied by y. In particular, if y is algebraic, then y and $-\bar{y}$ satisfy the same polynomial equation.

PROOF. By condition (ii), we have

$$\Omega \sum_{n<0} g(n)x^n = \sum_{n<-m} [P_m(n)g(n+m) + \cdots + P_0(n)g(n)]x^n$$

$$+ \sum_{n=-m}^{-1} [P_{-n-1}(n)g(-1) + P_{-n-2}(n)g(-2) + \cdots + P_0(n)g(n))x^n.$$

In particular,

$$\Omega \sum_{n<0} f(n)x^n = \sum_{n=-m}^{-1} [P_{-n-1}(n)f(-1) + \cdots + P_0(n)f(n)]x^n.$$

Call the right-hand side of the above equation $-q(x)$. If $z = \sum_{n<0} g(n)x^n$ satisfies $\Omega z = -q(x)$, then for all n satisfying $-m \leq n \leq -1$ we have

$$P_{-n-1}(n)f(-1) + \cdots + P_0(n)f(n) = P_{-n-1}(n)g(-1) + \cdots + P_0(n)g(n).$$

This is a triangular system of m equations in the m unknowns $g(-1), \ldots, g(-m)$. The coefficients on the main diagonal are $P_0(-1), \ldots, P_0(-m)$, which by condition (i) are non-zero. Hence $g(n) = f(n)$ for $-m \leq n \leq -1$. Thus by condition (i), $g(n) = f(n)$ for all $n < 0$. In other words, the series $\bar{y} = \sum_{n<0} f(n)x^n$ is the unique series of the form $z = \sum_{n<0} g(n)x^n$ satisfying $\Omega z = -q(x)$. On the other hand, by Theorem 3.1 we have $\Omega y = q(x)$. By analytic continuation, any branch y_1 (analytic in some open subset of $\mathbb{C}$) of the analytic function defined by y also satisfies $\Omega y_1 = q(x)$. In particular, $\Omega y^* = q(x)$. Hence $y^* = -\bar{y}$, and the proof follows.

Proposition 3.5 and its proof raise two questions.

(a) Conditions (i)–(iii) (especially (ii)) are extremely restrictive. For instance, they are satisfied by an algebraic function of y of degree two if, and only if, $y^2+(ax+b)y+c=0$ for complex numbers a, b, c with $ac\neq 0$. Can the conditions (i)–(iii) be relaxed? Toward this end we raise the following question.

Question

(a) Let $y=\sum_{n\geq 0} f(n)x^n \in \mathbb{C}[[x]]$ be algebraic, and suppose that y satisfies (3) for all $n\in\mathbb{N}$. Assume that for all $n\in\mathbb{Z}$ we have $P_0(n)\neq 0$ and $P_d(n)\neq 0$. Thus in particular we can uniquely define $f(n)$ for all $n\in\mathbb{Z}$ so that (3) holds. Is it true that the series $y^*=-\sum_{n<0} f(n)x^n$ is a branch of y at ∞? (The answer is negative if we only assume y to be analytic; e.g., let $f(n)$ satisfy $(n+\frac{1}{2})f(n+1)-f(n)=0, f(0)=1$. This suggests a negative answer also for algebraic functions, though it can be shown that if y is algebraic then y^* is a branch at ∞ of some element of the splitting field of y.)

(b) The proof of Proposition 3.5 used complex variable theory in an essential way. Can a purely algebraic (or formal) proof be given of this result when y is algebraic?

EXAMPLE 3.6. Suppose $2y^2+(3-x)y+1=0$. (See [22, p. 129] for the significance of this equation.) There are two power series $y=\sum_{n\geq 0} f(n)x^n$ satisfying this equation. Both of these series also satisfy

$$(n+2)f(n+2)-3(2n+1)f(n+1)+(n-1)f(n)=0 \tag{12}$$

$$(x^2-6x+1)y'-(x-3)y=-2.$$

Moreover, the branches y^* at ∞ satisfy $2xy^*(1/x)^2+(3x-1)y^*(1/x)+x=0$, so there is a (unique) branch $y^*=x^{-1}+3x^{-2}+11x^{-3}+\ldots$ of the form $\sum_{n<0} g(n)x^n$. Thus conditions (i)–(iii) of Proposition 3.5 are satisfied. It follows that *both* series $y=\sum_{n\geq 0} f(n)x^n$ must satisfy $\sum_{n<0} f(-n)x^n=-y^*$. Now the two series y begin $-1+x+\ldots$ and $-\frac{1}{2}-\frac{1}{2}x+\ldots$. Hence the recurrence (12), with either of the initial conditions $f(0)=-1$, $f(1)=1$ or $f(0)=f(1)=-\frac{1}{2}$ yield the same values of $f(n)$ for $n<0$.

4. FURTHER PROBLEMS AND EXAMPLES

We collect here a list of miscellaneous problems and examples concerning D-finite series.

(a) Develop general methods for determining when a power series is D-finite. A useful necessary condition for D-finiteness follows from the theory of differential equations. Namely, suppose that y is D-finite, satisfies (1) with $q_k(x)\neq 0$, and is analytic at $x=0$. Then y can be extended to an analytic function in any simply-connected region of the complex plane not containing a zero of $q_k(x)$. Thus for instance sec x is not D-finite (since it has infinitely many poles), and the partition generating function $\prod_{n\geq 1}(1-x^n)^{-1}$ is not D-finite, since it has the unit circle as a natural boundary.

(b) Suppose that $F(x_1,\ldots,x_k)=\sum f(a_1,\ldots,a_k)x_1^{a_1}\cdots x_k^{a_k}$ is a power series over $\mathbb{C}$ in k variables which represents a rational function of $x_1,\ldots,x_k$. Define the *diagonal power series* $\operatorname{diag} F=\sum_{n\geq 0} f(n,n,\ldots,n)x^n$. When $k=2$, diag F is known to be algebraic [11, p. 273], [9, Section 5], [22, Theorem 5.3]. When $k\geq 3$ it is known that diag F need not be algebraic (e.g., when $F(x_1,x_2,x_3)=(1-x_1-x_2-x_3)^{-1}$). Using the methods of [24] and some results from differential algebra, D. Zeilberger has shown (private communication) that diag F is D-finite for any k.

(c) Let $S_k(n)$ be the number of standard Young tableaux (e.g. [8, pp. 125–126]) with the n entries $1,2,\ldots,n$ and with $\leq k$ rows. Let $y_k=\sum_{n\geq 0} S_k(n)x^n$. It is well-known that y_1 and y_2 are algebraic, and it follows from [21, pp. 30–31] that y_3 is algebraic. (One can also give a more direct proof of this result.) It apparently is not known whether y_k is algebraic or D-finite for $k\geq 4$.

(d) Let $H_n(r)$ (respectively, $I_n(r)$) denote the number of $n\times n$ matrices (respectively, symmetric matrices) of non-negative integers such that every row and column sums to r. (Various modifications are possible; for instance, we could restrict the entries to be 0 or 1.) Are the power series $y_r=\sum_{n\geq 0} H_n(r)x^n$ and $z_r=\sum_{n\geq 0} I_n(r)x^n$ D-finite? It follows from [1] and [12] that the answer is affirmative for $r\leq 2$ (see also [21, Example 6.11]) but the case $r=3$ remains open. By a laborious computation Read [19, p. 351] [20, Section 3] has verified that if J_n denotes the number of $n\times n$ symmetric matrices of 0's and 1's with trace 0 and every row sum equal to 3, then $\sum_{n\geq 0} J_n x^n$ is D-finite. Can Read's techniques be extended to y_r and z_r?

(e) The recent proof by Apéry that $\zeta(3)$ is irrational (see [23] for a nice survey of this result) involves the recurrence

$$n^3u_n+(n-1)^3u_{n-2}=(34n^3-51n^2+27n-5)u_{n-1}, \qquad n\geq 2.$$

This recurrence is satisfied, for example, by

$$u_n=\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2=\sum_{k=0}^{n}\frac{(n+k)!^2}{k!^4(n-k)!^2}.$$

This leads to the question as to whether a function $g(n)=\sum_{k=0}^{n} f_1(k)f_2(n-k)f_3(n+k)$ is P-recursive provided each f_i is P-recursive. Of course more general questions of this nature can be raised.

(f) The recurrence [5, equation (21)]

$$\begin{aligned}(n+1)(n+2)(n+3)(3n-2)B(n)&=2(n+1)(9n^3+3n^2-4n+4)B(n-1)\\&\quad+(3n-1)(n-2)(15n^2-5n-14)B(n-2)\\&\quad+8(3n+1)(n-2)^2(n-3)B(n-3), \qquad n\geq 4,\end{aligned}$$

is satisfied by the function

$$B(n)=\binom{n+1}{1}^{-1}\binom{n+2}{2}^{-1}\sum_{k=1}^{n}\binom{n+1}{k-1}\binom{n+1}{k}\binom{n+1}{k+1}. \tag{13}$$

Here $B(n)$ is the number of "reduced Baxter permutations" of $\{1,2,\ldots,n\}$. One can easily see from (13) that $B(n)$ is P-recursive by an argument analogous to that of Example 2.4, though to get the actual recurrence requires a lot of computation. Can (13) be generalized to counting other classes of permutations?

(g) Given the differential equation (2), together with suitable initial conditions, satisfied by a D-finite power series y, give an algorithm suitable for computer implementation for deciding whether y is algebraic. One method for showing that a power series y such as $\sum_{n\geq 0}\binom{2n}{n}^2x^n$ is not algebraic is analytic, i.e., showing that the coefficients $\binom{2n}{n}^2$ have a rate of growth incompatible with the behaviour of algebraic functions (such as described in [13]). Alternatively, if the coefficients of $y=F(x)$ are rational (or even algebraic) numbers, then sometimes one can show that as a function of a complex variable, $F(x)$ is transcendental for some algebraic value of x [11, pp. 271–272]. Both these methods treat y as a function of a complex variable, not as a formal power series. Is there a more algebraic technique, and of greater generality, for deciding whether a power series is algebraic? (One purely formal criterion for algebraicity is Eisenstein's theorem [17, Part 8, Chapter 3, Section 2], but it is inapplicable to power series with integer coefficients.) In particular, while $\sum_{n\geq 0}\binom{2n}{n}^{2k}x^n$ is easily proved to be non-algebraic by standard analytic techniques for all integers $k\geq 1$, we are unable to decide whether $\sum_{n\geq 0}\binom{2n}{n}^{2k+1}x^n$ is algebraic for $k\geq 1$.

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Note added in proof

(i) An independent proof that diag F is D-finite (Section 4(b)) has been given by I. Gessel, Two theorems on rational power series, *Utilitas Math.*, to appear.

(ii) D. Zeilberger has shown (private communication) that the series y_k of Section 4(c) is D-finite for all $k \geqslant 1$.

(iii) D. Zeilberger has also shown, using his theory of "special functions", that many functions of the type considered in Section 4(e) are D-finite.

(iv) In connection with Section 4(g), A. van der Poorten has kindly brought to my attention the references: F. Baldessarri and B. Dwork, On second order linear differential equations with algebraic solutions, *Amer. J. Math.* **101** (1979), 42–76; and Y. Amice, Les Nombres *P*-adiques, Presses Universitaires de France, 1975, chap. 5.