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The Number of Faces of a Simplicial Convex Polytope*

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Let P be a simplicial convex d-polytope with $f_i = f_i(P)$ faces of dimension i. The vector $\mathbf{f}(P) = (f_0, f_1, \ldots, f_{d-1})$ is called the *f-vector* of P. In 1971 McMullen [6; 7, p. 179] conjectured that a certain condition on a vector $\mathbf{f} = (f_0, f_1, \ldots, f_{d-1})$ of integers was necessary and sufficient for $\mathbf{f}$ to be the f-vector of some simplicial convex d-polytope. Billera and Lee [1] proved the sufficiency of McMullen's condition. In this paper we prove necessity. Thus McMullen's conjecture is completely verified.

First we describe McMullen's condition. Given a simplicial convex d-polytope P with $\mathbf{f}(P) = (f_0, f_1, \ldots, f_{d-1})$, define

$$h_i = h_i(P) = \sum_{j=0}^{i} \binom{d-j}{d-i} (-1)^{i-j} f_{j-1},$$

where we set $f_{-1} = 1$. The vector $\mathbf{h}(P) = (h_0, h_1, \ldots, h_d)$ is called the *h-vector* of P [8]. The *Dehn–Sommerville equations*, which hold for any simplicial convex polytope, are equivalent to the statement that $h_i = h_{d-i}$, $0 \leqslant i \leqslant d$ [7, Sect. 5.1]. If k and i are positive integers, then k can be written uniquely in the form

$$k = \binom{n_i}{i} + \binom{n_{i-1}}{i-1} + \cdots + \binom{n_j}{j},$$

where $n_i > n_{i-1} > \cdots > n_j \geqslant j \geqslant 1$. Following [6, 8, 9], define

$$k^{\langle i \rangle} = \binom{n_i + 1}{i + 1} + \binom{n_{i-1} + 1}{i} + \cdots + \binom{n_j + 1}{j + 1}.$$

Also define $0^{\langle i \rangle} = 0$. Let us say that a vector $(k_0, k_1, \ldots, k_d)$ of integers is an *M-vector* (after F. S. Macaulay) if $k_0 = 1$ and $0 \leqslant k_{i+1} \leqslant k_i^{\langle i \rangle}$ for $1 \leqslant i \leqslant d-1$. McMullen's conjecture may now be stated as follows: A sequence $(h_0, h_1, \ldots, h_d)$ of integers is the h-vector of a simplicial convex d-polytope if and only if $h_0 = 1$, $h_i = h_{d-i}$ for $0 \leqslant i \leqslant d$, and the sequence $(h_0, h_1 - h_0, h_2 - h_1, \ldots, h_{[d/2]} - h_{[d/2]-1})$ is an M-vector. (McMullen [6, 7] writes g_i for our $h_{i+1} - h_i$.)

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We now show the necessity of this condition. By a result essentially due to Macaulay [5] (stated more explicitly in [9, Theorem 2.2]), a sequence $(k_0, ..., k_d)$ is an M-vector if and only if there exists a graded commutative algebra $R = R_0 \oplus R_1 \oplus \cdots \oplus R_d$ over a field $K = R_0$, generated (as an algebra with identity) by R_1, such that the *Hilbert function* $H(R, n) := \dim_K R_n$ is given by $H(R, n) = k_n$. Let P be a simplicial convex d-polytope in $\mathbb{R}^d$. Since P is simplicial, we do not change the combinatorial structure of P (including the f-vector) by making small perturbations of the vertices of P and the taking the convex hull of these new vertices. Hence we may assume that the vertices of P lie in $\mathbb{Q}^d$. Without loss of generality we may also assume that the origin is in the interior of P. For every proper face α of P, define σ_α to be the union of all rays whose vertex is the origin and which intersect α. Thus σ_α is a simplicial cone. The set $\{\sigma_\alpha\}$ of all such cones forms a *complete simplicial fan* Σ [2, Sect. 5]. To such a fan is associated a complete complex variety X_Σ [4; 2, Sect. 5; 13, p. 558]. The cohomology ring $A = H^*(X_\Sigma, \mathbb{Q})$ of this variety satisfies $H^{2i+1}(X_\Sigma, \mathbb{Q}) = 0$ [2, Sect. 10.9], and hence is commutative and may be graded by setting $A_i = H^{2i}(X_\Sigma, \mathbb{Q})$. With this grading we have that A is generated by A_1 and that $\dim_{\mathbb{Q}} A_i = h_i(P)$ [2, Theorem 10.8 and Remark 10.9].

Now define a function $\phi: \mathbb{R}^d \to \mathbb{R}$ by $\phi(x) = -\| x \|/\| x' \|$, where $\| \cdot \|$ denotes the Euclidean norm and where x' is the intersection of the boundary of P with the ray with vertex at the origin passing through x. Then ϕ is convex, continuous, linear on each cone σ_α, and a *different* linear function on each maximal cone σ_α. Hence by the criterion [4, Chap. II, Sect. 2; 2, Sect. 6.9; 13, p. 570] for projectivity of X_Ω, where Ω is a complete fan, we conclude that X_Σ is projective. It then follows by a result of Steenbrink [11, Theorem 1.13] that the hard Lefschetz theorem (see, e.g., [3, p. 122]) holds for X_Σ. This means that there is an element $\omega \in H^2(X, \mathbb{Q}) = A_1$ (the class of a hyperplane section) such that for $0 \leqslant i \leqslant [d/2]$ the map $A_i \to A_{d-i}$ given by multiplication by ω^{d-2i} is a bijection. In particular, the map $A_i \to A_{i+1}$ given by multiplication by ω is injective if $0 \leqslant i \leqslant [d/2]$. Now let I be the ideal of A generated by ω and $A_{[d/2]+1}$. It follows that the Hilbert function of the quotient ring $R = A/I$ is given by $H(R, i) = h_i - h_{i-1}$, $1 \leqslant i \leqslant [d/2]$. Hence $(h_0, h_1 - h_0, ..., h_{[d/2]} - h_{[d/2]-1})$ is an M-vector, and the proof is complete.

The above proof relies on two developments from algebraic geometry: the varieties X_Σ first defined in [13] and [4], and the hard Lefschetz theorem. The close connection between the varieties X_Σ and the combinatorics of convex polytopes has been apparent since [4, 13], while in fact a direct application of these varieties to combinatorics has been given by Teissier [12]. On the other hand, an application of the hard Lefschetz theorem to combinatorics appears in [10].

Let Δ be a triangulation of the sphere $\mathbb{S}^{d-1}$. We can define the f-vector and h-vector of Δ exactly as for simplicial convex polytopes, and it is natural to ask [6, p. 569] whether McMullen's conjecture extends to this situation. It is well known that the Dehn–Sommerville equations $h_i = h_{d-i}$ continue to hold for Δ, and in [8] it was shown that the h-vector $(h_0, h_1, \ldots, h_d)$ of Δ is an M-vector. However, it remains open whether $(h_0, h_1 - h_0, \ldots, h_{[d/2]} - h_{[d/2]-1})$ is always an M-vector.

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