Unimodality and Lie Superalgebras

*By Richard P. Stanley**

It is well-known how the representation theory of the Lie algebra sl(2, ℂ) can be used to prove that certain sequences of integers are unimodal and that certain posets have the Sperner property. Here an analogous theory is developed for the Lie superalgebra osp(1,2). We obtain new classes of unimodal sequences (described in terms of cycle index polynomials) and a new class of posets (the "superanalogue" of the lattice $L(m, n)$ of Young diagrams contained in an $m \times n$ rectangle) which have the Sperner property.

1. Introduction

Let m and n be integers with $m \leq n$. A sequence $a_m, a_{m+1}, \ldots, a_n$ of real numbers is *symmetric* [about $\frac{1}{2}(m+n)$] if $a_{m+i} = a_{n-i}$ for $0 \leq i \leq n-m$, and *unimodal* if $a_m \leq a_{m+1} \leq \cdots \leq a_j \geq a_{j+1} \geq \cdots \geq a_n$ for some j. We also call the Laurent polynomial $a_m q^m + a_{m+1} q^{m+1} + \cdots + a_n q^n$ symmetric or unimodal if its coefficients $a_m, a_{m+1}, \ldots, a_n$ have the corresponding property. It is well known how the representation theory of the Lie algebra sl(2) = sl(2, ℂ) can be used to prove certain sequences are symmetric and unimodal. This goes back to Dynkin [5, p. 332] and is further discussed, for example, in [1], [16], [18]. In particular, every finite-dimensional complex semisimple Lie algebra $\mathfrak{G}$ contains a copy of sl(2), known as a "principal three-dimensional subalgebra," which leads to a wide variety of unimodal sequences (explicitly described in [16]).

Here we derive an analogous theory for Lie superalgebras. The analogue of sl(2) is the orthosymplectic superalgebra osp(1,2) [also denoted by $B(0,1)$ or osp(2,4)]. It is no longer true that every finite-dimensional complex semisimple Lie superalgebra contains a principal osp(1,2). Indeed, the only general class of

Address for correspondence: Professor Richard P. Stanley, Room 2-375, M.I.T., Cambridge, MA 02139.

*Partially supported by a grant from the National Science Foundation and by a Guggenheim Fellowship.

Published by Elsevier Science Publishing Co., Inc. 0022-2526/85/$3.30

superalgebras which do contain a principal osp(1,2) are those denoted by $\mathrm{gl}(n+1/n)$. [Strictly speaking, $\mathrm{gl}(n+1/n)$ is not semisimple, and we should be dealing instead with $\mathrm{sl}(n+1/n)$, also denoted by $A(n, n-1)$ or $\mathrm{spl}(n+1, n)$. It is more convenient to work with $\mathrm{gl}(n+1/n)$, and this superalgebra is close enough to being semisimple to cause no difficulties.]

The Lie algebra sl(2) can also be used to prove that certain partially ordered sets have some desirable extremal properties, in particular the Sperner property. This use of sl(2) had its origins in [17] and was first explicitly formulated in [14]. In Section 8 we give a "superanalogue" in which sl(2) is replaced by osp(1,2).

2. Review of sl(2)

First we review the relevant background concerning sl(2), so that the analogy with osp(1,2) will be clear. The Lie algebra sl(2) is spanned by the three matrices

$$x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

with the bracket operation $[A, B] = AB - BA$. Let $\mathrm{gl}(n) = \mathrm{gl}(n, \mathbb{C})$ denote the Lie algebra of all $n \times n$ complex matrices, and let

$$\phi : \mathrm{sl}(2) \to \mathrm{gl}(N)$$

be a representation (= Lie-algebra homomorphism) of sl(2). Then ϕh is similar to a diagonal matrix with integer eigenvalues, say $\mathrm{diag}(j_1, \ldots, j_N)$, $j_i \in \mathbb{Z}$. We then define the *character* $\mathrm{ch}\,\phi$ of ϕ to be the polynomial

$$\mathrm{ch}\,\phi = q^{j_1} + \cdots + q^{j_N}.$$

(In the precise definition of $\mathrm{ch}\,\phi$ as given e.g. in [6, §22.5], the symbol q is regarded as a certain element in the group algebra of the dual to the weight lattice of sl(2), but for our purposes we may merely regard q as an indeterminate.)

The representation ϕ can be written as a direct sum of irreducible representations, and $\mathrm{ch}\,\phi$ can be uniquely written as a nonnegative integral linear combination of irreducible characters. The Lie algebra sl(2) has one irreducible representation ϕ_{n-1} (up to equivalence) of every dimension $n \geq 1$. [The image of sl(2) under ϕ_{n-1} is a "principal three-dimensional subalgebra" of $\mathrm{gl}(n)$.] The character of ϕ_{n-1} is given by

$$\mathrm{ch}\,\phi_n = q^{-n} + q^{-n+2} + q^{-n+4} + \cdots + q^n.$$

For these basic facts about sl(2), see e.g. [6, §7].

It follows that when we write $\mathrm{ch}\,\phi$ as a linear combination of irreducibles, we obtain, for certain nonnegative integers m_i,

$$\begin{aligned}\mathrm{ch}\,\phi &= m_0\,\mathrm{ch}\,\phi_0 + m_1\,\mathrm{ch}\,\phi_1 + \cdots \\ &= m_0 + m_1(q^{-1}+q) + m_2(q^{-2}+1+q^2) + \cdots \\ &= \sum_i b_i q^i,\end{aligned}$$

where $b_i = b_{-i}$ and $b_i - b_{i+2} = m_i \geq 0$ for $i \geq 0$. Hence we obtain:

THEOREM 2.1. *If* $\mathrm{ch}\,\phi = \Sigma b_i q^i$, *then the two sequences* $\ldots, b_{-4}, b_{-2}, b_0, b_2, b_4, \ldots$ *and* $\ldots, b_{-3}, b_{-1}, b_1, b_3, \ldots$ *are symmetric* (*about* 0) *and unimodal.*

3. Schur functions

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be a partition of length $l(\lambda) := \#\{i | \lambda_i \neq 0\} \leq n$, i.e., $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$, $\lambda_i \in \mathbb{Z}$. Write $|\lambda| = \lambda_1 + \cdots + \lambda_n$. Then λ indexes a certain irreducible representation $\psi_\lambda : \mathrm{gl}(n) \to \mathrm{gl}(N)$, whose description is essentially due to Schur [in the context of the Lie group $\mathrm{GL}(n)$]. We will not bother to define ψ_λ here, but will merely state the properties of interest to us. If $A \in \mathrm{gl}(n)$ has eigenvalues $\alpha_1, \ldots, \alpha_n$, then $\psi_\lambda(A)$ has all its eigenvalues of the form $a_1\alpha_1 + \cdots + a_n\alpha_n$, where $a_1, \ldots, a_n$ are nonnegative integers independent of $\alpha_1, \ldots, \alpha_n$, and $\Sigma a_i = |\lambda|$. Define the *character* $\mathrm{ch}\,\psi_\lambda$ of ψ_λ to be the polynomial

$$\mathrm{ch}\,\psi_\lambda = \sum x_1^{a_1} \cdots x_n^{a_n},$$

the sum being over all eigenvalues $\Sigma a_i\alpha_i$ of A. Then $\mathrm{ch}\,\psi_\lambda$ is a symmetric function of $x_1, \ldots, x_n$ denoted $s_\lambda(x) = s_\lambda(x_1, \ldots, x_n)$ and called a *Schur function.* The basic properties of Schur functions are discussed in [12] and [15].

Consider now the composite representation

$$\mathrm{sl}(2) \overset{\phi_{n-1}}{\to} \mathrm{gl}(n) \overset{\psi_\lambda}{\to} \mathrm{gl}(N)$$

of sl(2). Now $\phi_{n-1}(h)$ has eigenvalues $-n+1, -n+3, \ldots, n-1$, so $\psi_\lambda\phi_{n-1}(h)$ has eigenvalues $(-n+1)a_1 + (-n+3)a_2 + \cdots + (n-1)a_n$. Hence

$$\begin{aligned}\mathrm{ch}(\psi_\lambda\phi_{n-1}) &= \sum q^{-(n-1)a_1-(n-3)a_2+\cdots+(n-1)a_n} \\ &= q^{-(n+1)|\lambda|}\sum (q^2)^{a_1+2a_2+\cdots+na_n} \\ &= q^{-(n+1)|\lambda|}s_\lambda(q^2, q^4, \ldots, q^{2n}).\end{aligned} \tag{1}$$

We deduce from Theorem 2.1:

THEOREM 3.1. *For any partition λ of length $\leq n$, the polynomial $s_\lambda(q, q^2, \ldots, q^n)$ is symmetric and unimodal.*

Note: If $l(\lambda) > n$, then $s_\lambda(x_1, \ldots, x_n) = 0$, so the condition $l(\lambda) \leq n$ of the previous theorem is irrelevant.

A simple explicit formula for $s_\lambda(q, q^2, \ldots, q^n)$ appears in [12, Example 1, p. 27] or [15, Theorem 15.3]. Theorem 3.1 is implicit in [5, p. 332], is more explicit in [16, Example 2], and is also given in [12, Example 4, p. 67]. The coefficient of q^i in $s_\lambda(q, q^2, \ldots, q^n)$ has a combinatorial interpretation—it is the number of column-strict plane partitions (as defined in [12, Example 13, p. 48] and [15, §1]) of shape λ, largest part $\leq n$, and sum of parts equal to i.

Let us also mention that when $\lambda = (m)$, the partition with a single part equal to m, then

$$s_m(q, \ldots, q^n) = q^m \begin{bmatrix} n+m-1 \\ m \end{bmatrix},$$

where

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{(1-q^a)(1-q^{a-1}) \cdots (1-q^{a-b+1})}{(1-q^b)(1-q^{b-1}) \cdots (1-q)}$$

denotes a *q-binomial coefficient*.

There is a generalization of Theorem 3.1 pointed out to me by A. Kerber, and proved by him in a different way than that given below. Let $p_\lambda(x)$ denote the power-sum symmetric function [12, pp. 15–16], defined by

$$p_m(x) = \sum_i x_i^m, \qquad p_\lambda(x) = p_{\lambda_1}(x) p_{\lambda_2}(x) \cdots p_{\lambda_l}(x),$$

where $l = l(\lambda)$. If $w \in S_m$, the symmetric group on m letters, then $\rho(w)$ denotes the partition whose parts are equal to the cycle lengths of w. The irreducible (ordinary) characters χ^λ of S_m are indexed by partitions λ of m. We then have the famous formula of Frobenius (see [12, Chapter I.7])

$$s_\lambda(x) = \frac{1}{m!} \sum_{w \in S_m} \chi^\lambda(w) p_{\rho(w)}(x). \tag{2}$$

Thus

$$s_\lambda(q, q^2, \ldots, q^n) = \frac{1}{m!} \sum_{w \in S_m} \chi^\lambda(w) \prod_i (q^i + q^{2i} + \cdots + q^{ni})^{c_i(w)}, \tag{3}$$

where w has $c_i(w)$ cycles of length i.

Now if H is a subgroup of S_n and χ a character of H, then define [7, 5.1.27] the *generalized cycle index* of H with respect to χ by

$$\mathrm{Cyc}(H,\chi) = \frac{1}{|H|}\sum_{w\in H}\chi(w^{-1})\prod_i x_i^{c_i(w)}.$$

If $\chi = 1$ is the trivial character, we write Cyc(H) for Cyc(H,1). Equation (3) becomes

$$s_\lambda(q,q^2,\dots,q^n) = \mathrm{Cyc}(S_m,\chi^\lambda)(x_i \to q^i + q^{2i} + \cdots + q^{ni}), \tag{4}$$

where the notation indicates that we substitute $q^i + q^{2i} + \cdots + q^{ni}$ for x_i in $\mathrm{Cyc}(S_m, \chi^\lambda)$.

We may generalize (4) as follows. Let

$$f(q) = a_0 + a_1 q + \cdots + a_r q^r$$

be any polynomial with symmetric, unimodal, nonnegative integer coefficients. Let $n = f(1)$, and let $\phi: \mathrm{sl}(2) \to \mathrm{gl}(n)$ be the representation with character $\mathrm{ch}\,\phi = q^{-r}f(q^2)$. Let $\psi_\lambda: \mathrm{gl}(n) \to \mathrm{gl}(N)$ be as above, where $l(\lambda) \le n$ and $|\lambda| = m$. Then we obtain in the same way as (1) and (4) that

$$\mathrm{ch}(\psi_\lambda\phi) = q^{-mr}s_\lambda\Big(\underbrace{1,1,\dots,1}_{a_0},\underbrace{q^2,q^2,\dots,q^2}_{a_1},\dots,\underbrace{q^{2r},\dots,q^{2r}}_{a_r}\Big),$$

so from (2) we get that

$$\mathrm{Cyc}(S_m,\chi^\lambda)(x_i \to f(q^i)) \tag{5}$$

is symmetric (about $\frac{1}{2}mr$) and unimodal. The substitution $x_i \to f(q^i)$ is sometimes called the *Polya composition* with $f(q)$, denoted

$$\mathrm{Cyc}(S_m,\chi^\lambda)[f(q)].$$

Since the center of symmetry of the polynomial (5) is at $\frac{1}{2}mr$ (independent of λ), it follows that any nonnegative linear combination of polynomials (5), where m and f are fixed, will also be symmetric (about $\frac{1}{2}mr$) and unimodal. Hence we may replace χ^λ in (5) with any ordinary character χ of S_m. Thus we have proved:

THEOREM 3.2. *Let χ be an ordinary character of S_m, and let $f(q)$ be a polynomial with nonnegative integral unimodal coefficients, satisfying $q^r f(1/q) = f(q)$. Let*

$$g(q) = \mathrm{Cyc}(S_m,\chi)[f(q)]. \tag{6}$$

Then $g(q)$ has nonnegative integral unimodal coefficients, and $q^{mr}g(1/q) = g(q)$.

In particular, it is easily seen (e.g., by Frobenius reciprocity) that for any subgroup G of S_m and any ordinary character χ of G we have

$$\mathrm{Cyc}(G,\chi) = \mathrm{Cyc}\big(S_m, \mathrm{ind}_G^{S_m}\chi\big),$$

where $\mathrm{ind}_G^{S_m}\chi$ denotes the induction of χ to S_m. There follows the result of Kerber:

COROLLARY 3.3. *Let G be a subgroup of S_m, χ an ordinary character of G, and $f(q)$ as in Theorem* 3.2. *Then*

$$\mathrm{Cyc}(G,\chi)[f(q)]$$

satisfies the conclusions to Theorem 3.2.

Let us note that if $f(q)$ satisfies all the conditions of Theorem 3.2 except integrality, then $g(q)$ [as defined by (6)] need not have unimodal coefficients. For instance,

$$\mathrm{Cyc}(S_2)\left[\tfrac{1}{2}+\tfrac{1}{2}q\right] = \tfrac{1}{8}(3+2q+3q^2).$$

4. The superalgebra osp(1,2)

A Lie superalgebra is a vector space $\mathfrak{G}$ (which we will always take over $\mathbb{C}$), together with two subspaces $\mathfrak{G}_0$ and $\mathfrak{G}_1$ for which $\mathfrak{G} = \mathfrak{G}_0 \oplus \mathfrak{G}_1$, and a binary operation $[A, B]$ satisfying certain axioms. Rather than give the precise definition here, we will be content with defining examples of concern to us. Our basic reference is [8] and the useful summary [9]. All the results stated below without proof can essentially be found in these references.

Let V be a complex vector space of dimension $m+n$, and let V_0 and V_1 be subspaces satisfying $V = V_0 \oplus V_1$, $\dim V_0 = m$, $\dim V_1 = n$. The Lie superalgebra $\mathrm{gl}(m/n)$ is defined as follows. As a vector space it is given by $\mathrm{End}\,V$, the set of linear transformations $A: V \to V$. For $i = 0,1$, define

$$\mathrm{End}_i V = \{A \in \mathrm{End}\,V : AV_j \subseteq V_{i+j}\},$$

where the subscript $i+j$ is taken modulo 2. Thus

$$\mathrm{End}\,V = \mathrm{End}_0 V \oplus \mathrm{End}_1 V.$$

Define a binary operation $[A, B]$ on $\mathrm{End}\,V$ by

$$[A, B] = AB - (-1)^{ij} BA,$$

where $A \in \mathrm{End}_i V$, $B \in \mathrm{End}_j V$, and extending to all of $\mathrm{End}\,V$ by bilinearity

Choose an ordered basis for V whose first m elements form a basis for V_0 and last n for V_1. Then End V can be identified with the space of all $(m+n)\times(m+n)$ complex matrices

$$A = \begin{matrix} m\{ \\ n\{ \end{matrix}\left[\begin{array}{c|c} \overbrace{A_1}^{m} & \overbrace{A_2}^{n} \\ \hline A_3 & A_4 \end{array}\right],$$

where $\mathrm{End}_0 V$ consists of those matrices with $A_2 = 0$ and $A_3 = 0$, and $\mathrm{End}_1 V$ of those with $A_1 = 0$ and $A_4 = 0$.

Any subspace $\mathfrak{G}$ of $\mathrm{gl}(m/n)$ satisfying $\mathfrak{G} = [\mathfrak{G} \cap \mathrm{gl}(m/n)_0] \oplus [\mathfrak{G} \cap \mathrm{gl}(m/n)_1]$ and closed under the operation $[A, B]$ is itself a Lie superalgebra, with $\mathfrak{G}_i = \mathfrak{G} \cap \mathrm{gl}(m/n)_i$. In particular, define the *orthosymplectic* Lie superalgebra $\mathrm{osp}(1,2) \subset \mathrm{gl}(1/2)$ [sometimes denoted osp(2,4) or $B(0,1)$] to be the set of all 3×3 complex matrices of the form

$$\left[\begin{array}{c|cc} 0 & \alpha & \beta \\ \hline \beta & \gamma & \delta \\ -\alpha & \varepsilon & -\gamma \end{array}\right]$$

Thus $\dim \mathrm{osp}(1,2) = 5$.

A (finite-dimensional) *representation* of a Lie superalgebra $\mathfrak{G} = \mathfrak{G}_0 \oplus \mathfrak{G}_1$ is a linear transformation $\phi: \mathfrak{G} \to \mathrm{gl}(m/n) = \mathrm{End}_0 V \oplus \mathrm{End}_1 V$, such that $\phi \mathfrak{G}_i \subseteq \mathrm{End}_i V$ and $\phi[A, B] = [\phi A, \phi B]$. We write $A \cdot v$ for $(\phi A)(v)$ and think of $A \in \mathfrak{G}$ as acting on V. Two representations

$$\phi: \mathfrak{G} \to \mathrm{End}_0 V \oplus \mathrm{End}_1 V,$$

$$\psi: \mathfrak{G} \to \mathrm{End}_0 W \oplus \mathrm{End}_1 W$$

are *equivalent* if there is an isomorphism $\sigma: V \to W$ and a bijection $\pi: \{0,1\} \to \{0,1\}$ such that $\sigma V_i = W_{\pi i}$ $(i = 0,1)$ and $\sigma(\phi A) = (\psi A)\sigma$ for all $A \in \mathfrak{G}$. A representation $\phi: \mathfrak{G} \to \mathrm{End}_0 V \oplus \mathrm{End}_1 V$ is *irreducible* if V has no proper $\mathfrak{G}$-invariant subspace $W = W_0 \oplus W_1$ where $W_i = W \cap V_i$.

THEOREM 4.1.

(a) *Every representation* $\phi: \mathrm{osp}(1,2) \to \mathrm{End}\, V$ *is completely reducible, i.e., we can write* V *as a direct sum of irreducible invariant subspaces.*

(b) *For each integer* $n \geq 0$ *there is an irreducible representation* $\phi_n: \mathrm{osp}(1,2) \to \mathrm{gl}(n+1/n)$, *and this accounts for all inequivalent (finite-dimensional) irreducible representations of* osp(1,2).

Note: Unlike the situation for the Lie algebra gl(n), not every finite-dimensional representation of $\mathrm{gl}(m/n)$ is completely reducible.

Let $h = \mathrm{diag}(0, 1, -1) \in \mathrm{osp}(1,2)$. If $\phi: \mathrm{osp}(1,2) \to \mathrm{gl}(m/n)$ is a representation, then ϕh is similar to a diagonal matrix with integer eigenvalues $a_1, \ldots, a_{m+n}$. We then define the *character* of ϕ by

$$\mathrm{ch}\,\phi = q^{a_1} + \cdots + q^{a_{m+n}}.$$

THEOREM 4.2. *Let* $\phi_n : \mathrm{osp}(1,2) \to \mathrm{gl}(n+1/n)$ *be the irreducible representation of Theorem* 4.1(b). *Then*

$$\mathrm{ch}\,\phi_n = q^{-n} + q^{-n+1} + \cdots + q^n.$$

In fact, writing $\mathrm{gl}(n+1/n) = \mathrm{End}(V_0 \oplus V_1)$ *where* $\dim V_0 = n+1$ *and* $\dim V_1 = n$, *then* $\phi_n h$ *restricted to* V_0 *has eigenvalues* $-n, -n+2, \ldots, n$, *while* $\phi_n h$ *restricted to* V_1 *has eigenvalues* $-n+1, -n+3, \ldots, n-1$.

COROLLARY 4.3. *Let* $\phi : \mathrm{osp}(1,2) \to \mathrm{gl}(m/n)$ *be any representation. Then the Laurent polynomial*

$$\mathrm{ch}\,\phi = \sum_{i=-N}^{N} b_i q^i.$$

is unimodal and symmetric about 0 (*i.e.*, $b_i = b_{-i}$).

Proof: By Theorems 4.1 and 4.2, $\mathrm{ch}\,\phi$ is a nonnegative integer linear combination of the Laurent polynomials $\mathrm{ch}\,\phi_n = q^{-n} + q^{-n+1} + \cdots + q^n$, and the proof follows. □

5. Super-Schur functions

We now turn to the superalgebra analogue of Schur functions. Let $\Gamma(m,n)$ be the set of all partitions $\lambda = (\lambda_1, \lambda_2, \ldots)$ such that $\lambda_i \le n$ if $i \ge m+1$. Thus the Young diagram of λ lies inside a hook of arm height m and leg width n. Then λ indexes a certain irreducible representation $\psi_\lambda : \mathrm{gl}(m/n) \to \mathrm{gl}(M/N)$, as described, e.g., in [3].

Suppose $A \in \mathrm{End}_0 V$, where $\mathrm{gl}(m/n) = \mathrm{End}\,V$. Let $\alpha_1, \ldots, \alpha_m$ be the eigenvalues of A restricted to V_0, and $\beta_1, \ldots, \beta_n$ the eigenvalues of A restricted to V_1. Then the eigenvalues of $\psi_\lambda(A)$ have the form $a_1\alpha_1 + \cdots + a_m\alpha_m + b_1\beta_1 + \cdots + b_n\beta_n$, where the a_i's and b_j's are nonnegative integers independent of the α_i's and β_j's. Moreover, $\Sigma a_i + \Sigma b_j = |\lambda|$. Define the *character* $\mathrm{ch}\,\psi_\lambda$ of ψ_λ to be the polynomial

$$\mathrm{ch}\,\psi_\lambda = \sum x_1^{a_1} \cdots x_m^{a_m} y_1^{b_1} \cdots y_n^{b_n}, \tag{7}$$

the sum being over all eigenvalues $\Sigma a_i\alpha_i + \Sigma b_j\beta_j$ of $\psi_\lambda(A)$.

While the evaluation of the characters $\mathrm{ch}\,\psi_\lambda$ is included in the general theory of Kač, a combinatorial description appears in [2], [4], and most explicitly in [3]. We will use the notation $s_\lambda(x/y)$ for these characters and call them *super-Schur functions*; Berele and Regev denote them by $\mathrm{HS}_\lambda(x;y)$ and call them "hook Schur functions." They are polynomials which are symmetric in the x_i's and y_j's separately, and have the additional "cancellation property"

$$s_\lambda(x/y)|_{x_1 = -y_1} = s_\lambda(x/y)|_{x_1 = y_1 = 0}. \tag{8}$$

Such polynomials are essentially the "bisymmetric functions" of [13, §5]. [More precisely, $s_\lambda(x/-y)$ is a bisymmetric function.]

The super-Schur functions are given by the formula [4, (9); 3, Definition 6.3]

$$s_\lambda(x/y) = \sum_{\mu \subset \lambda} s_\mu(x) s_{\lambda'/\mu'}(y). \tag{9}$$

Here $s_{\lambda'/\mu'}$ denotes a skew Schur function [12, Chapter I.5] and $'$ denotes the conjugate partition. In the terminology of D. E. Littlewood [10; 11, Chapter 6.4], $s_\lambda(x/-y)$ is a Schur function of the series

$$\frac{\prod_{i=1}^{m}(1-x_i)}{\prod_{j=1}^{n}(1-y_j)}.$$

In the notation of λ-rings [12, pp. 26–27], the polynomial $s_\lambda(x/-y)$ corresponds to the operation $S^\lambda(X-Y)$, where $X = x_1 + \cdots + x_m$ and $Y = y_1 + \cdots + y_n$. If ω_y denotes the automorphism of the ring of symmetric functions in the variable $y = (y_1, y_2, \ldots)$ as described in [12, pp. 14–17] (regard ω_y as commuting with the x_i's), then

$$s_\lambda(x/y) = \omega_y s_\lambda(x, y), \tag{10}$$

where $s_\lambda(x, y)$ denotes the Schur function s_λ in the two sets of variables x and y. Since ω_y is an algebra automorphism preserving the standard scalar product [12, Chapter I.4] on symmetric functions, almost every formula involving Schur functions has a "superanalogue" obtained by applying ω_y.

The formula (9), together with the well-known combinatorial definition of skew Schur functions [12, p. 42; 15, §12], gives a combinatorial definition of $s_\lambda(x/y)$. Namely, set $x_1 < \cdots < x_m < y_1 < \cdots < y_n$, and fill the Young diagram of shape λ with x_i's and y_j's such that:

(a) The entries weakly increase in every row and column.
(b) The x_i's strictly increase in columns.
(c) The y_j's strictly increase in rows.

Let $p(T)$ be the product of the entries of the resulting tableau T. Then

$$s_\lambda(x/y) = \sum_T p(T), \tag{11}$$

summed over all tableaux of shape λ satisfying (a)–(c).

Example: To compute $s_{21}(x_1, x_2/y_1)$, we have

$$\begin{array}{lllll} x_1x_1 & x_1x_2 & x_1x_1 & x_1x_2 & x_2x_2 \\ x_2 & x_2 & y_1 & y_1 & y_1 \\ & x_1y_1 & x_1y_1 & x_2y_1 & \\ & x_2 & y_1 & y_1 & \end{array}$$

so $s_{21}(x_1, x_2/y_1) = x_1^2x_2 + x_1x_2^2 + x_1^2y_1 + 2x_1x_2y_1 + x_2^2y_1 + x_1y_1^2 + x_2y_1^2$.

It is perhaps of interest to note that we may choose *any* ordering of the x_i's and y_j's in the combinatorial description of $s_\lambda(x/y)$. More precisely:

THEOREM 5.1. *Fix an arbitrary linear ordering of the set* $\{x_1,\dots,x_m,y_1,\dots,y_n\}$. *Fill the Young diagram of shape* λ *with* x_i*'s and* y_j*'s such that:*

(i) *The entries weakly increase in every row and column.*
(ii) *Any* x_i *appears at most once in each column.*
(iii) *Any* y_j *appears at most once in each row.*

Then $s_\lambda(x/y)=\Sigma_T p(T)$, *summed over all tableaux of shape* λ *satisfying* (i)–(iii), *where* $p(T)$ *is the product of the entries of* T.

Proof: Let h_i and e_i denote the complete homogeneous and elementary symmetric functions, respectively, as defined in [12, Chapter I.2]. Thus $h_i=s_i$ and $e_i=s_{1^i}$. By (11) and the Littlewood-Richardson rule [12, Chapter I.9] (or by applying the automorphism ω_y to the scalar product $\langle s_\lambda, h_\mu\rangle$), the coefficient of $x_1^{a_1}\cdots x_m^{a_m}y_1^{b_1}\cdots y_n^{b_n}$ in $s_\lambda(x/y)$ is equal to the coefficient of s_λ when the product $h_{a_1}\cdots h_{a_m}e_{b_1}\cdots e_{b_n}$ is expanded as a linear combination of Schur functions. But the factors of this product can be written in any order, and by the Littlewood-Richardson rule this yields the desired result. □

There is an alternative combinatorial interpretation of $s_\lambda(x/y)$ which makes the cancellation property (8) obvious. We merely state the result without proof; it is not difficult to deduce it from Theorem 5.1 by letting $m=n=\infty$ and choosing the ordering $x_1<y_1<x_2<y_2<\cdots$.

THEOREM 5.2. *Fill in the Young diagram of shape* λ *with positive integers such that:*

(i) *The entries weakly increase in every row and column.*
(ii) *The entries strictly increase along any diagonal running from the upper left to lower right. (Equivalently, no* 2×2 *square has all its entries equal.)*

Let T *be the resulting tableau. Let* m_i *denote the number of entries of* T *equal to* i, r_i *the number of rows of* T *which contain an* i, *and* c_i *the number of columns of* T *which contain an* i. *Define*

$$q(T)=\prod_{i\geq 1} x_i^{m_i-r_i}y_i^{m_i-c_i}(x_i+y_i)^{r_i+c_i-m_i}.$$

Let $x=(x_1,x_2,\dots)$ *and* $y=(y_1,y_2,\dots)$. *Then*

$$s_\lambda(x/y)=\sum_T q(T),$$

where T *ranges over all tableaux satisfying* (i) *and* (ii).

6. Application to unimodality

Consider the composite representation

$$\mathrm{osp}(1,2)\xrightarrow{\phi_n}\mathrm{gl}(n+1/n)\xrightarrow{\psi_\lambda}\mathrm{gl}(M/N)$$

of osp(1,2). Let $h = \mathrm{diag}(0,1,-1) \in \mathrm{osp}(1,2)$ as in Section 4. Write $\mathrm{gl}(n+1/n) = \mathrm{End}(V_0 \oplus V_1)$. By Theorem 4.2, the eigenvalues of $\phi_n(h)$ restricted to V_0 are $-n, -n+2, \ldots, n$, while those restricted to V_1 are $-n+1, -n+3, \ldots, n-1$. Thus if

$$s_\lambda(x/y) = \sum x_1^{a_1} \cdots x_{n+1}^{a_{n+1}} y_1^{b_1} \cdots y_n^{b_n},$$

then by the definition (7) of $s_\lambda(x/y)$ there follows

$$\begin{aligned} \mathrm{ch}\,\psi_\lambda\phi_n &= \sum q^{-na_1 + (-n+2)a_2 + \cdots + na_{n+1} + (-n+1)b_1 + (-n+3)b_2 + \cdots + (n-1)b_n} \\ &= q^{-n|\lambda|} \sum q^{2a_2 + 4a_3 + \cdots + 2na_{n+1} + b_1 + 3b_2 + \cdots + (2n-1)b_n} \\ &= q^{-n|\lambda|} s_\lambda(1, q^2, \ldots, q^{2n}/q, q^3, \ldots, q^{2n-1}). \end{aligned}$$

Hence from Corollary 4.3, we conclude:

THEOREM 6.1. *Let* $\lambda \in \Gamma(n+1, n)$. *Then the polynomial* $s_\lambda(1, q^2, \ldots, q^{2n}/q, q^3, \ldots, q^{2n-1})$ *is symmetric about* $n|\lambda|$ *and is unimodal.*

Remark: By (9) we have

$$\begin{aligned} s_\lambda(1, q^2, \ldots, q^{2n}/q, q^3, \ldots, q^{2n-1}) &= \sum_{\mu \subset \lambda} s_\mu(1, q^2, \ldots, q^{2n}) s_{\lambda'/\mu'}(q, q^3, \ldots, q^{2n-1}) \\ &= \sum_{\mu \subset \lambda} f_\mu(q), \quad \text{say.} \end{aligned}$$

It is easily seen that $f_\mu(q) = q^{2n|\lambda|} f_\mu(1/q)$ and that $f_\mu(q)$ is even or odd depending on whether $|\lambda/\mu| = |\lambda| - |\mu|$ is even or odd. Moreover, it follows from Theorem 3.1 and the nonnegativity of the integers $c^\lambda_{\mu\nu}$ in the expansion $s_{\lambda/\mu} = \Sigma_\nu c^\lambda_{\mu\nu} s_\nu$ [12, Chapter I.9] that the coefficients of the even-degree (respectively, odd-degree) terms of $f_\mu(q)$ for $|\lambda/\mu|$ even (respectively, odd) are unimodal. Hence it is a consequence of the representation theory of sl(2) and gl(n) alone that the two polynomials

$$\sum_{\substack{\mu \subset \lambda \\ |\lambda/\mu| \text{ even}}} f_\mu(q) \quad \text{and} \quad \sum_{\substack{\mu \subset \lambda \\ |\lambda/\mu| \text{ odd}}} f_\mu(q), \tag{12}$$

respectively, have unimodal coefficients of their even-degree (respectively, odd-degree) terms; and both are symmetric about the same point (viz., $n|\lambda|$). The superalgebra structure has the effect of "unifying" (12) into a single polynomial with unimodal coefficients. This is analogous to (but far less profound than) the way in which superalgebra theory unifies particles with half-integer spin (fermions) and integer spin (bosons).

Example: Suppose λ consists of a single row of length l. Then

$$s_l(1, q^2, \ldots, q^{2n}/q, q^3, \ldots, q^{2n-1}) = \sum_{i=0}^{l} s_i(1, q^2, \ldots, q^{2n}) s_{1^{l-i}}(q, q^3, \ldots, q^{2n-1})$$

$$= \sum_{i=0}^{l} q^{(l-i)^2} \begin{bmatrix} n+i \\ i \end{bmatrix}_{q^2} \begin{bmatrix} n \\ l-i \end{bmatrix}_{q^2},$$

where $\begin{bmatrix} a \\ b \end{bmatrix}_{q^2}$ denotes the q-binomial coefficient in the variable q^2. If we denote the above expression by $P_{ln}(q)$, then

$$\sum_{l \geq 0} P_{ln}(q) t^l = \frac{(1+qt)(1+q^3t) \cdots (1+q^{2n-1}t)}{(1-t)(1-q^2t) \cdots (1-q^{2n}t)}.$$

Even though $P_{ln}(q)$ has a simple, elementary definition, we don't know how to prove its coefficients are unimodal without using superalgebras. If instead we took λ to consist of a single column of length l, then we would obtain the unimodality of the polynomials $P'_{ln}(q)$ defined by

$$\sum_{l \geq 0} P'_{ln}(q) t^l = \frac{(1+t)(1+q^2t) \cdots (1+q^{2n}t)}{(1-qt)(1-q^3t) \cdots (1-q^{2n-1}t)}.$$

Example: Suppose $\lambda = (\lambda_1, \lambda_2, \ldots)$ where $\lambda_{n+1} \geq n$. The representation ψ_λ of $\mathrm{gl}(n+1/n)$ is then called *typical*, and by [3, Theorem 6.20] or [9, §2.4] we have

$$s_\lambda(x_1, \ldots, x_{n+1}/y_1, \ldots, y_n) = s_\alpha(x) s_\beta(y) \prod_{i=1}^{n+1} \prod_{j=1}^{n} (x_i + y_j),$$

where $\alpha = (\lambda_1 - n, \lambda_2 - n, \ldots, \lambda_{n+1} - n)$ and $\beta' = (\lambda_{n+2}, \lambda_{n+3}, \ldots)$. In particular, if $\lambda = (n, n, \ldots, n)$ ($n+1$ times), then

$$s_\lambda(x_1, \ldots, x_{n+1}/y_1, \ldots, y_n) = \prod (x_i + y_j),$$

a result essentially due to Littlewood [10, Theorem XVIII; 11, Theorem XVIII, p. 115]. Thus by Theorem 6.1 the polynomial

$$\prod_{i=0}^{n} \prod_{j=1}^{n} (q^{2i} + q^{2j-1}) = q^{\frac{1}{6}n(n+1)(4n-1)} \prod_{i=1}^{n} (1 + q^{2i-1})^{2(n-i+1)}$$

is unimodal.

7. A cycle-index generalization

Theorem 6.1 can be generalized in the same way as Theorem 3.1 was extended to Theorem 3.2.

THEOREM 7.1. *Let* χ *be an ordinary character of the symmetric group* S_m, *and let* $f(q)$ *be a polynomial with nonnegative integral unimodal coefficients, satisfying* $q^{2r}f(1/q)=f(q)$ *for some integer* $r\geq 0$. *Let*

$$g(q) = \operatorname{Cyc}(S_m,\chi)\left(x_i\to f\left((-1)^{i-1}q^i\right)\right).$$

Then $g(q)$ *has nonnegative integral unimodal coefficients, and* $q^{2mr}g(1/q)=g(q)$.

Proof: Let $\phi:\operatorname{osp}(1,2)\to \operatorname{gl}(k/n)$ have character $q^{-r}f(q)$. (If c denotes the middle coefficient of $f(q)$, then $k=\frac{1}{2}[f(1)+c]$ and $n=\frac{1}{2}[f(1)-c]$.) Let $\chi=\Sigma c_\lambda\chi^\lambda$ be the decomposition of χ into irreducibles. Let $\psi:\operatorname{gl}(k/n)\to\operatorname{gl}(M/N)$ be given by $\psi=\Sigma c_\lambda\psi_\lambda$, so $\operatorname{ch}\psi=\Sigma c_\lambda s_\lambda(x/y)$. Suppose $f(q)=a_0+a_1q+\cdots+a_{2r}q^{2r}$. Then

$$\operatorname{ch}\psi\phi$$
$$= \sum c_\lambda s_\lambda\Big(\underbrace{1,\dots,1}_{a_0},\underbrace{q^2,\dots,q^2}_{a_2},\dots,\underbrace{q^{2r},\dots,q^{2r}}_{a_{2r}}/\underbrace{q,\dots,q}_{a_1},\dots,\underbrace{q^{2r-1},\dots,q^{2r-1}}_{a_{2r-1}}\Big). \tag{13}$$

Now it follows from (2), (10), and the fact that $\omega_y p_i(y)=(-1)^{i-1}p_i(y)$ [12, p. 16] that

$$s_\lambda(x/y) = \frac{1}{m!}\sum_{w\in S_m}\chi^\lambda(w)\prod_i\left[p_i(x)-(-1)^i p_i(y)\right]^{c_i(w)},$$

where $|\lambda|=m$ and w has $c_i(w)$ cycles of length i. Hence

$$\operatorname{ch}\psi\phi = \frac{1}{m!}\sum_\lambda c_\lambda\sum_{w\in S_m}\chi^\lambda(w)\prod_i\Big[a_0+a_2q^{2i}+\cdots+a_{2r}q^{2ri}$$
$$-(-1)^i\left(a_1q^i+\cdots+a_{2r-1}q^{(2r-1)i}\right)\Big]^{c_i(w)}$$
$$= \frac{1}{m!}\sum_{w\in S_m}\chi(w)\prod_i f\left((-1)^{i-1}q^i\right)^{c_i(w)}$$
$$= \operatorname{Cyc}(S_m,\chi)\left(x_i\to f\left((-1)^{i-1}q^i\right)\right).$$

Thus by Theorem 6.1, $g(q)$ has nonnegative integral unimodal coefficients. Each term in the sum on the right-hand side of (13) is easily seen to be symmetric about mr, so the same is true of $\operatorname{ch}\psi\phi$. □

COROLLARY 7.2. *Let G be a subgroup of* S_m, χ *an ordinary character of* G, *and* $f(q)$ *as in Theorem* 7.1. *Then the polynomial*

$$\operatorname{Cyc}(G,\chi)\left(x_i \to f\left((-1)^{i-1}q^i\right)\right)$$

has nonnegative integral unimodal coefficients and is symmetric about mr.

Proof: Exactly the same as the deduction of Corollary 3.3 from Theorem 3.2. □

As in Section 3, we note that if $f(q)$ is not required to have integral coefficients, then $g(q)$ need not be unimodal. For instance, let $f(q) = \frac{1}{2} + q + \frac{1}{2}q^2$. Then

$$\operatorname{Cyc}(S_2)\left(x_i \to f\left((-1)^{i-1}q^i\right)\right) = \tfrac{1}{8}(3+4q+2q^2+4q^3+3q^4).$$

8. The Sperner property

The Lie algebra sl(2) has been used [14] to show that certain posets have the Sperner property. We will briefly review these results here and indicate their analogues for osp(1,2). We follow [14] in notation and terminology.

A *ranked poset* P of length r is a partially ordered P together with a partition $P = \bigcup_{i=0}^r P_i$ into $r+1$ nonvoid ranks P_i, $0 \le i \le r$, such that elements in P_i cover only elements in P_{i-1}. Assuming P is connected; then the ranking of P, if it exists, is unique. We will assume P is finite, and we set $p_i = |P_i|$. A ranked poset P is *strongly Sperner* if for every $k \ge 1$ no union of k antichains of P contains more elements than does the union of the k largest ranks of P. In particular ($k=1$), a strongly Sperner poset is *Sperner*, i.e., no antichain of P contains more than $\max p_i$ elements. A ranked poset of length r is *rank-symmetric* if $p_i = p_{r-i}$ for $0 \le i \le r$. It is *rank-unimodal* if $p_0 \le p_1 \le \cdots \le p_k \ge p_{k+1} \ge \cdots \ge p_r$ for some $0 \le k \le r$. Finally, a ranked poset is *Peck* if it is rank-symmetric, rank-unimodal, and strongly Sperner.

If $P = \bigcup_{i=0}^r P_i$ is any ranked poset, define a graded complex vector space

$$\tilde{P} = \tilde{P}_0 \oplus \tilde{P}_1 \oplus \cdots \oplus \tilde{P}_r,$$

where $\tilde{P}_i$ is the complex vector space with basis P_i. A linear operator X on $\tilde{P}$ is a *lowering operator* if $X\tilde{P}_i \subseteq \tilde{P}_{i-1}$, and a *raising operator* if $X\tilde{P}_i \subseteq \tilde{P}_{i+1}$. A raising operator X is an *order-raising operator* if for all $a \in P_i$ we have

$$Xa = \sum_b \theta(a,b)b,$$

where $\theta(a,b)=0$ unless b covers a in P. For any ranked poset P of length r, define a linear operator H on $\tilde{P}$ by

$$Ha = (2i-r)a, \qquad a \in P_i.$$

We now say that a ranked poset P *carries a representation of* sl(2) if there exists a lowering operator Y and an order-raising operator X on $\tilde{P}$ such that $XY - YX = H$. This is equivalent to the statement that if x, h, y are the matrices spanning sl(2) defined in Section 2, then the linear transformation $\phi: \mathrm{sl}(2) \to \mathrm{gl}(\tilde{P})$ defined by $\phi(x)=X$, $\phi(h)=H$, $\phi(y)=Y$ is a homomorphism of Lie algebras.

We now state (as a single theorem) the results of [17, Lemma 1.1] and [14, Theorem 1].

THEOREM 8.1. *Let P be a ranked poset of length r. The following three conditions are equivalent*:

(i) *P is Peck.*
(ii) *There exists an order-raising operator X on* $\tilde{P}$ *such that*

$$X^{r-2i}|_{\tilde{P}_i}: \tilde{P}_i \to \tilde{P}_{r-i}$$

is an isomorphism of vector spaces for every $0 \le i < r/2$.
(iii) *P carries a representation of* sl(2).

Moreover, an order-raising operator X satisfies (ii) *if and only if P carries a representation of* sl(2) *whose order-raising operator is X.*

We wish to give a "super-analogue" of Theorem 8.1. Suppose P is ranked of *even* length $2r$. Write $\tilde{P} = \tilde{P}^0 \oplus \tilde{P}^1$, where $\tilde{P}^0 = \tilde{P}_0 \oplus \tilde{P}_2 \oplus \cdots \oplus \tilde{P}_{2r}$ and $\tilde{P}^1 = \tilde{P}_1 \oplus \tilde{P}_3 \oplus \cdots \oplus \tilde{P}_{2r-1}$. Define the following three elements of osp(1,2):

$$h = \left[\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & 0 & -1 \end{array}\right], \qquad x = \left[\begin{array}{c|cc} 0 & 0 & 1 \\ \hline 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right], \qquad y = \left[\begin{array}{c|cc} 0 & 1 & 0 \\ \hline 0 & 0 & 0 \\ -1 & 0 & 0 \end{array}\right]. \tag{14}$$

Then osp(1,2) has a vector space basis consisting of h, x, y, and

$$x^2 = \left[\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right], \qquad y^2 = \left[\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & -1 & 0 \end{array}\right].$$

The superalgebra structure on osp(1,2) is defined by the relations

$$[h,x] = hx - xh = x,$$

$$[h,y] = hy - yh = -y,$$

$$[x,y] = xy + yx = h,$$

$$[x,x] = 2x^2, \qquad [y,y] = 2y^2.$$

DEFINITION 8.2. Let P be a ranked poset of length $2r$. Define a linear operator H on $\tilde{P}$ by

$$Ha = (i-r)a, \qquad a \in P_i. \tag{15}$$

Then P is said to *carry a representation of* osp(1,2) if there exists a lowering operator Y and an order-raising operator X such that $XY+YX=H$. Equivalently, the linear transformation

$$\phi : \mathrm{osp}(1,2) \to \mathrm{gl}(\tilde{P}^0/\tilde{P}^1)$$

defined by $\phi(h)=H$, $\phi(x)=X$, $\phi(y)=Y$, $\phi(x^2)=X^2$, $\phi(y^2)=Y^2$ is a homomorphism of Lie superalgebras.

THEOREM 8.3. *Let P be a ranked poset of length $2r$. Then P carries a representation of* osp(1,2) *if and only if P is Peck.*

Proof: Assume P carries a representation of osp(1,2) with order-raising operator X. While a proof that P is Peck could be given along the lines of [14, Theorem 1], it is easier to appeal to Theorem 8.1. One easily checks that h, x^2, y^2 span a sub-superalgebra of osp(1,2) isomorphic to the Lie algebra sl(2). Hence the posets

$$P^0 = P_0 \oplus P_2 \oplus \cdots \oplus P_{2r}$$

and

$$P^1 = P_1 \oplus P_3 \oplus \cdots \oplus P_{2r-1}$$

each carry representations of sl(2) with order-raising operator X^2. By Theorem 8.1, the linear transformations

$$(X^2)^{r-2i}|_{\tilde{P}_{2i}} : \tilde{P}_{2i} \to \tilde{P}_{2(r-i)}, \qquad 0 \le i \le \frac{r}{2},$$

$$(X^2)^{r-1-2i}|_{\tilde{P}_{2i+1}} : \tilde{P}_{2i+1} \to \tilde{P}_{2(r-i)-1}, \qquad 0 \le i < \frac{r-1}{2},$$

are isomorphisms. But then for any $0 \le i < r$, the linear transformations

$$X^{2r-2i}|_{\tilde{P}_i} : \tilde{P}_i \to \tilde{P}_{2r-i}$$

are isomorphisms. By Theorem 8.1, P is Peck.

The converse is proved entirely analogously to the corresponding result for sl(2) [14, p. 277]; the details are omitted. □

It follows from Theorems 8.1 and 8.3 that if P carries a representation of osp(1,2), then it also carries a representation of sl(2). However, in certain cases there may be a "natural" way to define the osp(1,2) representation while a direct construction of the sl(2) representation appears intractable. The next result gives such an example; it describes a new class of Peck posets (or even of posets with the Sperner property), and may be regarded as the "super-analogue" of the fact [17, Section 4] that certain posets $L(m, n)$ are Peck.

THEOREM 8.4. *Let k and r be positive integers, and define $K(k,2r)$ (respectively, $\overline{K}(k,2r)$) to be the set of all Young diagrams Y contained in a $k \times 2r$ rectangle, such that no two rows of Y have the same odd (respectively, even) length (including, in the case of $\overline{K}$, no two rows of zero length, where Y is regarded as having exactly k rows). Partially order $K(k,2r)$ and $\overline{K}(k,2r)$ by inclusion of Young diagrams. Define the* rank *of a Young diagram Y to be its number of squares, so that $K(k,2r)$ and $\overline{K}(k,2r)$ become ranked posets. Then $K(k,2r)$ and $\overline{K}(k,2r)$ are Peck posets.*

Proof: First consider the case $K(k,2r)$. We index the rows and columns of a matrix A in $\mathrm{gl}(r+1/r)$ by the numbers $0,2,4,\ldots,2r,1,3,\ldots,2r-1$, in the order given. Let $E_{ij} \in \mathrm{gl}(r+1/r)$ denote the matrix with a 1 in position (i, j) and 0's elsewhere. Define a linear transformation $\psi_r : \mathrm{osp}(1,2) \to \mathrm{gl}(r+1/r)$ as follows, where h, x, y are given by (14):

$$\begin{aligned}
\psi_r(h) &= rE_{00} + (r-1)E_{11} + \cdots - rE_{2r,2r},\\
\psi_r(x) &= E_{01} + E_{12} + \cdots + E_{2r-1,r},\\
\psi_r(y) &= rE_{10} + (r-1)E_{32} + (r-2)E_{54} + \cdots + E_{2r-1,2r-2}\\
&\quad - E_{21} - 2E_{43} - 3E_{64} - \cdots - rE_{2r,2r-1},\\
\psi_r(x^2) &= \psi_r(x)^2, \qquad \psi_r(y^2) = \psi_r(y)^2. \qquad (16)
\end{aligned}$$

One easily checks by direct computation that ψ_r is a homomorphism of Lie superalgebras. [In fact, ψ_r is the irreducible representation ϕ_r of osp(1,2), but it is irrelevant here that ψ_r is irreducible.]

The representation ψ_r defines an action of osp(1,2) on a vector space $V_0 \oplus V_1$, where $\dim V_0 = r+1$, $\dim V_1 = r$. Define the kth *supersymmetric power* $\hat{S}^k(V_0 \oplus V_1)$ of the pair (V_0, V_1) to be the kth tensor power $T^k(V_0 \oplus V_1)$ modulo the subspace generated by all relations

$$v \otimes w - (-1)^{ij} w \otimes v, \qquad (17)$$

where $v \in V_i$, $w \in V_j$. [We may identify $\hat{S}^k(V_0 \oplus V_1)$ with the space

$$\coprod_{j=0}^{k} S^j(V_0) \otimes \Lambda^{k-j}(V_1),$$

where S^i and Λ^i denote the ith symmetric and exterior power, respectively.] The action ψ_r of osp(1,2) on $V_0 \oplus V_1$ induces an action of osp(1,2) on $T^k(V_0 \oplus V_1)$ by

$$A \cdot (v_0 \otimes v_1 \otimes \cdots \otimes v_{2r}) = \sum_{j=0}^{2r} v_0 \otimes v_1 \otimes \cdots \otimes Av_j \otimes \cdots \otimes v_{2r}. \tag{18}$$

The subspace defined by (17) is stable under this action, so osp(1,2) acts on $\hat{S}^k(V_0 \oplus V_1)$ by the same rule (18). [We identify $v_0 \otimes \cdots \otimes v_{2r}$ with its image in $\hat{S}^k(V_0 \oplus V_1)$.]

Now let $x_{2r}, x_{2r-2}, \ldots, x_0$ and $x_{2r-1}, x_{2r-3}, \ldots, x_1$ be the ordered bases of V_0 and V_1, respectively, which define the matrices $A \in \mathrm{gl}(r+1/r)$. In other words, $x_{2r}, \ldots, x_0, x_{2r-1}, \ldots, x_1$ are the unit coordinate vectors for $V_0 \oplus V_1$ in the given order. A basis for $\hat{S}^k(V_0 \oplus V_1)$ consists of all tensors $x = x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_k}$ for which $2r \geq i_1 \geq i_2 \geq \cdots \geq i_k \geq 0$ and for which no two odd i_j's are equal. Hence we may identify x with the Young diagram Y in $K(k,2r)$ whose rows are of length $i_1, i_2, \ldots, i_k$. Thus we may identify the vector space $\tilde{K}(k,2r)$ with $\hat{S}^k(V_0 \oplus V_1)$.

It is then immediate from (16) and (18) that $\psi_r(x)$ is an order-raising operator, $\psi_r(y)$ is a lowering operator (in fact, an order-lowering operator), and $\psi_r(h)$ acts as in (15). Hence $K(k,2r)$ carries a representation of osp(1,2), and the proof follows from Theorem 8.3.

The proof for $\overline{K}(k,2r)$ is entirely analogous. We replace $\hat{S}^k(V_0 \oplus V_1)$ by the kth *superexterior power* $\hat{\Lambda}^k(V_0 \oplus V_1)$ defined by taking $T^k(V_0 \oplus V_1)$ modulo the subspace generated by all relations

$$v \otimes w - (-1)^{(i-1)(j-1)} w \otimes v,$$

or equivalently $\hat{\Lambda}^k(V_0 \oplus V_1) = \hat{S}^k(V_1 \oplus V_0)$. The details are omitted. □

Acknowledgments

I am grateful to V. Kač for providing technical assistance with the theory of superalgebras used in this paper.

References

1. G. Almkvist, Representations of SL(2, C) and unimodal polynomials, preprint, Univ. of Lund, 1983.
2. A. B. Balantekin and I. Bars, Representations of supergroups, *J. Math. Phys.* 22:1810–1818 (1981).
3. A. Berele and A. Regev, Hook Young diagrams with applications to combinatorics and to representations of Lie superalgebras, to appear.
4. P. H. Dondi and P. D. Jarvis, Diagram and superfield techniques in the classical superalgebras, *J. Phys. A* 14:547–563 (1981).
5. E. B. Dynkin, Maximal subgroups of the classical groups, *Trudy Moskov. Math. Obshch.* 1:39–166 (1952); *Amer. Math. Soc. Transl.* 6:245–378 (1957).

6. J. E. HUMPHREYS, *Introduction to Lie Algebras and Representation Theory*, Springer, New York, 1972.
7. G. JAMES and A. KERBER, *The Representation Theory of the Symmetric Group*, Addison-Wesley, Reading, Mass., 1981.
8. V. G. KAČ, Lie superalgebras, *Adv. in Math.* 26:8–96 (1977).
9. V. G. KAČ, Representations of classical Lie superalgebras, in *Differential Geometry Methods in Mathematical Physics II* (K. Bleuler, H. R. Petry, and A. Reetz, Eds.), Lecture Notes in Math., No. 676, Springer, New York, 1978, pp. 597–626.
10. D. E. LITTLEWOOD, Some properties of S-functions, *Proc. London Math. Soc.* (*2*) 40:49–70 (1936).
11. D. E. LITTLEWOOD, *The Theory of Group Characters*, 2nd ed., Oxford, U.P., Oxford 1950.
12. I. G. MACDONALD, *Symmetric Functions and Hall Polynomials*, Oxford U.P., Oxford, 1979.
13. N. METROPOLIS, G. NICOLETTI, and G.-C. ROTA, A new class of symmetric functions, Mathematical Analysis and Applications, Part B, *Adv. in Math. Suppl. Stud.* 7B:563–575 (1981).
14. R. A. PROCTOR, Representations of $\mathrm{sl}(2,\mathbb{C})$ on posets and the Sperner property, *SIAM J. Algebraic Discrete Methods* 3:275–280 (1982).
15. R. STANLEY, Theory and application of plane partitions, Parts 1, 2, *Stud. Appl. Math.* 50:167–188, 259–279 (1971).
16. R. STANLEY, Unimodal sequences arising from Lie algebras, in *Combinatorics, Representation Theory and Statistical Methods in Groups* (T. V. Narayana, R. M. Mathsen, and J. G. Williams, Eds.), Lecture Notes in Pure and Applied Mathematics, Vol. 57, Marcel Dekker, New York, 1980, pp. 127–136.
17. R. P. STANLEY, Weyl groups, the hard Lefschetz theorem, and the Sperner property, *SIAM J. Algebraic Discrete Methods* 1:168–184 (1980).
18. R. STANLEY, $\mathrm{GL}(n,\mathbb{C})$ for combinatorialists, in *Surveys in Combinatorics* (E. K. Lloyd, Ed.), London Math Soc. Lecture Note Series #82, Cambridge U.P., Cambridge, 1983, pp. 187–199.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

(Received July 20, 1984)