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On the Number of Reduced Decompositions of Elements of Coxeter Groups

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Let $r(w)$ denote the number of reduced decompositions of the element w of a Coxeter group W. Using the theory of symmetric functions, a formula is found for $r(w)$ when W is the symmetric group S_n. For the element $w_0 \in S_n$ of longest length and certain other $w \in S_n$, the formula for $r(w)$ is particularly simple. For the hyperoctahedral group B_n some conjectures are made in analogy to the S_n case. The situation for other W remains unclear.

1. INTRODUCTION

Let W be a Coxeter group with simple reflections $S = \{\sigma_1, \cdots, \sigma_m\}$ [5]. Given $w \in W$, let $R(w)$ denote the set of all reduced decompositions of w, i.e., the set of l-tuples $\rho = (\tau_1, \ldots, \tau_l)$, $\tau_i \in S$, for which $w = \tau_1 \cdots \tau_l$ and $l = l(w)$, the length of w. When no confusion will result we will simply write $\rho = \tau_1 \cdots \tau_l$. Let $r(w) = \text{card } R(w)$, the number of reduced decompositions of w. Our main object here is to compute the number $r(w)$. For the symmetric group $W = S_n$ (the Weyl group of type A_{n-1}), we can give a formula for $r(w)$ in terms of standard Young tableaux (SYT). In many cases, $r(w)$ is just the number f^λ of SYT of a certain shape λ. In particular, when $w = w_0$, the element of longest length $\binom{n}{2}$, we have

$$r(w_0) = f^{(n-1,n-2,\ldots,1)} = \binom{n}{2}! \Big/ 1^{n-1}3^{n-2}5^{n-3} \cdots (2n-3)^1.$$

In the course of our argument we obtain some remarkable connections between the set $R(w)$ and the theory of symmetric functions. Alternative approaches to determining $r(w)$ for $w \in S_n$ have subsequently been found by Edelman and Greene [7] and by Lascoux and Schützenberger [11].

When W is the hyperoctahedral group B_n, we offer some conjectures analogous to the S_n (or A_{n-1}) case. These conjectures were inspired in part by an idea of Robert Proctor. In particular, for the element $w_0 \in B_n$ of longest length n^2, we conjecture that

$$r(w_0) = f^\lambda = (n^2)!1!2! \cdots (n-1)!/n!(n+1)! \cdots (2n-1)!$$

where $\lambda = (n, n, \ldots, n)$ (n times).

It is very natural to try to carry over the results for A_{n-1} and conjectures for B_n to the other Coxeter groups. However, all plausible conjectures seem to fail in the case D_4.

The sets $R(w)$ and numbers $r(w)$ can be interpreted in terms of a certain partial ordering of W which we call the *weak Bruhat order.* If $w, w' \in W$, then define $w \leq w'$ if there exist $\tau_1, \ldots, \tau_j \in S$ such that $w' = w\tau_1 \cdots \tau_j$ and $l(w') = l(w) + j$. (Compare this definition with the usual Bruhat order (e.g. [4] [16]), in which the τs are allowed to be *conjugates* of elements of S.) The resulting partially ordered set has a unique minimal element $\hat{0}$, the identity of W, and every maximal chain in the interval $[\hat{0}, w]$ has length $l(w)$. If W is finite then there is a unique maximal element w_0 whose length is the sum of the exponents [5, V.6.2] of W. There is an obvious correspondence between maximal chains in $[\hat{0}, w]$ and reduced decompositions of w. In particular, $r(w)$ is equal to the number of maximal chains in $[\hat{0}, w]$; and when W is finite $r(w_0)$ is equal to the total number of maximal chains in W (always regarded as being partially ordered by the weak Bruhat order).

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When $W = S_n$, the weak Bruhat order was first systematically studied by Yanagimoto and Okamoto [17], who showed (Theorem 2.1) that it formed a lattice. The same result was also given by Guilbaud and Rosenstiehl [9, thm. 3, p. 96] [1, pp. 138–139], who called the weak Bruhat order of S_n a 'planted permutohedron'. Subsequently Björner [2], [3] investigated the weak Bruhat order of an arbitrary Coxeter group W, showing in particular that it always formed a meet semilattice (and a lattice when W is finite).

This paper had its origins in a communication from Paul Edelman, who computed that S_3 has 2 maximal chains, S_4 has $16 = 2^4$, and S_5 has $768 = 2^8 \cdot 3$. It turned out that these numbers had previously been computed by Jacob Goodman and Richard Pollack, who also found that S_6 had $292{,}864 = 2^{11} \cdot 11 \cdot 13$ maximal chains. Their interest in this problem stems from a connection [8] between the weak Bruhat order of S_n and the classification of finite configurations of points in the plane. I noticed from this data that the number of maximal chains in S_n was $f^{(n-1,n-2,\ldots,1)}$ for $1 \leqslant n \leqslant 6$, from which this paper eventually arose.

2. The Symmetric Group

Let $W = S_n$, acting on the symbols $1, 2, \ldots, n$. Choose σ_i to be the adjacent transposition $(i, i+1)$. Let $w \in S_n$ and $\rho = \sigma_{i_1} \cdots \sigma_{i_l} \in R(w)$. Define the *descent set* $D(\rho) = \{j: i_j > i_{j+1}\}$. Thus $D(\rho)$ is a subset of $\{1, 2, \ldots, l-1\}$. We will give not simply a formula for $r(w)$, but rather a description of the number of elements $\rho \in R(w)$ with a given descent set $S \subset \{1, 2, \ldots, l-1\}$.

Let $S \subset \{1, 2, \ldots, l-1\}$, and let $x = (x_1, x_2, \ldots)$ be a countably infinite set of indeterminates. Following some unpublished work of Ira Gessel, define the *fundamental quasi-symmetric function* $Q_{S,l}(x) = Q_{S,l}(x_1, x_2, \ldots)$ to be the formal power series $\sum x_{a_1} x_{a_2} \cdots x_{a_l}$, where the sum ranges over all integer sequences $1 \leqslant a_1 \leqslant \cdots \leqslant a_l$ such that $a_j < a_{j+1}$ if $j \in S$. If no confusion will result we simply write $Q_S(x)$ for $Q_{S,l}(x)$. Now if $w \in S_n$ and $l = l(w)$, then define

$$F_w(x) := \sum_{\rho \in R(w)} Q_{D(\rho),l}(x). \tag{1}$$

Thus $F_w(x)$ is a formal power series in the variables $x_1, x_2, \ldots$, with nonnegative integral coefficients and homogeneous of degree l.

Example. Let $w = 4132 \in S_4$. (We are regarding $w \in S_n$ as a *word* in the symbols $1, 2, \ldots, n$.) Then $R(w) = \{\sigma_3\sigma_2\sigma_3\sigma_1, \sigma_3\sigma_2\sigma_1\sigma_3, \sigma_2\sigma_3\sigma_2\sigma_1\}$ (where we multiply from right to left, i.e. each σ_i acts on the *positions* in w, not the *symbols*). Then

$$F_w(x) = Q_{13}(x) + Q_{12}(x) + Q_{23}(x).$$

Our first main result is the following.

2.1 Theorem. *$F_w(x)$ is a* symmetric *function of the x_i's.*

Proof. For any monomial $u = x_1^{b_1} x_2^{b_2} \cdots$ and power series $F(x)$, let $F(x)|_u$ denote the coefficient of u in $F(x)$. Pick some $j \geqslant 1$, and let $u' = x_1^{b_1} x_2^{b_2} \cdots x_j^{b_{j+1}} x_{j+1}^{b_j} \cdots$. It suffices to show that $F_w(x)|_u = F_w(x)|_{u'}$. For any $S \subset \{1, \ldots, l-1\}$ and any monomial $u = x_1^{b_1} x_2^{b_2} \cdots$, we have

$$Q_S(x)|_u = \begin{cases} 1, & \text{if } S \subset \{b_1, b_1 + b_2, b_1 + b_2 + b_3, \ldots\}, \\ 0, & \text{otherwise.} \end{cases}$$

Let $R_u(w)=\{\rho\in R(w)\colon Q_{D(\rho)}(x)|_u=1\}$ and $r_u(w)=\text{card } R_u(w)$. Thus $F_w(x)|_u=r_u(w)$, and it suffices to show that $r_u(w)=r_{u'}(w)$. To do this we construct a bijection between the sets $R_u(w)$ and $R_{u'}(w)$.

Suppose $\rho=\tau_1\tau_2\cdots\tau_l\in R_u(w)$, where $u=x_1^{b_1}x_2^{b_2}\cdots$ as usual. Set $c_i=b_1+b_2+\cdots+b_i$. Let ρ_* be the factor $\tau_{c_{j-1}+1}\tau_{c_{j-1}+2}\cdots\tau_{c_{j+1}}$ of ρ, and set $\rho=\rho_1\rho_*\rho_2$. Thus $\rho_*\in R(w_*)$ for some $w_*\in S_n$. Let $u_*=x_j^{b_j}x_{j+1}^{b_{j+1}}$ and $u'_*=x_j^{b_{j+1}}x_{j+1}^{b_j}$. Suppose we have a bijection $\phi_*\colon R_{u_*}(w_*)\to R_{u'_*}(w_*)$. Then the map $\phi\colon R_u(w)\to R_{u'}(w)$ defined by $\phi(\rho)=\rho_1\phi_*(\rho_*)\rho_2$ is a bijection. Hence it suffices to assume that $u=x_1^b x_2^c$ and that $\rho=\sigma_{i_1}\cdots\sigma_{i_b}\sigma_{j_1}\cdots\sigma_{j_c}$ where $i_1<i_2<\cdots<i_b$ and $j_1<j_2\cdots<j_c$.

We proceed to define $\phi(\rho)$ for $\rho\in R_u(w)$ as above. Let w_1 denote the permutation $\sigma_{i_1}\cdots\sigma_{i_b}$. For instance, suppose

$$\rho=(\sigma_1\sigma_2\sigma_3\sigma_7\sigma_8\sigma_9\sigma_{10}\sigma_{11}\sigma_{13}\sigma_{16}\sigma_{17}\sigma_{18}\sigma_{19}\sigma_{21}\sigma_{22})$$
$$\cdot(\sigma_1\sigma_4\sigma_5\sigma_6\sigma_7\sigma_8\sigma_{12}\sigma_{13}\sigma_{14}\sigma_{15}\sigma_{16}\sigma_{20}\sigma_{21}),$$

so

$$w=3,\ 2,\ 4,\ 5,\ 6,\ 8,\ 9,\ 10,\ 1,\ 11,\ 12,\ 14,\ 13,\ 15,\ 17,\ 18,\ 7,\ 19,\ 20,\ 22,\ 23,\ 16,\ 21.$$

Then

$$w_1=2,\ 3,\ 4,\ 1,\ 5,\ 6,\ 8,\ 9,\ 10,\ 11,\ 12,\ 7,\ 14,\ 13,\ 15,\ 17,\ 18,\ 19,\ 20,\ 16,\ 22,\ 23,\ 21.$$

Let $C_1,\ldots,C_\beta$ (where $b+\beta=n$) denote the cycles in the disjoint cycle decomposition of w_1, arranged so their least elements c_i are increasing. Note that the numbers $c_1,\ldots,c_\beta$ are just the *left-to-right minima* of w_i, i.e. $c_1=1$ and c_{i+1} is the least element of w_1 to the right of c_i. In the above example, $(c_1,\ldots,c_8)=(1,5,6,7,13,15,16,21)$.

Next observe that every left-to-right minimum of w is also a left-to-right minimum of w_1. Let $c_{m_1}<c_{m_2}<\cdots<c_{m_s}$ be the left-to-right minima of w, and let $n_1<\cdots<n_s$ denote the positions they occupy. For the above example, $(m_1,\ldots,m_4)=(1,4,7,8)$ and $(n_1,\ldots,n_4)=(9,17,22,23)$.

Claim. *There is a unique sequence* $1\leqslant t_1<t_2<\cdots<t_\beta\leqslant n$ *with the following two properties:*

(a) $t_{m_{i+1}-1}=n_i$ for $1\leqslant i\leqslant s$, *where we set* $m_{s+1}=\beta+1$,

(b) *If for each* $1\leqslant j\leqslant\beta$ *we take the element in position* t_j *of* w *and move it* $t_j-t_{j-1}-1$ *positions to the left* (*setting* $t_0=0$), *then the resulting permutation* w'_1 *can be written* (*uniquely*) *as an* increasing *product of adjacent transpositions* (*necessarily* c *of them*), *i.e.,* $w'_1=\sigma_{d_1}\sigma_{d_2}\cdots\sigma_{d_c}$ *where* $d_1<d_2<\cdots<d_c$.

For instance, in the above example we have $(t_1,\ldots,t_8)=(2,3,9,10,11,17,22,23)$. If in w we move 2 one space left, leave 4 the same, move 1 five spaces left, etc., then we obtain

$$w'_1=2,\ 3,\ 4,\ 1,\ 5,\ 6,\ 8,\ 9,\ 10,\ 11,\ 12,\ 7,\ 14,\ 13,\ 15,\ 17,\ 18,\ 16,\ 19,\ 20,\ 22,\ 23,\ 21$$
$$=\sigma_1\sigma_2\sigma_3\sigma_7\sigma_8\sigma_9\sigma_{10}\sigma_{11}\sigma_{13}\sigma_{16}\sigma_{17}\sigma_{21}\sigma_{22}. \tag{2}$$

We proceed to the proof of the above claim. Consider the element c_{m_1} (which is always 1) of w, which occupies position n_1. To the left of c_{m_1} appear $c_{m_1+1},\ldots,c_{m_2-1}$, while c_{m_2} is to the right. Thus w has the form

$$w=3,4,5,\ldots,q,2,\ldots,c_{m_1+1},\ldots,c_{m_2-1},\ldots,c_{m_1},\ldots$$

where possibly $q=2$. In order for w'_1 to be an increasing product of adjacent transpositions, we must either move 1 to the beginning, in which case $m_2=2$ and $t_1=n_1$, or 2 to the beginning, in which case $t_1=q-1$. In the latter case after moving 2 we now have

$$2, 3, 4, \ldots, q, q+2, q+3, \ldots, q', q'+1, \ldots, c_{m_1+1}, \ldots.$$

If $m_2=3$ we must move 1 directly to the right of q; otherwise we must move $q+1$ directly to the right of q. We continue in this way until we have moved m_2-2 elements to the left, and then we move 1 to the left. The only problem that can arise is that there are no elements besides 1 which we can move before moving m_2-2 other elements. But to reach a situation in which only 1 can be moved, we must have previously moved $c_{m_1+1}=c_2$, $c_{m_1+2}=c_3, \ldots, c_{m_2-1}$, so we have indeed moved m_2-2 other elements.

After 1 gets moved to the left our permutation has the form

$$2, 3, 4, \ldots p, 1, p+1, p+2, \ldots, x, x+2, \ldots, c_{m_2+1}, \ldots, c_{m_3-1}, \ldots, c_{m_2}, \ldots$$

We now apply the same procedure as above to the interval $x+2, \ldots, c_{m_2}$, i.e. we move m_3-m_2 elements to the left, the last one being m_2. Continuing this way, we see there is a *unique* way of converting w to an increasing product of adjacent transpositions, subject to the requirements of (a) and (b). Thus the claim is proved.

Let $w_1'=\sigma_{d_1}\sigma_{d_2}\cdots\sigma_{d_c}$ be the unique expression of w_1' as an increasing product of adjacent transpositions. The above procedure shows that $w=w_1'w_2'$ where w_2' is a (unique) increasing product of b adjacent transpositions, say $w_2'=\sigma_{e_1}\sigma_{e_2}\cdots\sigma_{e_b}$. Hence we can define $\phi(\rho)=\sigma_{d_1}\cdots\sigma_{d_c}\sigma_{e_1}\cdots\sigma_{e_b}$. This uniquely defines ϕ on any reduced expression ρ of w with one descent. (If ρ has no descents, define $\phi(\rho)=\rho$.) It remains to verify that $\phi: R_u(w)\to R_{u'}(w)$ is a bijection. We will in fact show that $\phi^2(\rho)=\rho$ for all $\rho\in R_u(w)$. Let $C_1', C_2', \ldots, C_\alpha'$ (where $a+\alpha=n$) denote the cycles in the disjoint cycle decomposition of w_1', arranged so their least elements c_i' are increasing. Let $c'_{m_1'}<c'_{m_2'}<\cdots<c'_{m_s'}$ be the left-to-right minima of w (so $c'_{m_i'}=c_{m_i}$). In order to show that $\phi^2(\rho)=\rho$, we need to establish the following: $m'_{i+1}-m'_i$ is equal to the number of elements of w_1 which have to be moved to the left or which have no element moved passed them to obtain w, and which end up to the left of c_{m_i} but to the right of $c_{m_{i-1}}$ (with $m'_{s+1}=\alpha+1$).

EXAMPLE. Continuing the above example, we see from (2) that $c_1'=1$, $c_2'=5$, $c_3'=6$, $c_4'=7$, $c_5'=13$, $c_6'=15$, $c_7'=16$, $c_8'=19$, $c_9'=20$, $c_{10}'=21$, so $m_1'=1$, $m_2'=4$, $m_3'=7$, $m_4'=10$, $m_5'=\alpha+1=11$. To obtain w from w_1, move 3 one space to the left, fix 4, move 10 five spaces left, fix 11 and 12, move 18 five spaces left, fix 19, and 20, move 23 two spaces left, and fix 21. To the left of $c_{m_1}=1$ but to the right of $c_{m_0}=0$ we have moved or fixed 3, 4 and 10, for a total of $3=m_2'-m_1'$ elements. To the left of $c_{m_2}=7$ but to the right of $c_{m_1}=1$ we have moved or fixed 11, 12, and 18, for a total of $3=m_3'-m_2'$ elements, etc.

In general, write $w=w_1w_2$ with $w_1=C_1C_2\cdots$ and $w_2=D_1D_2\cdots$ in disjoint cycle notation (with increasing cycles). Let D_{r+1} be the cycle of w_2 containing $c_{m_1}=1$. D_{r+1} will then contain $C_2, C_3, \ldots, C_{m_1-1}$ and will intersect C_{m_1}. Let $p=m_2-m_1$. Thus we can write w as a juxtaposition of words $v_1, v_2, \ldots, v_{r+p}, v$ such that (a) $v_1, \ldots, v_r$ have the form $v_1=3, 4, \ldots, b_1, b_2$; $v_2=b_1+2, b_1+3, \ldots, b_2, b_1+1$; $v_3=b_2+2, b_2+3, \ldots, b_3, b_2+1; \ldots$; $v_r=b_{r-1}+2, b_{r-1}+3, \ldots, b_r, b_{r-1}+1$, where $b_r=|C_1|-1$, (b) $v_{r+1}, \ldots, v_{r+p-1}$ have the form $v_{r+1}=b_r+2, b_r+3, \ldots, b_{r+1}, b_r+2; \ldots; v_{r+p-1}=b_{r+p-2}+2, b_{r+p-2}+3, \ldots, b_{r+p-1}, b_{r+p-2}+1$, and (c) $v_{r+p}=b_{r+p-1}+1, b_{r+p-1}+2, \ldots, b_{r+p}, 1$, where b_{r+p} lies in D_{r+1}. The first step in obtaining w_1' from w is to take the last element of each word $v_1, v_2, \ldots, v_{p-1}$ and move it to the beginning of the word, and then to move 1 to the beginning of v_p. Writing $\bar v_i$ for the word obtained from v_i by moving the last element to the beginning, it follows that w_1 looks like $w_1=\bar v_1\bar v_2\cdots\bar v_{p-1}1v_pv_{p+1}\cdots v_{p+r-1}v'_{p+r}$, where v'_{p+r} denotes v_{p+r} with the last element 1 removed. Then C_1' contains the elements of $\bar v_1\cdots\bar v_{p-1}1$, C_2' of v_p, C_3' of $v_{p+1}, \ldots, C'_{r+1}$ of v_{p+r-1}, and C'_{r+2} contains c_{m_2}. Hence $r=m_2'-m_1'$. But by definition r

is exactly the number of elements of w_1 which have to be moved to the left or which have no element moved passed them to obtain w, and which end up to the left of c_{m_2} but to the right of $c_{m_1}=1$. Thus $m_2'-m_1'$ has the desired interpretation.

The same argument (using, instead of D_{r+1}, the cycle of w_2 containing $c_{m_{i+1}}$) works for interpreting $m_{i+1}'-m_i'$, so the proof is complete.

3. Consequences of Symmetry

Since $F_w(x)$ is a symmetric function, we can expand it as a unique integer linear combination of the Schur functions $s_\lambda(x)$, where λ ranges over all partitions of $l=l(w)$ (written $\lambda\vdash l$). Say

$$F_w(x)=\sum_{\lambda\vdash l}\alpha_{w\lambda}s_\lambda(x),\qquad \alpha_{w\lambda}\in\mathbb{Z}. \tag{3}$$

See [12] or [15] for information on Schur functions. Let us just recall for now that the coefficient of $x_1\cdots x_l$ in $s_\lambda(x)$ is the number f^λ of *standard Young tableaux* (SYT) of shape λ [12, p. 5], i.e. the number of left-justified arrays of integers, each integer $1,2,\ldots,l$ appearing exactly once, with λ_i entries in row i and every row and column increasing. For instance, the five SYT of shape $(3,2)$ are

123	124	125	134	135
45	35	34	25	24

The famous 'hook length formula' of Frame-Robinson-Thrall (e.g. [12, p. 43, ex. 2]) gives an explicit formula for f^λ, viz.,

$$f^\lambda=l!\Big/\prod_{i,j}h_{ij}(\lambda),$$

where i, j range over all positive integers for which $h_{ij}(\lambda):=\lambda_i+\lambda_j'-i-j+1>0$. Here $\lambda'=(\lambda_1',\lambda_2',\ldots)$ denotes the conjugate partition to λ[12, p. 2].

3.1. Corollary. *If $\alpha_{w\lambda}$ is given by* (3), *then*

$$r(w)=\sum_{\lambda\vdash l}\alpha_{w\lambda}f^\lambda.$$

Proof. For any $S\subset\{1,\ldots,l-1\}$, the coefficient of $x_1\cdots x_l$ in $Q_{S,l}(x)$ is one. Hence by (1), the coefficient of $x_1\cdots x_l$ in $F_w(x)$ is $r(w)$, and the proof follows from (3).

It is convenient (especially when trying to extend our results to $W=B_n$ in the next section) to interpret (3) more combinatorially. A *multiset* is (informally) a set with repeated elements. A *virtual multiset* allows elements of negative multiplicity. One may think of a multiset M as a function $M:S\to\mathbb{N}$ for some set S, where $\mathbb{N}=\{0,1,2,\ldots\}$, while a virtual multiset is a function $M:S\to\mathbb{Z}$. In either case, $M(x)$ is regarded as the multiplicity of $x\in S$. Now define for each $w\in S_n$ a virtual multiset M_w whose elements are partitions λ of $l=l(w)$, with multiplicity $M_w(\lambda)=\alpha_{w\lambda}$. Thus Corollary 3.1 can be rewritten

$$r(w)=\sum_{\lambda\in M_w}f^\lambda, \tag{4}$$

where it is understood that each $\lambda\in M_w$ is counted as many times as its multiplicity (which conceivably could be negative).

The following result follows from the approach of Edelman and Greene [7] toward this subject. We have been unable to derive it from our techniques.

3.2. THEOREM. *For all $w \in S_n$ and $\lambda \vdash l = l(w)$, we have $\alpha_{w\lambda} \geqslant 0$. Equivalently, the virtual multiset M_w is a genuine multiset.*

We now give a refinement of Corollary 3.1 involving the descent sets of $\rho \in R(w)$. Let $\mathscr{S}_\lambda$ denote the set of all SYT of shape $\lambda \vdash l$. Given $\tau \in \mathscr{S}_\lambda$, define the *descent set*

$$D(\tau) := \{1 \leqslant i \leqslant l-1 \colon i+1 \text{ appears in a lower row than } i\}.$$

It follows, e.g., from [14, p. 81] that

$$s_\lambda(x) = \sum_{\tau \in \mathscr{S}_\lambda} Q_{D(\tau),l}(x). \tag{5}$$

For instance, if $\lambda = (3, 2)$ then we have

τ:	123 45	124 35	125 34	134 25	135 24
$D(\tau)$:	3	2, 4	2	1, 4	1, 3.

Hence $s_{32}(x) = Q_3(x) + Q_{24}(x) + Q_2(x) + Q_{14}(x) + Q_{13}(x)$.

If we compare (1), (3), and (5), we deduce the following equality of multisets:

$$\{D(\rho) \colon \rho \in R(w)\} = \bigcup_{\lambda \in M_w} \{D(\tau) \colon \tau \in \mathscr{S}_\lambda\}.$$

EXAMPLE. Let $w = 31524 \in S_5$. The elements of $R(w)$ are

ρ	$D(\rho)$
$\sigma_2\sigma_1\sigma_4\sigma_3$	1, 3
$\sigma_2\sigma_4\sigma_1\sigma_3$	2
$\sigma_4\sigma_2\sigma_1\sigma_3$	1, 2
$\sigma_2\sigma_4\sigma_3\sigma_1$	2, 3
$\sigma_4\sigma_2\sigma_3\sigma_1$	1, 3

On the other hand, consider all SYT of shapes of 211 and 22:

τ	$D(\tau)$
12 3 4	2, 3
13 2 4	1, 3
14 2 3	1, 2
12 34	2
13 24	1, 3

Since the multiset of $D(\rho)$s coincides with that of $D(\tau)$s, it follows from the uniqueness of (3) that $F_w(x)=s_{211}(x)+s_{22}(x)$, so $r(w)=f^{211}+f^{22}=3+2=5$.

We now discuss several properties of the symmetric functions $F_w(x)$ before turning to their actual computation. If $\lambda=(\lambda_1,\lambda_2,\ldots)\vdash l$, then let $c(\lambda)$ denote the set of all partitions obtained by subtracting one from some λ_i (so that the parts are still descending, i.e. $\lambda_i-1\geqslant\lambda_{i+1}$). E.g. $c(4421)=\{4321,4411,442\}$. Define

$$s_{\lambda/1}(x)=\sum_{\mu\in c(\lambda)} s_\mu(x).$$

Then $s_{\lambda/1}(x)$ is an instance of a *skew Schur function* [12, ch. I.5] [15, Section 21]. Extend the definition to $F_{w/1}(x)$ by setting

$$F_{w/1}(x)=\sum_\lambda \alpha_{w\lambda}s_{\lambda/1}(x).$$

Given $w\in S_n$, let $c(w)$ denote the set of elements v which w *covers* in (the weak Bruhat order of) S_n, i.e., $v<w$ and $l(v)=l(w)-1$.

3.3. THEOREM. *For all $w>\hat{0}$ in S_n, we have*

$$F_{w/1}(x)=\sum_{v\in c(w)} F_v(x).$$

Equivalently,

$$\bigcup_{v\in c(w)} M_v=\bigcup_{\lambda\in M_w} c(\lambda). \tag{6}$$

PROOF. For all $\rho=\sigma_{i_1}\cdots\sigma_{i_l}\in R(w)$, let $\rho'=\sigma_{i_1}\cdots\sigma_{i_{l-1}}$. Since ρ' and w uniquely determine ρ, it follows that $\{\rho'\colon \rho\in R(w)\}=\bigcup_{v\in c(w)}R(v)$. Hence

$$\bigcup_{v\in c(w)}\{D(\sigma)\colon \sigma\in R(v)\}=\{D(\rho)-\{l-1\}\colon \rho\in R(w)\},$$

where $D(\rho)-\{l-1\}$ denotes that $l-1$ is to be removed from $D(\rho)$ when $l-1\in D(\rho)$.

On the other hand, if $\lambda\vdash l$ and $\tau\in\mathcal{S}_\lambda$, then let τ' denote the SYT obtained by deleting l from τ. Clearly

$$\{\tau'\colon \tau\in\mathcal{S}_\lambda\}=\bigcup_{\mu\in c(\lambda)}\mathcal{S}_\mu.$$

Hence by (5) and the definition of $s_{\lambda/1}$ we have

$$s_{\lambda/1}(x)=\sum_{\tau\in\mathcal{S}_\lambda} Q_{D(\tau)-\{l-1\},l-1}(x).$$

The proof follows from (3).

Now suppose that $w=a_1a_2\cdots a_n\in S_n$ and that for some $1\leqslant j\leqslant n$ we have $\{a_1,a_2,\ldots,a_j\}=\{1,2,\ldots,j\}$ (as sets). Let $w_1=a_1a_2\cdots a_j\in S_j$ and $w_2=a_{j+1}-j, a_{j+2}-j,\ldots,a_n-j\in S_{n-j}$.

3.4. THEOREM. *With w as above, we have*

$$F_w(x)=F_{w_1}(x)F_{w_2}(x).$$

PROOF. Let $\rho_1 \in R(w_1)$ and $\rho_2 \in R(w_2)$. Let ρ_2^* denote the reduced expression obtained by replacing each σ_i in ρ_2 by σ_{i+j}. Let ρ be any *shuffle* of ρ_1 and ρ_2^*, i.e. any permutation of the letters in ρ_1 and ρ_2^* such that the letters of ρ_1 and ρ_2^* appear in the same order as they do in ρ_1 and ρ_2^*. Then $\rho \in R(w)$, and every $\rho \in R(w)$ occurs uniquely in this way.

Hence it suffices to prove the following: Let π be a permutation of $\{1, \ldots, r\}$ and ρ of $\{r+1, \ldots, r+s\}$. Let $\mathrm{Sh}(\pi, \rho)$ denote the set of shuffles of π and ρ. Then

$$Q_{D(\pi),r}(x) \cdot Q_{D(\rho),s}(x) = \sum_{\sigma \in \mathrm{Sh}(\pi,\rho)} Q_{D(\sigma),r+s}(x). \tag{7}$$

While (7) is not hard to prove directly, it is also an immediate consequence of the theory of P-partitions developed in [14]. Specifically, let C_π and C_ρ be chains with r and s elements, respectively. If $\pi = a_1 a_2 \cdots a_r$, then label the elements of C_π from bottom to top by $a_1, \ldots, a_r$, and similarly for C_ρ. Call these labelings ω_1 and ω_2. Let P be the disjoint union of C_π and C_ρ with the same labels. Denote by ω this labeling of P. Then in the notation of [14], we have $G(C_\pi, \omega_1; x) = Q_{D(\pi),r}(x)$, $G(C_\rho, \omega_2; x) = Q_{D(\rho),s}(x)$ and $\mathscr{L}(P, \omega) = \mathrm{Sh}(\pi, \rho)$ (in [14, p. 27], the generating function G is defined only for finitely many variables, but this is irrelevant). Now on the one hand $G(P, \omega; x) = G(C_\pi, \omega_1; x) G(C_\rho, \omega_2; x)$ and on the other

$$G(P, \omega; x) = \sum_{\sigma \in \mathscr{L}(P,\omega)} Q_{D(\sigma),r+s}(x).$$

The proof follows.

Since the expansion of a product of symmetric functions in terms of Schur functions can be rather complicated [12, ch. I. 9], Theorem 3.4 suggests that no simple direct description of the expansion (3) (or of the multiset M_w) is possible. For instance, if $n = 2m$ and $w = 214365 \cdots 2n, 2n-1$, then $F_w(x) = s_1(x)^n = \sum_{\lambda \vdash n} f^\lambda s_\lambda(x)$. More generally, it will follow from Corollary 4.2 that for any $\lambda \vdash l$, there exists some n and $w \in S_n$ for which $F_w(x) = s_\lambda(x)$. Thus for any finite sequence $\lambda, \mu, \nu, \ldots$ of partitions there is a w for which $F_w(x) = s_\lambda(x) s_\mu(x) s_\nu(x) \cdots$.

Next we note that the poset S_n has a unique nontrivial automorphism which we will denote by $*$, viz.,

$$(a_1 a_2 \cdots a_n)^* = n+1-a_n, n+1-a_{n-1}, \ldots, n+1-a_1.$$

Let ω be the linear transformation (actually an algebra automorphism) satisfying $\omega s_\lambda(x) = s_{\lambda'}(x)$, where λ' denotes the conjugate partition to λ (see [12, pp. 14, 26, 35]).

3.5 THEOREM. *For any $w \in S_n$, we have $F_{w^*}(x) = \omega F_w(x)$.*

PROOF. Clearly $\{D(\rho) : \rho \in R(w)\} = \{\overline{D(\rho)} : \rho \in R(w^*)\}$, where $\overline{D(\rho)}$ denotes the complement $\{1, 2, \ldots, l-1\} - D(\rho)$. Now for any $\lambda \vdash l$ and $\tau \in \mathscr{S}_\lambda$, let $\tau' \in \mathscr{S}_{\lambda'}$ denote the transpose of τ. It is easy to see that $D(\tau') = \overline{D(\tau)}$. Thus from (5),

$$\omega s_\lambda(x) = s_{\lambda'}(x) = \sum_{\tau \in \mathscr{S}_\lambda} Q_{\overline{D(\tau)},l}(x).$$

The proof follows from the definition (1) of $F_w(x)$.

4. THE DOMINANT TERMS OF $F_w(x)$

In this section we will obtain some information on the numbers $\alpha_{w\lambda}$ which in some cases will completely determine $F_w(x)$ and therefore $r(w)$. If $\lambda = (\lambda_1, \lambda_2, \ldots)$ and $\mu =$

$(\mu_1, \mu_2, \ldots)$ are partitions of the same integer l, then define $\lambda \leqslant \mu$ if $\lambda_1+\lambda_2+\cdots+\lambda_i \leqslant \mu_1+\mu_2+\cdots+\mu_i$ for all i. This defines a partial ordering (actually a lattice) of all partitions of l, called the *dominance order.*

Next, given $w = a_1 a_2 \cdots a_n \in S_n$, define

$$r_i(w) = \text{card}\,\{j\colon j < i \text{ and } a_j > a_i\},\ 1 \leqslant i \leqslant n$$

$$s_i(w) = \text{card}\,\{j\colon j > i \text{ and } a_j < a_i\},\ 1 \leqslant i \leqslant n.$$

Thus $\sum_i r_i(w) = \sum_i s_i(w) = l(w)$. Let $\lambda(w) = (\lambda_1(w), \lambda_2(w), \ldots)$ denote the partition obtained by arranging the numbers $r_1(w), \ldots, r_n(w)$ in descending order (and ignoring any 0's). Let $\mu(w) = (\mu_1(w), \mu_2(w), \ldots)$ denote the *conjugate* to the partition $\mu'(w)$ obtained by arranging the numbers $s_1(w), \ldots, s_n(w)$ in descending order.

EXAMPLE. Let $w = 3417625 \in S_7$. Then $(r_1(w), \ldots, r_7(w)) = (0, 0, 2, 0, 1, 4, 2)$ and $(s_1(w), \ldots, s_7(w)) = (2, 2, 0, 3, 2, 0, 0)$. Thus $\lambda(w) = (4, 2, 2, 1)$, $\mu'(w) = (3, 2, 2, 2)$, $\mu(w) = (4, 4, 1)$.

Our second main result on S_n is the following.

4.1. THEOREM. *Let $w \in S_n$, and let $\alpha_{w\lambda}$ be given by* (3). *Then:*

(a) *If $\alpha_{w\lambda} \neq 0$ then $\lambda(w) \leqslant \lambda \leqslant \mu(w)$.*
(b) $\alpha_{w\lambda(w)} = \alpha_{w,\mu(w)} = 1.$

PROOF. (a) Let $\nu = (\nu_1, \nu_2, \ldots, \nu_l) \vdash l$. Suppose that the monomial $x^\nu = x_1^{\nu_1} x_2^{\nu_2} \cdots x_l^{\nu_l}$ appears in $F_w(x)$ with a nonzero coefficient. We claim that $\nu \leqslant \mu(w)$. To say that x^ν appears in $F_w(x)$ means that some $\rho = \sigma_{i_1} \cdots \sigma_{i_l} \in R(w)$ satisfies

$$i_1 < i_2 < \cdots < i_{\nu_1},\ i_{\nu_1+1} < i_{\nu_1+2} < \cdots < i_{\nu_1+\nu_2}, \ldots, i_{\nu_1+\cdots+\nu_{l-1}+1} < \cdots < i_{\nu_1+\cdots+\nu_l}.$$

If we begin with the identity permutation $12 \cdots n$ and successively multiply by σ_{i_1}, $\sigma_{i_2}, \ldots, \sigma_{i_{\nu_1}}$, we increase by one each time the number of elements which have a smaller element somewhere to the right. Multiplication by subsequent σ_{i_j} can never decrease this number since ρ is reduced. Hence ν_1 cannot exceed the total number of elements of w which have a smaller element somewhere to the right. This latter number is precisely $\mu_1(w)$, so $\nu_1 \leqslant \mu_1(w)$.

Now successively multiply by $\sigma_{i_{\nu_1+1}}, \ldots, \sigma_{i_{\nu_2}}$. After each multiplication we increase by one the value $k_1 + 2k_2$, where k_i denotes the number of elements having exactly i smaller elements somewhere to the right. Moreover, no element will have more than two smaller elements somewhere to the right. Multiplication by subsequent σ_{i_j} can never decrease the value of $k_1 + 2k_2$. Hence $\nu_1 + \nu_2$ cannot exceed the value of $k_1 + 2k_2$ for w itself, which is equal to $\mu_1(w) + \mu_2(w)$. Thus $\nu_1 + \nu_2 \leqslant \mu_1(w) + \mu_2(w)$. Continuing this line of reasoning, we obtain $\nu \leqslant \mu(w)$ as claimed.

Now recall [12, p. 57] that the transition matrix $(K_{\lambda\mu})$ between the monomial symmetric functions and Schur functions of degree l is upper triangular with 1's on the diagonal with respect to any ordering of the partitions of l which extends dominance order. It then follows from the previous paragraph that if $\alpha_{w\lambda} \neq 0$ then $\lambda \leqslant \mu(w)$, and moreover that $\alpha_{w,\mu(w)}$ is the coefficient of $x^{\mu(w)}$ in $F_w(x)$.

A dual type of argument can be used to show that if $\alpha_{w\lambda} \neq 0$ then $\lambda(w) \leqslant \lambda$. Alternatively, the automorphism * satisfies $\mu(w^*) = \lambda'(w)$. Hence by Theorem 3.5 and what we have just proved, we have that $\lambda \leqslant \lambda'(w)$ whenever $\alpha_{w\lambda'} \neq 0$. Since conjugation is an anti-automorphism of dominance order (e.g. [12, p. 6, (1.11)]) we deduce that $\lambda(w) \leqslant \lambda'$ whenever $\alpha_{w\lambda'} \neq 0$, as desired.

(b) Let $\mu = \mu(w)$. By the above it suffices to show that the coefficient of x^μ in $F_w(x)$ is one, i.e., there is a unique $\rho \in R(w)$ such that $\rho = \sigma_{i_1} \cdots \sigma_{i_l}$ where

$$i_1 < i_2 \cdots < i_\mu,\ i_{\mu_1+1} < \cdots < i_{\mu_1+\mu_2}, \ldots.$$

Let X be the set of elements of w which have a smaller element somewhere to the right, so $|X| = \mu_1$. Since each σ_{i_j}, $1 \leqslant j \leqslant \mu_1$, adds an element to X which cannot be deleted by any subsequent σ_{i_r}s, it follows that if the elements of X are $t_1 < t_2 < \cdots < t_{\mu_1}$ and if $1 \leqslant j \leqslant \mu_1$, then σ_{i_j} must transpose the element t_j with the element on its left. Thus $i_1, \ldots, i_{\mu_1}$ exist and are unique. Let $w' = (\sigma_{i_1} \cdots \sigma_{i_{\mu_1}})^{-1} w$. Then $\mu(w') = (\mu_2(w), \mu_3(w), \ldots)$, and the proof follows by induction on $l(w)$.

4.2. Corollary. *If $\lambda(w) = \mu(w)$ then $F_w(x) = s_{\lambda(w)}(x)$ and $r(w) = f^{\lambda(w)}$. If $\mu(w)$ covers $\lambda(w)$ in the dominance order (i.e., no elements lie in between), then $F_w(x) = s_{\lambda(w)}(x) + s_{\mu(w)}(x)$ and $r(w) = f^{\lambda(w)} + f^{\mu(w)}$.*

In particular, letting $w_0 = n, n-1, \ldots, 1$, the maximal element of S_n under weak Bruhat order, we see that $\lambda(w) = \mu(w) = (n-1, n-2, \ldots, 1)$. Hence:

4.3. Corollary. *The number of maximal chains in the weak Bruhat order of S_n is given by*

$$r(w_0) = f^{(n-1, n-2, \ldots 1)}$$

$$= \binom{n}{2}! \Big/ 1^{n-1} 3^{n-2} \cdots (2n-3)^1,$$

the latter equality by the hook-length formula.

Let us remark that $\lambda(w) = \mu(w)$ whenever $l(w) \geqslant \binom{n}{2} - 3$. Moreover, $\lambda(w) = \mu(w)$ for 23 elements of S_4 (the exception being 2143) and for 103 elements of S_5. Of the remaining 17 elements of S_5, 15 of them have the property that $\mu(w)$ covers $\lambda(w)$. The two exceptions are 32154 and 21543. Let us also note that the converse to Theorem 4.1(a) is false, i.e. if $\lambda(w) \leqslant \lambda \leqslant \mu(w)$ we need not have $\alpha_{w\lambda} \neq 0$. For instance, if $w = 3215674 \in S_7$, then $\lambda(w) = (3, 2, 1)$, $\mu(w) = (5, 1)$, and $F_w(x) = s_{321}(x) + s_{411}(x) + s_{42}(x) + s_{51}(x)$. Thus $\alpha_{w,33} = 0$ yet $321 \leqslant 33 \leqslant 51$.

5. The Hyperoctahedral Group

Now let W be the hyperoctahedral group (or Weyl group of type B_n) of order $2^n n!$. On the basis of considerable computational evidence it appears that much of what we did for S_n carries over to B_n. In this case, however, we have been unable to supply any proofs. First we give an analogue of (6).

5.1. Conjecture. *Let $w \in B_n$. Then there exists a (unique) multiset M_w of partitions of $l(w)$ into* distinct *parts satisfying:*

(a) $M_{\mathrm{id}} = \{\varnothing\}$

(b) $M_{w_0} = \{(2n-1, 2n-3, \ldots, 5, 3, 1)\}$, *where w_0 is the element of B_n of longest length $l = n^2$.*

(c) *Let $c(w)$ denote the elements which w covers in the weak Bruhat of B_n,* i.e.

$$c(w) = \{v \in B_n \colon v < w \text{ and } l(v) = l(w) - 1\}.$$

For any partition $\mu = (\mu_1, \mu_2, \ldots)$ *into distinct parts, let* $c(\mu)$ *denote the set of all partitions into* distinct *parts obtained by subtracting one from some* μ_i. *Then*

$$\bigcup_{v \in c(w)} M_v = \bigcup_{\mu \in M_w} c(\mu) \qquad (\textit{multiset union}).$$

5.2. Corollary (to Conjecture 5.1). *If* μ *is a partition into distinct parts, then let* g^μ *denote the number of* shifted *standard tableau of shape* μ [12, p. 135]. *Then*

$$r(w) = \sum_{\mu \in M_w} g^\mu.$$

In particular, the number of maximal chains in the weak Bruhat order of B_n *is*

$$\begin{aligned} r(w_0) &= g^{(2n-1, 2n-3, \ldots, 1)} \\ &= (n^2)!1!2! \cdots (n-1)!/n!(n+1)! \cdots (2n-1)! \end{aligned} \tag{8}$$

Proof. The function $r(w)$ is uniquely defined by

$$r(w) = \sum_{v \in c(w)} r(v), \qquad r(id) = 1. \tag{9}$$

On the other hand, the function g^μ clearly satisfies

$$g^\mu = \sum_{\nu \in c(\mu)} g^\nu, \qquad g^\varnothing = 1.$$

Hence if M_w satisfies conditions (a) and (c) of Conjecture 5.1, then the function $s(w) = \sum_{\mu \in M_w} g^\mu$ also satisfies the recurrence (9), so $s(w) = r(w)$. It remains only to verify (8), but this follows from the hook-length formula for shifted tableaux (e.g. [12, p. 135]).

To illustrate Conjecture 5.1, a list of all 48 elements w of B_3 together with the multiset M_w is given in Table 1. We abbreviate a multiset such as $\{(4,2,1), (4,2,1), (5,2)\}$ as 421, 421, 52. A notation such as $\bar{1}3\bar{2}$ for an element of B_3 denotes the linear transformation $\mathbb{R}^3 \to \mathbb{R}^3$ satisfying $\varepsilon_1 \to -\varepsilon_1$, $\varepsilon_2 \to \varepsilon_3$, $\varepsilon_3 \to -\varepsilon_2$, where $\varepsilon_1, \varepsilon_2, \varepsilon_3$ is the standard basis for $\mathbb{R}^3$.

Table 1

The 48 elements w of B_3

w	M_w	w	M_w	w	M_w
123	$\varnothing$	$2\bar{1}3$	4	$\bar{1}32$	51
213	1	$32\bar{1}$	4, 31	$3\bar{1}\bar{2}$	42
132	1	$3\bar{2}1$	31	$\bar{2}3\bar{1}$	42
$12\bar{3}$	1	$\bar{2}13$	4	$\bar{1}2\bar{3}$	51
231	2	$2\bar{3}\bar{1}$	31	$\bar{3}\bar{1}2$	42
312	2	$\bar{3}21$	4, 31	$\bar{3}\bar{2}\bar{1}$	321
$13\bar{2}$	2	$\bar{3}1\bar{2}$	31	$\bar{2}\bar{3}1$	42
$21\bar{3}$	2, 2	$1\bar{2}\bar{3}$	4	$\bar{1}3\bar{2}$	52
$1\bar{3}2$	2	$\bar{1}23$	5	$\bar{2}\bar{1}3$	43
$23\bar{1}$	3	$3\bar{1}2$	41	$\bar{1}\bar{3}2$	52
321	3, 21	$3\bar{2}\bar{1}$	32	$\bar{3}\bar{1}\bar{2}$	421
$31\bar{2}$	3, 21	$\bar{2}31$	41	$\bar{2}\bar{3}\bar{1}$	421
$1\bar{2}3$	3	$2\bar{1}\bar{3}$	41	$\bar{1}\bar{2}3$	53
$2\bar{3}1$	3, 21	$\bar{3}2\bar{1}$	41, 32	$\bar{1}\bar{3}\bar{2}$	521
$\bar{3}12$	3	$\bar{3}\bar{2}1$	32	$\bar{2}\bar{1}\bar{3}$	431
$1\bar{3}\bar{2}$	21	$\bar{2}1\bar{3}$	41	$\bar{1}\bar{2}\bar{3}$	531

Let us now turn to an analogue of Theorem 2.1, i.e. an analogue of the symmetric function $F_w(x)$. In order to define $F_w(x)$ for $w \in B_n$ in analogy to the S_n case, we must first order the set of simple reflections so we can define the descent set $D(\rho)$ for $\rho \in R(w)$. However, even for B_2 the power series $F_w(x)$ need not be symmetric for any ordering of S. For instance, there is an element $w \in B_2$ whose unique reduced decomposition is $\rho = \sigma_1\sigma_2\sigma_1$. Since $D(\rho) = \{2\}$, we get $F_w(x) = Q_{\{2\},3}(x) = \sum_{a \leqslant b < c} x_a x_b x_c$, which is not symmetric. In view of this example the following conjecture is rather mysterious. It has been checked for $n \leqslant 3$.

5.3. CONJECTURE. *As above, let $w_0 \in B_n$ be the element of length n^2. Order the set S by the following scheme:*

(where we have identified S with the nodes of the Dynkin diagram for B_n in the standard way). Let $\nu = (n, n, \ldots, n) \vdash n^2$, the partition of n^2 with n parts equal to n. Then $F_{w_0}(x) = s_\nu(x)$.

It follows from Conjecture 5.3, just as Corollary 3.1 follows from Theorem 2.1, that $r(w_0) = f^\nu$. In view of Corollary 5.2 this means that $f^{(n,n,\ldots,n)} = g^{(2n-1,2n-3,\ldots,1)}$. This equality can easily be verified using the appropriate hook length formulas.

6. A COMBINATORIAL DIGRESSION

We wish to consider some extensions of the 'strange identity' $f^{(n,n,\ldots,n)} = g^{(2n-1,2n-3,\ldots,1)}$. Given any partition $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash l$, let

$$P_\lambda = \{(i,j) \in \mathbb{Z}^2 \colon 1 \leqslant i \leqslant k, 1 \leqslant j \leqslant \lambda_i\}.$$

Let P_λ inherit the standard product ordering of $\mathbb{Z}^2$. We can identify P_λ with the Young diagram of λ in an obvious way. A linear extension of P_λ (i.e. an order-preserving bijection $P_\lambda \to \{1, \ldots, l\}$) may be identified with an SYT of shape λ. Thus if $e(P)$ denotes the number of linear extensions of a poset P, then $e(P_\lambda) = f^\lambda$. Similarly, given a partition $\mu = (\mu_1, \ldots, \mu_k)$ into distinct parts, define the poset

$$Q_\mu = \{(i,j) \in \mathbb{Z}^2 \colon 1 \leqslant i \leqslant k, i \leqslant j \leqslant \mu_i + i - 1\}.$$

Then Q_μ may be identified with the shifted Young diagram of shape μ, and $e(Q_\mu) = g^\mu$. Write $P_n = P_{(n,n,\ldots,n)}$ and $Q = Q_{(2n-1,2n-3,\ldots,1)}$. Then the equality $f^{(n,n,\ldots,n)} = g^{(2n-1,2n-3,\ldots,1)}$ is equivalent to $e(P_n) = e(Q_n)$, so we may ask what other properties P_n and Q_n have in common.

6.1. THEOREM (J. Stembridge and R. Proctor [13], independently). *For any k, the number of chains in P_n of length k is equal to the number of chains in Q_n of length k.*

The next result was proved by R. Proctor [13] using branching rules for the symplectic group.

6.2. THEOREM (R. Proctor). *Recall that an* order ideal *of a poset P is a subset I satisfying $x \in I$, $y < x \Rightarrow y \in I$. Then for any k, the number of chains of order ideals of P_n of length k (ordered by containment) is the same as for Q_n. (In the terminology of* [14], *P_n and Q_n have the same* order polynomial.)

In fact, the following stronger condition (based on a suggestion of P. Edelman) may be true. Let $\underline{k}$ denote a k-element chain. For any j and k, the number of j-element antichains in the product $P_n \times \underline{k}$ is the same as for $Q_n \times \underline{k}$.

Proctor has extended the above results to certain other pairs P_λ and Q_μ. See [13] for further details.

7. Root Systems and Other Groups

If R is a root system, then let R^+ denote the set of positive roots of R, partially ordered by the usual ordering on roots [10, p. 47], i.e., $\alpha > \beta$ if $\beta - \alpha$ is a sum of positive roots. R. Proctor observed, at a time when Corollary 4.3 was only a conjecture, that A_{n-1}^+ is isomorphic to the poset $P_{(n-1,n-2,\ldots,1)}$ defined in the previous section. Thus for $w_0 \in S_n$, we have $r(w_0) = e(A_{n-1}^+)$. Proctor suggested that this might be true of other Weyl groups as well. Indeed, it is the case that $B_n^+ = Q_n$, so the conjecture (8) takes the form $r(w_0) = e(B_n^+)$. It was this observation that led to Conjecture 5.1. (Previously I had only conjectured that $r(w_0) = f^{(n,n,\ldots,n)}$. Procter's observation shows that the 'correct' formulation is $r(w_0) = g^{(2n-1,2n-3,\ldots,1)}$. It is remarkable that Conjecture 5.3 'vindicates' the original formulation $r(w_0) = f^{(n,n,\ldots,n)}$.)

More generally, let J be any subset of the set S of simple reflections of a Weyl group W with root system R. Let W_J be the corresponding parabolic subgroup of W, and let w_0^J be the longest minimum length coset representative of W_J (i.e. the maximum element of the quotient Bruhat order $W^J = W/W_J$ [4] [16]). Let $\{\lambda_i : i \in S\}$ be the fundamental weights corresponding to R, and set $\lambda = \sum_{i \notin J} \lambda_i$. Define a subposet R_J^+ of R^+ by $R_J^+ = \{\alpha : (\alpha, \lambda) > 0\}$. Proctor suggested (again when all the results for S_n were only conjectures) that

$$r(w_0^J) = e(R_J^+). \tag{10}$$

It indeed follows from Corollary 4.2 that (10) is valid for $R = A_{n-1}$ and any J. Proctor has verified (10) for $R = B_n$ and any J when $n \leq 4$.

In view of the situation for A_{n-1} and B_n, it is certainly natural to expect that (10) continues to hold for any Weyl group W. Unfortunately, for $w_0 \in D_4$ we have $r(w_0) = 2316 = 2^2 \cdot 3 \cdot 193$ while $e(D_4^+) = 2400$. Moreover, (10) fails for certain J when W is of type F_4, though it has not yet been checked whether $r(w_0) = e(F_4^+)$. In all known cases for which (10) fails, we have $r(w_0^J) < e(R_J^+)$. As pointed out by Proctor, it is easy to verify (10) when J corresponds to a minuscule weight. However, we are unable to offer any conjecture as to, say, the value of $r(w_0)$ for $w_0 \in D_n$. Perhaps a good example to look at would be the affine Weyl group $\tilde{A}_{n-1}$ (there is no w_0 but one could still consider each $w \in \tilde{A}_{n-1}$).

Note Added in Proof

The conjecture concerning $P_n \times \underline{k}$ and $Q_n \times \underline{k}$ mentioned at the end of Section 6 has been proved in the case $k = 1$ by J. Stembridge.

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