Advanced Studies in Pure Mathematics 11, 1987
Commutative Algebra and Combinatorics
pp. 187-213

Generalized H-Vectors, Intersection Cohomology of Toric Varieties, and Related Results

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§ 1. Background

Let $\mathscr{P}$ be a simplicial convex d-polytope, i.e., a d-dimensional convex polytope all of whose proper faces are simplices. Let $f_i = f_i(\mathscr{P})$ denote the number of i-dimensional faces of $\mathscr{P}$. The vector $f(\mathscr{P}) = (f_0, f_1, \cdots, f_{d-1})$ is called the *f-vector* of $\mathscr{P}$, and in [B-L] and [S₅] such vectors are completely characterized. A survey of this subject appears in [S₈]. Here we will be interested in extending the ideas of [S₈] to the non-simplicial case. While we come nowhere near a characterization of f-vectors for non-simplicial polytopes $\mathscr{P}$, we do discuss an interesting numerical sequence associated with $\mathscr{P}$. We also discuss some extensions of this work, as well as many conjectures and open problems. For instance, Conjectures 4.2 (b), 4.3 and 5.5 (a) extend the result that the h-vector of a Cohen-Macaulay complex is nonnegative.

Let us first review some material related to simplicial complexes, most of it to be found in [S₈]. Let Δ be an abstract $(d-1)$-dimensional simplicial complex on the n-element vertex set V, with $f_i = f_i(\Delta)$ i-dimensional faces (or faces with $(i+1)$-elements). Here the empty set $\emptyset$ is regarded as a face of dimension -1, so $f_{-1} = 1$. Define a vector $h(\Delta) = (h_0, h_1, \cdots, h_d)$, called the *h-vector* of Δ, by the condition

$$\sum_{i=0}^{d} f_{i-1}(x-1)^{d-i} = \sum_{i=0}^{d} h_i x^{d-i}. \tag{1}$$

In particular,

$$h_0 = 1,\ h_1 = f_0 - d,\ h_d = (-1)^{d-1}\tilde{\chi}(\Delta), \quad \sum h_i = f_{d-1}, \tag{2}$$

where $\tilde{\chi}(\Delta)$ denotes the reduced Euler characteristic of Δ, i.e., $\tilde{\chi}(\Delta) = \sum_{i\geq -1}(-1)^i f_i$. The *Dehn-Sommerville equations* for the simplicial polytope $\mathscr{P}$ assert that $h_i = h_{d-i}$ when Δ is the boundary complex $\Delta(\mathscr{P})$ of $\mathscr{P}$ (so $f_i(\mathscr{P}) = f_i(\Delta)$, and we set $h_i(\mathscr{P}) = h_i(\Delta)$).

Received November 9, 1985.
* Partially supported by NSF Grant #MCS-8104855.

We consider several other classes of simplicial complexes which include $\Delta(\mathscr{P})$. Call Δ a *sphere complex* if the geometric realization $|\Delta|$ of Δ is homeomorphic to a $(d-1)$-sphere $\mathscr{S}^{d-1}$. Fix once and for all a coefficient field K. Call Δ a *Gorenstein* complex* if for all faces $F \in \Delta$ (including $F=\emptyset$) we have

$$\dim \tilde{H}_i(lk\ F)=\begin{cases}1, & i=\dim(lk\ F)\\ 0, & \text{otherwise,}\end{cases}$$

where $\tilde{H}$ denotes reduced simplicial homology over K and $lk\ F=\{G \in \Delta\colon G \cap F=\emptyset \text{ and } G \cup F \in \Delta\}$, the *link* of F. (The reason for this terminology is that the Gorenstein complexes (over K) Γ of [S_8, pp. 220–221] or [S_7, p. 74] are precisely those of the form $\sigma * \Delta$, where σ is a simplex, * denotes simplicial join, and Δ is a Gorenstein* complex.) Finally call Δ an *Eulerian complex* if for all faces $F \in \Delta$ (including $F=\emptyset$) we have

$$(-1)^{\dim(lk\ F)}\tilde{\chi}(lk\ F)=1.$$

We have the hierarchy

$$\begin{aligned}&\text{boundary complex of simplicial } \mathscr{P} \Rightarrow \text{sphere complex}\\ &\Rightarrow \text{Gorenstein* complex} \Rightarrow \text{Eulerian complex,}\end{aligned}$$

and all three implications are strict. Moreover, the Dehn-Sommerville equations $h_i=h_{d-i}$ continue to hold for Eulerian complexes (see [S_9, Ch. 3.14]).

Let us also note that sphere complexes, Gorenstein* complexes, and Eulerian complexes are *topological* concepts, i.e., depend only on the geometric realization $X=|\Delta|$ of Δ. For sphere complexes this fact is obvious from the definition; in the other two cases one has that:

(a) Δ is Gorenstein* $\Leftrightarrow$ for all $p \in X$ we have

$$\dim \tilde{H}_i(X)=\dim H_i(X, X-p)=\begin{cases}1, & i=\dim X\\ 0, & \text{otherwise.}\end{cases}$$

(b) Δ is Eulerian $\Leftrightarrow$ for all $p \in X$, we have

$$\tilde{\chi}(X)=\chi(X, X-p)=(-1)^{\dim X}.$$

Here $\tilde{H}_i(X)$ denotes reduced singular homology (over K), $H_i(X, X-p)$ denotes relative homology (over K), $\tilde{\chi}(X)$ denotes the reduced Euler characteristic, and $\chi(X, X-p)$ the relative Euler characteristic, i.e.,

$$\chi(X, X-p)=\sum_i(-1)^i \dim H_i(X, X-p).$$

We also define a simplicial complex Δ to be *Cohen-Macaulay* (over K) if

$$\dim \tilde{H}_i(lk\ F)=0, \quad i<\dim(lk\ F),$$

for all $F \in \Delta$. This concept is also topological [Mu, Corollary 3.4], [W, Theorem 8.3]; Δ is Cohen-Macaulay if and only if

$$\dim \tilde{H}_i(X)=\dim H_i(X, X-p)=0, \quad i<\dim X,$$

for all $p \in X=|\Delta|$. Clearly Gorenstein* complexes (and hence sphere complexes and boundary complexes of simplicial polytopes) are Cohen-Macaulay, but Eulerian complexes need not be Cohen-Macaulay.

We now refer the reader to any of [B-L] [S_5] [S_7] [S_8] for the definition of *M-vector* (also called "O-sequence"). A necessary and sufficient condition for a vector $(h_0, h_1, \cdots, h_d)$ to be the h-vector of some simplicial d-polytope is that $h_i=h_{d-i}$ and $(h_0, h_1-h_0, \cdots, h_m-h_{m-1})$ is an M-vector, where $m=[d/2]$. It is open whether this condition characterizes the h-vector of sphere complexes or Gorenstein* complexes (though it certainly doesn't for Eulerian complexes). It is known, however, that if Δ is Gorenstein* then $h(\Delta)$ is an M-vector (but not in general if Δ is Eulerian). More generally, it is known [S_3, Theorem 6] [B-F-S] that $h(\Delta)$ is the h-vector of some Cohen-Macaulay Δ if and only if $h(\Delta)$ is an M-vector. In particular, $h(\Delta)\geq 0$ (i.e., each $h_i\geq 0$) when Δ is Cohen-Macaulay.

§ 2. The *h*-vector of an Eulerian poset

We now want to extend the above discussion to the non-simplicial case. If Δ is a simplicial complex (or more generally, a polyhedral complex, CW complex, etc.) then $P(\Delta)$ denotes the poset of faces of P, ordered by inclusion. Call $P(\Delta)$ the *face poset* of Δ. If P is any poset, then $\hat{P}$ denotes P with a maximal element $\hat{1}$ adjoined. A poset P is *graded of rank* d if every maximal chain of P has $d+1$ elements. Suppose P has a unique minimal element $\hat{0}$ and every interval $[\hat{0}, t]$ is graded. If $[\hat{0}, t]$ has rank k then we write $\rho(t)=k$ and call k the *rank* of t. An *Eulerian poset* is a finite graded poset P with $\hat{0}$ and $\hat{1}$ such that for all $x\leq y$ in P we have

$$\mu(x, y)=(-1)^{\rho(y)-\rho(x)},$$

where μ denotes the Möbius function of P. If Δ is a simplicial complex, then $\hat{P}(\Delta)$ is Eulerian if and only if Δ is Eulerian. (The Eulerian posets $\hat{P}(\Delta)$ are characterized as being Eulerian lattices for which every interval $[\hat{0}, t]$ with $t<\hat{1}$ is a boolean algebra.) See [S_6, Sect. 2] [S_9, Ch. 3.14] for further information on Eulerian posets.

Let $\hat{P}$ be an arbitrary Eulerian poset, and let $P=\hat{P}-\{\hat{1}\}$ as above. As will become apparent in Section 4, it is convenient to regard the h-vector we are about to define as a function of P, rather than of $\hat{P}$. If $t \in P$ then write $P_t=[\hat{0}, t)=\{s \in P: \hat{0} \leq s < t\}$. In particular, $P_{\hat{0}}=\emptyset$. Define two polynomials $f(P, x)$ and $g(P, x)$ by the following rules:

(a) $f(\emptyset, x)=g(\emptyset, x)=1$

(b) If $\hat{P}$ has rank $d+1 \geq 1$ and if $f(P, x)=k_0+k_1x+\cdots$, then

$$g(P, x)=\sum_{i=0}^{m}(k_i-k_{i-1})x^i, \tag{3}$$

where $m=[d/2]$ (and we set $k_{-1}=0$).

(c) If $\hat{P}$ has rank $d+1 \geq 1$, then

$$f(P, x)=\sum_{t \in P} g(P_t, x)(x-1)^{d-\rho(t)}. \tag{4}$$

It is clear that conditions (a)–(c) uniquely define by induction the polynomials $f(P, x)$ and $g(P, x)$, and that $f(P, x)$ has degree d (since in (4) the term with $t=\hat{0}$ has degree d and all other terms have smaller degree). If $f(P, x)=k_0+k_1x+\cdots+k_dx^d$, then we set $h_i=k_{d-i}$ and call $h(P)=(h_0, h_1, \cdots, h_d)$ the *h-vector* of P. Note that from the definitions, we have $h_0=1$ and (by induction)

$$h_d=\sum_{t \in P}(-1)^{d-\rho(t)}=1, \tag{5}$$

since $\hat{P}$ is Eulerian. Moreover, if P has a single element $\hat{0}$, then $f(\hat{0}, t)=g(\hat{0}, t)=1$, so from (4) we have

$$h_1=f_0-d, \tag{6}$$

where $\hat{P}$ has f_0 elements of rank 1.

2.1. Proposition. *Let $\hat{B}_{d+1}$ denote the boolean algebra of* rank $d+1 \geq 1$. *Then*

$$f(B_{d+1}, x)=1+x+\cdots+x^d$$
$$g(B_{d+1}, x)=1.$$

Proof. Induction on d. Clear for $d=0$. Assuming its validity for rank $\leq d$, we have by (4),

$$f(B_{d+1}, x)=\sum_{i=0}^{d}\binom{d+1}{i}(x+1)^{d-i}$$

$$=(x-1)^{-1}\left[\sum_{i=0}^{d+1}\binom{d+1}{i}(x-1)^{d+1-i}-1\right]$$
$$=(x-1)^{-1}(x^{d+1}-1)$$
$$=1+x+\cdots+x^{d},$$

and the proof follows. □

2.2. Corollary. *Let $\hat{P}$ be Eulerian of rank $d+1$, and suppose P is* simplicial, *i.e., for all $t \in P$ the interval* $[\hat{0}, t]$ *is a boolean algebra. Let f_i be the number of elements of P* of *rank $i+1$. Then*

$$f(P, x)=\sum_{i=0}^{d} f_{i-1}(x-1)^{d-i}.$$

In particular, if Δ is an Eulerian complex then by (1) *we have $h(\Delta)=h(P(\Delta))$.*

Proof. By Proposition 2.1 we have $g(P_t, x)=1$ for all $t \in P$, and the proof follows from (4). □

For a slight generalization of Corollary 2.2, define the poset P of rank $\geq k$ to be *k-simplicial* if every interval $[\hat{0}, t]$ with $\rho(t)=k$ is a boolean algebra. Then Corollary 2.2 clearly extends to:

2.3. Corollary. *Let P be k-simplicial* (*with $\hat{P}$ Eulerian of rank $d+1$ as usual*) *with f_{i-1} elements of rank i. Then for $0\leq j\leq k$, $h_{d-j}(P)$ is equal to the coefficient of x^{d-j} in $\sum_{i=0}^{d} f_{i-1}(x-1)^{d-i}$.* □

If $\hat{P}$ is Eulerian (or just graded with $\hat{0}$ and $\hat{1}$) of rank $d+1$ and $S=\{a_1, \cdots, a_j\}$ is a subset of $\{1, 2, \cdots, d\}$ with $a_1<\cdots<a_j$, then define $\alpha_S=\alpha_S(\hat{P})$ to be the number of chains $t_1<t_2<\cdots<t_j$ in $\hat{P}$ (or in P) with $\rho(t_i)=a_i$. M. Bayer (private communication) has given a formula expressing the numbers $h_i(P)$ as linear combinations of the numbers α_S for $S\subseteq\{1, 2, \cdots, d\}$. (This linear combination is not unique since the α_S's satisfy certain linear relations [B-B, Theorem 7.4].) In particular, suppose that $\hat{P}-\{\hat{0}\}$ is *dual simplicial*, i.e., every interval $[t, \hat{1}]$ with $t>\hat{0}$ is a boolean algebra. Then the numbers $f_0, f_1, \cdots, f_d$ determine the numbers α_S, viz., with S as above,

$$\alpha_S=f_{a_1-1}(d+1-a_1)!/(a_2-a_1)!(a_3-a_2)!\cdots(d+1-a_j)!.$$

Hence for all Eulerian posets $\hat{P}$ of rank $d+1$ for which $\hat{P}-\{\hat{0}\}$ is dual simplicial, the numbers $h_i(P)$ are linear combinations (independent of P) of the numbers $f_j(P)$, as first observed by G. Kalai. This linear combination is not unique, and a simple explicit formula expressing $h_i(P)$ in terms of the $f_i(P)$'s is not known. An example is given by $d=4$ and $h_0=h_4=1$, $h_1=h_3=f_0-4$, $h_2=6+5f_0-3f_2$.

We now come to the main result on the h-vector of an Eulerian poset, viz., the Dehn-Sommerville equations are still satisfied. A proof of this result appears in [S_9, Theorem 3.14.9], but for the sake of completeness we will also give it here.

2.4. Theorem (generalized Dehn-Sommerville equations). *Let $\hat{P}$ be Eulerian of rank $d+1\geq 1$ and let $h(P)=(h_0, h_1, \cdots, h_d)$. Then $h_i=h_{d-i}$, $0\leq i\leq d$.*

Proof. Induction on d, the case $d=0$ being trivial. Assume true for all $\hat{Q}$ of rank $\leq d$. Now by (4) we have for all $t\in\hat{P}$,

$$g(P_t, x)+(x-1)f(P_t, x)=\sum_{s\leq t} g(P_s, x)(x-1)^{\rho(t)-\rho(s)}. \tag{7}$$

In general, suppose Q is an Eulerian poset of rank $d+1$, and let $u, v\colon Q\to \boldsymbol{Z}[y]$. By Möbius inversion on Q (see [S_9, Theorem 3.7.1]) the identity

$$u(t)=\sum_{s\leq t} v(s)y^{\rho(t)-\rho(s)}, \qquad \text{for all } t\in Q,$$

is equivalent to

$$v(t)=\sum_{s\leq t} u(s)(-y)^{\rho(t)-\rho(s)}, \qquad \text{for all } t\in Q.$$

Thus from (7) we conclude (putting $t=\hat{1}$ and noting that $\hat{P}_{\hat{1}}=\hat{P}$)

$$g(P, x)=\sum_{s\in\hat{P}} (g(P_s, x)+(x-1)f(P_s, x))(1-x)^{d+1-\rho(s)}. \tag{8}$$

By the induction hypothesis,

$$g(P_s, x)+(x-1)f(P_s, x)=x^{\rho(s)}g(P_s, 1/x), \quad s<\hat{1}.$$

Hence from (8),

$$\begin{aligned}(1-x)f(P, x)&=\sum_{s\in\hat{P}} x^{\rho(s)}g(P_s, 1/x)(1-x)^{d+1-\rho(s)}\\ &=x^d\sum_{s\in\hat{P}} g(P_s, 1/x)\left(\frac{1}{x}-1\right)^{d-\rho(s)}(1-x),\end{aligned}$$

so

$$\begin{aligned}f(P, x)&=x^d\sum_{s\in\hat{P}} g(P_s, 1/x)\left(\frac{1}{x}-1\right)^{d-\rho(s)}\\ &=x^d f(P, 1/x)\end{aligned}$$

by (4), and the proof follows. □

In general, when P is not simplicial it seems quite difficult to compute $f(P, x)$ or $g(P, x)$ without using the laborious defining recurrence (3) and (4). We give two examples where these polynomials can be explicitly computed. These results are also given in [S_9, Exercises 3.70(c) and 3.71(f)].

2.5. Proposition. *Let $\hat{P}_d$ denote the Eulerian poset of rank $d+1$ with exactly two elements of rank i, $1 \leq i \leq d$, with the partial order relation defined by the condition that every element of rank $i+1$ is greater than every element of rank i, for $0 \leq i \leq d$. Write $f_d = f(P_d, x)$ and $g_d = g(P_d, x)$, and set $m=[d/2]$. Then*

$$g_d = \sum_{k=0}^{m} (-1)^k \left[\binom{d-1}{k} - \binom{d-1}{k-2}\right] x^k \tag{9}$$

$$f_d = \sum_{k=0}^{m} (-1)^k \left[\binom{d-1}{k} - \binom{d-1}{k-1}\right] (x^k + x^{d-k}). \tag{10}$$

Moreover,

$$g_{2j} = (1-x) g_{2j-1} \tag{11}$$

$$f_{2j+1} = (1-x)^{2j}(1+x). \tag{12}$$

Proof. The formulas (9) and (10) satisfy $f_0 = g_0 = 1$, as well as (3). Hence we need to verify (4), which takes the form

$$f_d = (x-1)^d + 2 \sum_{i=0}^{d-1} g_i (x-1)^{d-1-i}.$$

This is a straightforward verification using the binomial theorem, and (11) and (12) are immediate consequences of (9) and (10). □

2.6. Proposition. *Let $\hat{L}_d$ denote the lattice of faces of a d-dimensional cube, ordered by inclusion, and set $m=[d/2]$. Then*

$$g(L_d, x) = \sum_{k=0}^{m} \frac{1}{d-k+1} \binom{d}{k} \binom{2d-2k}{d} (x-1)^k. \tag{13}$$

Sketch of proof. (I. Gessel). Write $g_d(x)$ for the right-hand side of (13). We claim that

$$x^{d+1} g_d(1/x) = (x-1)^{d+1} + \sum_{k=0}^{d} 2^{d-k} \binom{d}{k} (x-1)^{d-k} g_k(x). \tag{14}$$

When we expand the left-hand side (LHS) of (14) in powers of $x-1$, we obtain

$$\mathrm{LHS}=\sum_{k,j}\frac{1}{k+1}\binom{d}{k}\binom{2k}{d}\binom{k+1}{j}(-1)^{d-k}(x-1)^{d+1-j}.$$

Similarly, the right-hand side (RHS) of (14) is given by

$$\mathrm{RHS}=(x-1)^{d+1}+\sum_{k,\ell}2^{d-k}\binom{d}{k}\frac{1}{\ell+1}\binom{2\ell}{\ell}\binom{\ell}{k-\ell}(x-1)^{d-\ell}.$$

Equating coefficients of $(x-1)^{d-\ell}$ in LHS and RHS, we must prove

$$\sum_{k\geq 0}\frac{1}{k+1}\binom{d}{k}\binom{2k}{d}(-1)^{d-k}=1, \tag{15}$$

and

$$\sum_{k\geq 0}\frac{1}{k+1}\binom{d}{k}\binom{2k}{d}\binom{k+1}{\ell+1}(-1)^{d-k}=\sum_{k\geq 0}2^{d-k}\binom{d}{k}\frac{1}{\ell+1}\binom{2\ell}{\ell}\binom{\ell}{k-\ell}. \tag{16}$$

Equation (16) simplifies to

$$\sum_{k}\binom{d-\ell}{k-\ell}\binom{2k}{d}(-1)^{d-k}=\sum_{k}2^{d-k}\binom{d-\ell}{d-k}\binom{2\ell}{k}. \tag{17}$$

The identities (15) and (17) can be proved by standard techniques (and are undoubtedly consequences of known identities); we omit the details.

Now set

$$\begin{aligned}f_d(x)&=(x-1)^d+\sum_{k=0}^{d-1}2^{d-k}\binom{d}{k}(x-1)^{d-1-k}g_d(x)\\&=(x-1)^d-(x-1)^{-1}g_d(x)+\sum_{k=0}^{d}2^{d-k}\binom{d}{k}(x-1)^{d-1-k}g_k(x).\end{aligned}$$

It follows that

$$(1-x)f_d(x)=g_d(x)-x^{d+1}g_d(x^{-1}).$$

Hence $f_d(x)$ and $g_d(x)$ satisfy the recurrences (3) and (4), and the proof follows. □

Note that from Proposition 2.6 there follows $g(L_d,1)=\frac{1}{d+1}\binom{2d}{d}$ (the d-th Catalan number) and $f(L_d,1)=2d\cdot g(L_{d-1},1)=2\binom{2(d-1)}{d-1}$.

L. Shapiro (see [S_9, Exercise 3.71(g)]) has deduced from Proposition 2.6 that $g(L_d, x)=\sum a_i x^i$, where a_i is the number of plane trees with $d+1$ vertices such that exactly i vertices have ≥ 2 sons.

One further example of Eulerian posets P for which it might be worthwhile to look at $f(P, x)$ or $g(P, x)$ is the following. Let $\hat{Q}_d$ be the face-lattice of the dual $\mathscr{P}_d$ to the first barycentric subdivision of a $(d-1)$-simplex. $\mathscr{P}_d$ is a simple (d$-$1)-polytope with $d!$ vertices and is sometimes called the *permutohedron*. Every face of $\mathscr{P}_d$ is a product of smaller $\mathscr{P}_i$'s. We have computed that (writing $g_d=g(\mathscr{P}_d, x)$),

$$
\begin{aligned}
g_1&=1\\
g_2&=1+3x\\
g_3&=1+20x\\
g_4&=1+115x+40x^2\\
g_5&=1+714x+735x^2.
\end{aligned}
$$

We conclude this section with a problem suggested by Theorem 2.4. Let $\mathscr{C}_k$ be the set of all non-isomorphic Eulerian posets of rank k (or, if desired, face lattices of convex polytopes of dimension $k-1$). Let V_d be the vector space of all functions $\mathscr{C}_0 \cup \mathscr{C}_1 \cup \cdots \cup \mathscr{C}_{d-1} \rightarrow \boldsymbol{R}$ of finite support. If $\hat{P} \in \mathscr{C}_d$ then define $f_P \in V_d$ by

$$f_P(Q)=\#\{t \in P\colon [\hat{0}, t] \cong Q\}.$$

Let W_d be the subspace of V_d spanned by all f_P with $\hat{P} \in \mathscr{C}_d$. It follows from Theorem 2.4 that

$$[V_d\colon W_d] \geq [d/2],$$

where $[V\colon W]$ denotes the codimension of W in V.

Problem. Compute $[V_d\colon W_d]$. Is it possible that $[V_d\colon W_d]=[d/2]$?

If we let V_d' and W_d' denote the analogues of V_d and W_d for Eulerian posets $\hat{P}$ with P *simplicial*, then it follows from [Gr, 9.2.1] [B-B, Theorem 3.6] (or the much stronger [S_5]) that $[V_d'\colon W_d']=[d/2]$.

§ 3. Toric varieties and intersection cohomology

The basic tool in the proof of the characterization of h-vectors of simplicial d-polytopes (discussed at the end of the Section 1) is the theory of toric varieties. In general, let $\mathscr{P}$ be an arbitrary convex d-polytope. If the vertices of $\mathscr{P}$ have rational coordinates, then we call $\mathscr{P}$ a *rational* polytope. If in addition $\mathscr{P}$ lies in $\boldsymbol{R}^d$ and contains the origin in its interior, then let us call $\mathscr{P}$ a *snug* polytope. Clearly an arbitrary d-polytope $\mathscr{P}$ is combinatorially equivalent to a d-polytope $\mathscr{P}'$ contained in $\boldsymbol{R}^d$ with the origin in its interior. It is known, however, that not every polytope is combinatorially equivalent to a rational polytope [Gr, 5.5.6]. On the other hand, every simplicial polytope *is* combinatorially equivalent to a rational one.

Given a snug d-polytope $\mathscr{P}$, one can associate with $\mathscr{P}$ a d-dimensional irreducible complex projective variety $X(\mathscr{P})$, the *toric variety* associated with $\mathscr{P}$ [D] [S_8, p. 218]. (Toric varieties are more general than the $X(\mathscr{P})$; in general they need not be projective or even complete.) When $\mathscr{P}$ is simplicial, the cohomology ring (over $\boldsymbol{C}$, say) $H^*(X(\mathscr{P}))$ has some very nice properties which can be used to characterize h-vectors of simplicial polytopes. In the non-simplicial case $H^*(X(\mathscr{P}))$ is poorly behaved. However, the recent *intersection (co)homology* theory of Goresky-MacPherson [G-M_1] [G-M_2] is better behaved and allows us to give some results on the h-vector $h(\mathscr{P})$ for arbitrary rational $\mathscr{P}$ analogous to (but not nearly as strong as) the case where $\mathscr{P}$ is simplicial.

We state without proof the following basic result on the intersection cohomology of projective toric varieties. Independent proofs have been given by J.N. Bernstein, A.G. Khovanskii, R.D. MacPherson, and perhaps others, but none are yet published.

3.1. Theorem. *Let $\mathscr{P}$ be a snug d-polytope, with (Eulerian) face lattice $\hat{P}$. Then the intersection cohomology $IH^*(X(\mathscr{P}))$ (over $\boldsymbol{C}$) of the toric variety $X(\mathscr{P})$ satisfies*

$$IH^*(X(\mathscr{P}))=IH^0(X(\mathscr{P}))\oplus IH^2(X(\mathscr{P}))\oplus\cdots\oplus IH^{2d}(X(\mathscr{P})),$$

where $\dim IH^{2i}(X(\mathscr{P}))=h_i(P)$, *as defined in Section* 2 (*so* $f(P,x)=\sum h_i(P)x^i$). $\square$

Thus as an immediate corollary we get $h_i\geq 0$ for rational polytopes $\mathscr{P}$. But an even stronger result is true (which for simplicial polytopes is part of the Generalized Lower Bound Conjecture; see [S_8, pp. 217–218]):

3.2. Corollary. *Let $\mathscr{P}$ be a rational d-polytope with face lattice $\hat{P}$, and set $m=[d/2]$.*

Then

$$1=h_0\leq h_1\leq\cdots\leq h_m$$

(*so by Theorem* 2.4 *the h-vector $h(P)$ is* unimodal, *i.e., increases to a maximum and then decreases*).

Proof. In general, the intersection cohomology $IH^*(X)=IH^0(X)\oplus IH^1(X)\oplus\cdots\oplus IH^{2d}(X)$ of an irreducible complex projective d-variety X is a module over the singular cohomology ring $H^*(X)$ and satisfies the *hard Lefschetz theorem* [B-B-D, Theorem 5.4.10]. This means that for some element $\omega\in H^2(X)$ (the class of a hyperplane section) and for $0\leq i\leq d$, the linear transformation $\omega^{d-i}: IH^i(X)\to IH^{2d-i}(X)$ (defined as multiplication by ω^{d-i}) is an isomorphism of vector spaces. In particular, for $0\leq i<d$ the map $\omega: IH^i(X)\to IH^{i+2}(X)$ is injective, so $\dim IH^i(X)\leq \dim IH^{i+2}(X)$, $0\leq i<d$. Applying this result to $X=X(\mathscr{P})$ and invoking Theorem 3.1 yields the desired conclusion. □

For general irreducible complex projective d-varieties X, intersection cohomology satisfies *Poincaré duality*. In particular, this means

$$\dim IH^i(X)=\dim IH^{2d-i}(X).$$

When applied to $X=X(\mathscr{P})$, we obtain Theorem 2.4 (the generalized Dehn-Sommerville equations) for rational polytopes. From the historical point of view, Theorem 3.1 was proved prior to Theorem 2.4. This suggested the problem of defining $f(P, x)$ for more general objects than face lattices $\hat{P}$ of rational polytopes and of finding more elementary (and more general) proofs of Theorem 2.4 and Corollary 3.2. Eulerian posets seem to be the most general natural class of objects which satisfy Theorem 2.4, but Corollary 3.2 remains open for non-rational polytopes. In fact, even the inequality $h_i\geq 0$ is known only for rational polytopes, and only by the use of intersection cohomology. In Section 4 we will discuss several additional open problems related to showing that $h_i\geq 0$ for non-rational polytopes. Let us also mention here that even for rational $\mathscr{P}$ it is not known in general whether $(h_0, h_1-h_0, \cdots, h_m-h_{m-1})$ is an M-vector, as was the case for simplicial $\mathscr{P}$. The basic obstacle seems to be that $IH^*(X(\mathscr{P}))$ is not a ring, but only a module over $H^*(X(\mathscr{P}))$. As pointed out by MacPherson, one really only needs to show that the primitive intersection cohomology $IH^*(X(\mathscr{P}))/(\omega)$ has the structure of a ring and is generated (as a C-algebra) by elements of degree 2 (i.e., by $IH^2(X(\mathscr{P}))/\omega IH^0(X(\mathscr{P}))$).

In general it is difficult to understand what Corollary 3.2 is saying about the combinatorial structure of $\mathscr{P}$. It is easy to deduce from Corollary 3.2 that if $\mathscr{P}$ is a rational 2-simplicial (i.e., every 2-face is a triangle) d-polytope with n vertices, then $f_1(\mathscr{P})\geq dn-\binom{d+1}{2}$. This result was conjectured by G. Kalai for all 2-simplicial polytopes, and he suggested that Corollary 3.2 may indeed be relevant. However, Kalai [Ka, Sect. 10] subsequently proved his conjecture in general, so that the use of Corollary

3.2 to prove a special case becomes much less interesting.

For another possible application of Corollary 3.2, suppose that there are polytopes $\mathscr{P}_1, \mathscr{P}_2, \cdots, \mathscr{P}_{d-1}$ such that every k-face of $\mathscr{P}$ is combinatorially equivalent to $\mathscr{P}_k$. Then the numbers $h_i(\mathscr{P})$ are linear combinations (depending only on i and $\mathscr{P}_1, \mathscr{P}_2, \cdots, \mathscr{P}_{i-1}$) of the $f_j(\mathscr{P})$'s, so Corollary 3.2 establishes linear inequalities among the f_j's (when $\mathscr{P}$ is rational). When $\mathscr{P}$ is simplicial these inequalities become the inequalities of the Generalized Lower Bound Conjecture, but in other cases they are not well understood. Suppose, for instance, $\mathscr{P}$ is cubical (i.e., each $\mathscr{P}_i$ is combinatorially equivalent to an i-cube). Then Corollary 3.2 yields

$$f_0 \geq d+1$$

$$f_1+f_2+\binom{d+1}{2} \geq df_0$$

$$\binom{d}{2}f_0+4f_3+2f_4 \geq (d-1)f_1+(d-3)f_2+\binom{d+1}{3},$$

etc. These inequalities are far from tight (e.g., it is known that $f_0 \geq 2^d$), and it is unclear whether they are of any significance. Moreover, they remain open in general for nonrational cubical polytopes (though it does not seem known at present whether every cubical polytope is rational).

§ 4. Generalizations and open problems

We will now extend the definition of $f(P, x)$ (but not $g(P, x)$) to more general posets. Call a finite poset P *lower Eulerian* if (a) P contains a unique minimal element $\hat{0}$, and (b) the interval $[\hat{0}, t]$ is Eulerian for all $t \in P$. Given $t \in P$, we still let $P_t = \{s \in P : s < t\}$. Since $\hat{P}_t = [\hat{0}, t]$ is Eulerian, the polynomial $g(P_t, x)$ is still defined, and (4) still makes sense. Thus when P is lower Eulerian with longest chain of length d, we define

$$f(P, x) = \sum_{t \in P} g(P_t, x)(x-1)^{d-\rho(t)}, \tag{18}$$

where $\rho(t)$ is the length of a maximal chain in $\hat{P}_t$. Thus $f(P, x)$ is a polynomial of degree d, say

$$f(P, x) = h_d + h_{d-1}x + \cdots + h_0 x^d,$$

and we define as before the *h-vector* $h(P) = (h_0, \cdots, h_d)$.

Let us note a few elementary consequences of the definition (18). First, $h_0 = 1$ and, just as we obtained (5) and (6), we get $h_1 = f_0 - d$ and $(-1)^{d-1}h_d = \tilde{\chi}(P) := \sum_{i \geq -1}(-1)^i f_i$, where P has f_{i-1} elements of rank i. If Δ is a simplicial complex, then the face poset $P(\Delta)$ of Δ is lower

Eulerian. In fact, $P(\Delta)$ is a simplicial (i.e., every interval $[\hat{0}, t]$ is a boolean algebra) meet-semilattice (i.e., every two elements of $P(\Delta)$ have a greatest lower bound). We see from (1) and Proposition 2.1 that $=h(\Delta)h(P(\Delta))$, i.e., the two notions of h-vector coincide. Finally, suppose $\hat{P}$ is Eulerian (i.e., is lower Eulerian with $\hat{1}$). Then from (18),

$$
\begin{aligned}
f(\hat{P}, x) &= g(P_{\hat{1}}, x) + \sum_{t \neq \hat{1}} g(P_t, x)(x-1)^{d-\rho(t)} \\
&= g(P, x) + (x-1) \sum_{t \in P} g(P_t, x)(x-1)^{d-1-\rho(t)} \\
&= g(P, x) + (x-1) f(P, x) \\
&= x^{d+1} g(P, 1/x),
\end{aligned} \tag{19}
$$

by (3) and Theorem 2.4. In particular, if $\hat{P}$ is the face lattice of a rational d-polytope then by Corollary 3.2 $f(\hat{P}, x)$ has nonnegative coefficients.

We now discuss some problems and conjectures concerning the polynomial $f(P, x)$. If P is a poset, then define the *order complex* $\Delta(P)$ to be the simplicial complex whose faces are the chains of P. If P has a $\hat{0}$, then we write $\Delta_0(P)$ for $\Delta(P-\{\hat{0}\})$. If P is the face poset of a finite regular CW-complex Γ, then $\Delta_0(P)$ is isomorphic (as an abstract simplicial complex) to the first barycentric subdivision *sd* Γ of Γ; hence the geometric realizations $|\Gamma|$ and $|\Delta_0(P)|$ are homeomorphic (see [Bj]), denoted $|\Gamma| \approx |\Delta(P)|$. The most general conjecture of any plausibility which we can think of concerning h-vectors of lower Eulerian posets is the following:

4.1. Conjecture. *Let P be a lower Eulerian meet-semilattice. Then there exists a simplicial complex Λ such that $|\Delta_0(P)| \approx |\Lambda|$ and $h(\Lambda) = h(P)$.* □

While this conjecture may be too optimistic, there are many consequences of it which seem more plausible and which may be more tractable. The following is a list of some of these weaker conjectures. In what follows, we say that a poset P with $\hat{0}$ has a certain topological property (such as Cohen-Macaulay) if $\Delta_0(P)$ has that property.

4.2. Conjecture (all consequences of Conjecture 4.1). *Let P be a lower Eulerian meet-semilattice of rank* (= *length of longest chain*) *d. Then*:

(a) *If P is Cohen-Macaulay, then $h(P)$ is an M-vector.*

(b) (*follows from* (a)) *If P is Cohen-Macaulay, then $h(P) \geq 0$* (*i.e., each $h_i \geq 0$*).

(c) *If P is Gorenstein*, then $(h_0, h_1 - h_0, \cdots, h_m - h_{m-1})$ is an M-vector, where $m = [d/2]$.*

(d) (*follows from* (c)) *If P is Gorenstein*, then $h_0 \leq h_1 \leq \cdots \leq h_m$.*

Perhaps some special cases of Conjecture 4.2 are more tractable. For instance, in (a)–(d) one could assume P is the face poset of the boundary of a convex polytope $\mathscr{P}$ (in which case (b) and (d) are true for $\mathscr{P}$ rational, but (a) and (c) are open even in this case, as discussed after the proof of Corollary 3.2).

Let us note that the above conjectures are certainly false if we drop the requirement that P is a meet-semilattice. For instance, the poset P_d of Proposition 2.5 is Gorenstein*, but $f(P_3, x)=x^3-x^2-x+1$.

We now consider the possibility of generalizing definition (18) by replacing $g(P_t, x)$ with other functions. Let P be a lower Eulerian poset with rank function ρ. For each $t\in P$ associate a polynomial $\gamma(t, x)\in \boldsymbol{R}[x]$. Call γ *acceptable* if for all $t\in P$ we have

$$\sum_{s\leq t}\gamma(s, x)(x-1)^{\rho(t)-\rho(s)}=x^{\rho(t)}\gamma(t, 1/x). \tag{20}$$

Clearly if γ is acceptable, then $\deg \gamma(t, x)\leq\rho(t)$. One easily sees that $g(P, x)$ is the unique acceptable function γ satisfying (a) $\gamma(\hat{0}, x)=1$ (a trivial normalization), and (b) $\deg \gamma(t, x)\leq[(\rho(t)-1)/2]$. In this respect the definition of $g(P_t, x)$ is analogous to the definition of Kazhdan-Lusztig polynomials [K-L, (1.1c)]. This analogy is perhaps not too surprising since certain Kazhdan-Lusztig polynomials can be computed from intersection (co)homology.

Given a lower Eulerian poset P of rank d and acceptable $\gamma\colon P\to\boldsymbol{R}[x]$, define

$$\begin{aligned}\gamma^*(P, x)&=\sum_{t\in P}\gamma(t, x)(x-1)^{d-\rho(t)}\\ \gamma(P, x)&=x^d\gamma^*(P, 1/x).\end{aligned} \tag{21}$$

Note that when P has a $\hat{1}$ (i.e., is Eulerian), then by (20) the definition of $\gamma(P, x)$ coincides with $\gamma(\hat{1}, x)$.

4.3. Conjecture. *Let P be a lower Eulerian Cohen-Macaulay meet-semilattice of rank d. Suppose $\gamma\colon P\to\boldsymbol{R}[x]$ is acceptable and $\gamma(t, x)$ has nonnegative coefficients for all $t\in P$. Then $\gamma(P, x)$ has nonnegative coefficients.* □

Conjecture 4.3 is true when P is simplicial and $\gamma(t, x)=1$ for all t, for in this case P is the face poset of a Cohen-Macaulay complex Δ for which $\gamma(P, x)=\sum h_i(\Delta)x^i$. Conjecture 4.2 is also true by Corollary 3.2 and (19) when P is the face lattice of a rational convex d-polytope $\mathscr{P}$ (including $\mathscr{P}$ itself as a face) and $\gamma(t, x)=g(P_t, x)$.

There is one other case for which we can prove Conjecture 4.3 which makes the entire conjecture much more plausible. Let $\mathscr{P}\subset\boldsymbol{R}^n$ be a convex d-polytope with integer vertices. If n is a positive integer then let $i(\mathscr{P}, m)$ denote the number of points $\alpha \in \mathscr{P}$ for which $m\alpha \in \boldsymbol{Z}^n$. Then $i(\mathscr{P}, m)$ is known to be a polynomial in m of degree d, so $i(\mathscr{P}, m)$ is defined for all $m \in \boldsymbol{Z}$. In particular, one can show $i(\mathscr{P}, 0)=1$. The polynomial $i(\mathscr{P}, m)$ was first considered by E. Ehrhart and is called the *Ehrhart polynomial of* $\mathscr{P}$. Its basic properties are discussed in [S_4] and [S_9, Ch. 4.6]. In particular, define

$$\omega(\mathscr{P}, x)=(1-x)^{d+1} \sum_{m \geq 0} i(\mathscr{P}, m)x^m.$$

Then $\omega(\mathscr{P}, x)$ is a polynomial of degree $\leq d$ with nonnegative integral coefficients [S_4, Theorem 2.1]. Let us note that the coefficients of $\omega(\mathscr{P}, x)$ need not form an M-vector. For instance, if $\mathscr{P}$ is the tetrahedron with vertices (0, 0, 0), (1, 1, 0), (1, 0, 1), (0, 1, 1), then $\omega(\mathscr{P}, x)=1+x^2$. Moreover, for any integral d-polytope $\mathscr{P}$, let $\bar{i}(\mathscr{P}, m)$ denote the number of points $\alpha \in \mathscr{P}-\partial\mathscr{P}$ for which $m\alpha \in \boldsymbol{Z}^n$. Then Ehrhart's "law of reciprocity" (e.g. [S_1, Proposition 6.1] [S_9, Theorem 4.6.26]) is equivalent to the formula

$$x^{d+1}\omega(\mathscr{P}, 1/x)=(1-x)^{d+1} \sum_{m \geq 0} \bar{i}(\mathscr{P}, m)x^m. \tag{22}$$

Now let Γ be an integral polyhedral complex, i.e., a (finite) collection of polytopes in $\boldsymbol{R}^n$ with integer vertices which form a polyhedral complex. If P denotes the face poset of Γ, then P is a lower Eulerian meet-semilattice, and it follows from (22) that ω is an acceptable function on P.

4.4. Theorem. *Let Γ be an integral polyhedral complex whose face poset $P=P(\Gamma)$ is Cohen-Macaulay of rank $d+1=1+\dim \Gamma=1+\max\{\dim \mathscr{P} : \mathscr{P} \in \Gamma\}$.*

Define

$$\omega^*(\Gamma, x)=\sum_{\mathscr{P} \in \Gamma} \omega(\mathscr{P}, x)(x-1)^{d-\dim \mathscr{P}}.$$

Then $\omega^*(\Gamma, x)$ has nonnegative coefficients.

Note. Let $|\Gamma|=\bigcup_{\mathscr{P} \in \Gamma} \mathscr{P}$, the underlying space of Γ. In analogy with $i(\mathscr{P}, m)$, define for $m>0$ $i(\Gamma, m)$ to be the number of points $\alpha \in |\Gamma|$ for which $m\alpha \in \boldsymbol{Z}^n$, and set $i(\mathscr{P}, 0)=1$. Let

$$\omega(\Gamma, x)=(1-x)^{d+1} \sum_{m \geq 0} i(\Gamma, m)x^m. \tag{23}$$

It follows easily from (22) that

$$\omega(\Gamma, x)=x^{d+1}\omega^*(\Gamma, 1/x),$$

where d is as in Theorem 4.4.

The proof of Theorem 4.4 is based on some algebraic considerations. Let σ be a rational pointed polyhedral cone in $\boldsymbol{R}^n$, i.e., σ is defined by finitely many homogeneous linear inequalities with rational coefficients, and σ does not contain a line. Define $K[\sigma]$ to be the K-algebra spanned by all monomials $x^a = x_1^{a_1} \cdots x_n^{a_n}$ for which $a = (a_1, \cdots, a_n) \in \boldsymbol{Z}^n \cap \sigma$. Then $K[\sigma]$ is noetherian and by a theorem of Hochster [H, Theorem 1] is Cohen-Macaulay. Since σ doesn't contain a line, it is easy to give $K[\sigma]$ the structure $K[\sigma]_0 \oplus K[\sigma]_1 \oplus \cdots$ of a connected (i.e., $K[\sigma]_0 = K$) graded K-algebra for which each monomial x^a is homogeneous. The following lemma is not difficult to verify.

4.5. Lemma. *Let σ be as above, and suppose* $\dim \sigma = d$. *For each face τ of σ let x_τ denote any fixed monomial x^a for which $a \in \boldsymbol{Z}^n \cap (\mathrm{relint}\ \tau)$ where* relint τ *denotes the relative interior of τ. Define*

$$\theta_i = \sum_\tau x_\tau^{m_\tau}, \qquad 1 \leq i \leq d, \tag{24}$$

where τ ranges over all $(d+1-i)$-dimensional faces of σ, and the m_τ's are positive integers chosen so that for fixed i, all the terms $x_\tau^{m_\tau}$ of (24) *have the same degree. Then $\theta_1, \cdots, \theta_d$ forms a homogeneous system of parameters for $K[\sigma]$.* □

Now let Σ be a (finite, rational, pointed) *fan* [D], i.e., a finite set of rational pointed polyhedral cones in $\boldsymbol{R}^n$, such that (i) if $\sigma \in \Sigma$ and τ is a face of σ, then $\tau \in \Sigma$, and (ii) if $\sigma, \sigma' \in \Sigma$, then $\sigma \cap \sigma'$ is a common face of σ and σ'. Define a K-algebra $K[\Sigma]$ as follows. As a vector space $K[\Sigma]$ has a basis consisting of all monomials x^a such that $a \in \boldsymbol{Z}^n \cap |\Sigma|$, where $|\Sigma| = \bigcup_{\sigma \in \Sigma} \sigma$. Multiplication of two monomials x^a and x^b is defined by

$$x^a x^b = \begin{cases} x^{a+b}, & \text{if } a, b \in \sigma \text{ for some } \sigma \in \Sigma \\ 0, & \text{otherwise.} \end{cases}$$

We now define a partial ordering of Σ by ordering the cones in Σ by *reverse* inclusion. In particular, Σ contains a unique maximal element $\{0\}$.

4.6. Lemma. *Suppose the poset Σ is Cohen-Macaulay. Then $K[\Sigma]$ is a Cohen-Macaulay ring.*

Proof. There are several ways to prove this lemma; here we make use of a recent result of Yuzvinsky. For each $\sigma \in \Sigma$, associate the ring $\Sigma_\sigma = K[\sigma]$, and for each $\sigma \leq \tau$ in Σ define a homomorphism $\rho_{\tau\sigma}\colon \Sigma_\sigma \to \Sigma_\tau$

by

$$\rho_{\tau\sigma}(x^a)=\begin{cases}x^a, & \text{if } a\in\tau\\ 0, & \text{if } a\notin\tau.\end{cases}$$

Then the rings Σ_σ and homomorphisms $\rho_{\tau\sigma}$ form a *sheaf* $\mathscr{A}$ of rings on Σ, as defined in [Y]. One sees immediately that $\mathscr{A}$ is a sharp flasque sheaf of integral domains, as defined in [Y]. Moreover, the section ring $\Gamma(\mathscr{A})$ of $\mathscr{A}$ is isomorphic to $K[\Sigma]$.

It is easy to see that $K[\Sigma]$ can be made into a connected graded K-algebra, $K[\Sigma]=K[\Sigma]_0\oplus K[\Sigma]_1\oplus\cdots$, such that each monomial x^a is homogeneous. For each $\tau\in\Sigma$ let x_τ be a monomial x^a in $K[\Sigma]$ such that $a\in$ relint τ, and define $\theta_1,\cdots,\theta_d$ as in (24), except now τ ranges over all $(d+1-i)$-dimensional cones in Σ. By Lemma 4.5 it follows that $\theta_1,\cdots,\theta_d$ is a *standard* system of elements of the irrelevant maximal ideal $K[\Sigma^+]$ of $K[\Sigma]$ (here Σ^+ is spanned by all monomials $x^a\in K[\Sigma]$ with $a\neq 0$), in the sense of [Y].

We now wish to invoke Theorem 5.1 of [Y]. The hypothesis that Σ is Cohen-Macaulay is equivalent to the statement that Σ is *K-spherical*, as defined in [Y]. It follows from Theorem 5.1 of [Y] that $\theta_1,\cdots,\theta_d$ is a regular sequence for $K[\Sigma]$. (The hypothesis in [Y, Theorem 5.1] that each Σ_σ is local is not significant here; we could either localize at the irrelevant ideal to begin with, or carry through the arguments of [Y] for the graded case.)

It is easily seen that the Krull dimension of $K[\Sigma]$ is equal to $d=\dim\Gamma$. Since a d-dimensional graded algebra is Cohen-Macaulay if it possesses a homogeneous regular sequence of degree d, it follows that $K[\Sigma]$ is Cohen-Macaulay. □

Proof of Theorem 4.4. For each $\mathscr{P}\in\Gamma$, define $\sigma_{\mathscr{P}}$ to be the cone of all vectors $(a_1,\cdots,a_n,b)\in\boldsymbol{R}^{n+1}$ such that either $(a_1,\cdots,a_n,b)=0$, or else $b>0$ and $(a_1/b,\cdots,a_n/b)\in\mathscr{P}$. Then $\Sigma=\{\sigma_{\mathscr{P}}:\mathscr{P}\in\Gamma\}$ is a fan, and $P(\Gamma)$ is isomorphic to the dual Σ^* of Σ, as posets. Hence by hypothesis Σ is Cohen-Macaulay, so by the previous lemma $K[\Sigma]$ is Cohen-Macaulay.

Now give $K[\Sigma]$ the structure of a graded K-algebra,

$$K[\Sigma]=K[\Sigma]_0\oplus K[\Sigma]_1\oplus\cdots,$$

by defining $\deg x_1^{a_1}\cdots x_n^{a_n}y^b=b$. The Hilbert function of $K[\Sigma]$ is clearly given by

$$H(K[\Sigma],m)=\begin{cases}i(\Gamma,m), & m>0\\ 1, & m=0.\end{cases}$$

Let $K[\Sigma]'$ be the subalgebra of $K[\Sigma]$ spanned by all monomials $x^a y$ for which a is a vertex of some $\mathscr{P} \in \Gamma$. It is easily seen that $K[\Sigma]$ is integral over $K[\Sigma]'$. Hence when K is infinite (which we can assume without loss of generality) then $K[\Sigma]'$, and hence $K[\Sigma]$, contains a maximal regular sequence of degree one. By standard properties of Cohen-Macaulay Hilbert functions [S_2, Corollary 3.11] and (23), it follows that $\omega^*(\Gamma, x)$ has nonnegative coefficients. $\square$

It is possible to prove Theorem 4.4 in somewhat more generality (e.g., by taking certain functions more general than $i(\mathscr{P}, m)$ or by replacing Γ with a more general complex), but we don't see how to use these methods to prove Conjecture 4.3 in its full generality.

§ 5. Relative lower Eulerian posets

Let Δ be a pure (i.e., all maximal faces have the same dimension) simplicial complex of dimension $d-1$. A linear ordering $F_1, F_2, \cdots, F_r$ of the maximal faces of Δ is called a *shelling* if the set of faces of F_i which intersect the subcomplex generated by $F_1, \cdots, F_{i-1}$ is a pure simplicial complex of dimension $d-2$, for $1 \leq i \leq r$. If Δ has a shelling, then Δ is called *shellable*. Every shellable complex Δ is Cohen-Macaulay [S_3, Sect. 5], and there is a simple combinatorial interpretation of its h-vector (essentially due to McMullen [Mc, p. 182]) which may be described as follows. Given a shelling $F_1, \cdots, F_r$, define $\delta(j)$ for $1 \leq j \leq r$ to be the cardinality of the smallest face of F_j (which is necessarily unique) which is not contained in the subcomplex generated by $F_1, \cdots, F_{j-1}$. In particular, $\delta(1)=0$. Then

$$f(\Delta, x) := \sum_{i=0}^{d} h_i x^i = \sum_{j=1}^{r} x^{\delta(j)}.$$

Thus we think that we build up the polynomial $f(\Delta, x) = \Sigma\, h_i x^i$ from the shelling one face at a time, the face F_j contributing $x^{\delta(j)}$ to $f(\Delta, x)$. We wish to generalize this construction to Eulerian posets. In this section we develop the necessary background; in particular we will introduce the concept of relative lower Eulerian posets, which even in the case of simplicial complexes leads to new algebraic and combinatorial results. In the next section we apply these considerations to shellability.

An *order ideal* of a poset P is a subset I of P such that if $x \in I$ and $y < x$ in P, then $y \in I$. A *relative poset* is a pair (P, I), where P is a poset and I an order ideal of P. If P is lower Eulerian of rank d then we call (P, I) a *relative lower Eulerian poset* and define the *h-vector* $h(P, I) = (h_0, h_1, \cdots, h_d)$ of (P, I) by

$$f(P, I, x) := \sum_{t \in P-I} g(P_t, x)(x-1)^{d-\rho(t)} \tag{25}$$
$$= h_d + h_{d-1}x + \cdots + h_0 x^d,$$

a straightforward generalization of (18).

Now consider the special case when P is the face poset of a (finite) simplicial complex Δ, so I is the face poset of some subcomplex Γ. We call the pair (Δ, Γ) a *relative* (*simplicial*) *complex*. Given the pair (Δ, Γ), let $K[\Delta]$ denote the *face ring* (or Stanley-Reisner ring) of Δ over the field K, as defined, e.g., in [S_8] [S_7, p. 62]. If Δ has vertices $x_1, \cdots, x_n$, then define the *face ideal* $K[\Delta/\Gamma]$ of the pair (Δ, Γ) to be the ideal of $K[\Delta]$ spanned by all monomials in the x_i's whose support lies in $\Delta - \Gamma$. We call (Δ, Γ) a *Cohen-Macaulay relative complex* (over K) if $K[\Delta/\Gamma]$ is a Cohen-Macaulay $K[\Delta]$-module. This means that we can choose homogeneous algebraically independent polynomials $\theta_1, \cdots, \theta_e$ in $K[\Delta]$ (where $e-1 = \dim(\Delta, \Gamma)$, the dimension of the largest face of $\Delta - \Gamma$) such that $K[\Delta/\Gamma]$ is a finitely-generated free $K[\theta_1, \cdots, \theta_e]$-module. We can choose the module generators $\eta_1, \cdots, \eta_s$ to be homogeneous, and we write

$$K[\Delta/\Gamma] = \coprod_{i=1}^{s} \eta_i K[\theta_1, \cdots, \theta_e]. \tag{26}$$

When K is infinite we can choose each θ_i to have degree one. As before we regard the field K as fixed once and for all, and all our rings, modules, vector spaces, etc., are defined over K. In particular, all coefficient groups in the computation of homology groups are taken to be K.

We define the f-vector $f(\Delta, \Gamma)$ and h-vector $h(\Delta, \Gamma)$ of (Δ, Γ) exactly as for ordinary simplicial complexes. Namely, $f_i = f_i(\Delta, \Gamma)$ is the number of i-dimensional faces F of Δ which do not lie in Γ (i.e., $F \in \Delta - \Gamma$). In particular, $f_{-1} = 0$ unless $\Gamma = \emptyset$, in which case $f_{-1} = 1$. Similarly define $h_i = h_i(\Delta, \Gamma)$ by

$$\sum_{i=0}^{e} f_{i-1}(x-1)^{e-i} = \sum_{i=0}^{e} h_i x^{e-i},$$

where $e = \dim(\Delta, \Gamma)$. The h-vector $h(\Delta, \Gamma)$ of (Δ, Γ) then coincides with the h-vector $h(P(\Delta), P(\Gamma))$ of the relative lower Eulerian poset $(P(\Delta), P(\Gamma))$ as defined by (25), where $P(\cdot)$ denotes face poset.

5.1. Proposition. *If (Δ, Γ) is Cohen-Macaulay then $h_i(\Delta, \Gamma) \geq 0$ for all i.*

Proof. Let $K[\Delta/\Gamma]_m$ denote the m-th degree subspace of $K[\Delta/\Gamma]$. Set $H(K[\Delta/\Gamma], m) = \dim_K K[\Delta/\Gamma]_m$, the Hilbert function of $K[\Delta/\Gamma]$, and define the Hilbert series

$$F(K[\Delta/\Gamma], \lambda)=\sum_{m\geq 0} H(K[\Delta/\Gamma], m)\lambda^m.$$

Just as for simplicial complexes [S_3, Corollary to Theorem 3] [S_7, p. 67] we have

$$F(K[\Delta/\Gamma], \lambda)=(1-\lambda)^{-e}(\sum_i h_i\lambda^i).$$

Choosing each θ_i in (26) to have degree one (tensoring with an extension field of K if necessary) and setting $d_i=\deg \eta_i$, we also have

$$F(K[\Delta/\Gamma], \lambda)=(1-\lambda)^{-e}\sum_{i=1}^{s}\lambda^{d_i},$$

and the proof follows. □

It is natural to ask what other conditions the h-vector of a relative Cohen-Macaulay complex must satisfy. Given a vector $v=(\alpha_1, \alpha_2, \cdots)$, define

$$Ev=(0, \alpha_1, \alpha_2, \cdots),$$

and set $E^{i+1}v=E(E^iv)$.

5.2. Proposition. *Suppose the minimal faces $F_1, \cdots, F_\nu$ of $\Delta-\Gamma$ have cardinalities $a_1, a_2, \cdots, a_\nu$. If (Δ, Γ) is Cohen-Macaulay with h-vector $h=(h_0, h_1, \cdots)$, then there exist M-vectors $v_1, \cdots, v_\nu$ such that*

$$h=E^{a_1}v_1+\cdots+E^{a_\nu}v_\nu. \tag{27}$$

Proof. Let

$$M_1=K[\Delta/\Gamma]/(\theta_1K[\Delta/\Gamma]+\cdots+\theta_eK[\Delta/\Gamma]).$$

Then M_1 is a $K[\Delta]$-module with Hilbert series $h_0+h_1\lambda+\cdots+h_e\lambda^e$, and minimal generators $y_1, \cdots, y_\nu$ of degrees $a_1, \cdots, a_\nu$. The submodule $y_1K[\Delta]$ has a Hilbert series of the form $\lambda^{a_1}(h_{01}+h_{11}\lambda+\cdots)$, where $v_1=(h_{01}, h_{11}, \cdots)$ is an M-vector. Now let $M_2=M_1/y_1K[\Delta]$. Then $y_2K[\Delta]$ (where we identify $y_2\in M_1$ with its image in M_2) is a submodule of M_2 with Hilbert series $\lambda^{a_2}(h_{02}+h_{12}\lambda+\cdots)$, where $v_2=(h_{02}, h_{12}, \cdots)$ is an M-vector. Continuing in this way, we obtain the desired vectors $v_1, \cdots, v_\nu$. □

A. Björner has asked whether the condition (27) actually characterizes h-vectors of relative Cohen-Macaulay complexes (with $a_1, \cdots, a_\nu$ specified in advance). We can also ask if there is a simple numerical criterion (or at least an efficient algorithm), similar to the numerical description of M-vectors (e.g., [S_8, p. 217]) for checking whether a vector h has the form (27) (again with $a_1, \cdots, a_\nu$ specified in advance).

We now come to the problem of deciding when a relative complex (Δ, Γ) is Cohen-Macaulay. The basic result is proved in exactly the same way as the non-relative proof which appears in [S_7, Theorem 4.1], so it will be omitted. First a word on notation. Let $F \in \Delta$. We let $lk_\Gamma F$ denote the link in Γ of the face F of Δ, i.e.,

$$lk_\Gamma F = \{G \in \Gamma : G \cap F = \emptyset,\ G \cup F \in \Gamma\}.$$

In particular, $lk_\Gamma F = \emptyset$ if $F \notin \Gamma$. Do not confuse this with the case where F is a maximal face of Γ; here $lk_\Gamma F = \{\emptyset\}$, the simplicial complex whose only face is $\emptyset$ (not the same as the empty simplicial complex!). We denote by $\tilde{H}_*(\Delta, \Gamma)$ reduced relative simplicial homology of the pair (Δ, Γ) (with coefficient field K). In particular,

$$\tilde{H}_i(\Delta, \emptyset) \cong \tilde{H}_i(\Delta) \tag{28}$$

$$\tilde{H}_i(\Delta, \{\emptyset\}) \cong H_i(\Delta). \tag{29}$$

5.3. Theorem. *The pair (Δ, Γ) is Cohen-Macaulay if and only if for all $F \in \Delta$ (including $F = \emptyset$), we have*

$$\tilde{H}_i(lk_\Delta F, lk_\Gamma F) = 0, \qquad \text{if } i < \dim (lk_\Delta F). \qquad \square$$

Let us note a few elementary consequences of Theorem 5.3. First, by definition of relative homology the question of whether (Δ, Γ) is Cohen-Macaulay depends only the difference $\Delta - \Gamma$. (This is also clear from our original definition that $K[\Delta/\Gamma]$ is a Cohen-Macaulay module.) Hence without loss of generality we may assume that every maximal face of Δ is not a face of Γ. Moreover:

5.4. Corollary. (i) *Let (Δ, Γ) be Cohen-Macaulay, and suppose F is a maximal face of Γ. Then* $\dim (lk_\Delta F) = -1$ *or* 0 *(i.e., F is a maximal face of Δ or a codimension one face of a maximal face of Δ).*

(ii) *If Δ triangulates a $(d-1)$-ball and Γ triangulates a $(d-2)$-ball contained in $\partial\Delta$ (the boundary of Δ), then (Δ, Γ) is Cohen-Macaulay.*

Proof. (i) We have $lk_\Gamma F = \{\emptyset\}$, so by (29) and Theorem 5.3, $H_i(lk_\Delta F) = 0$ for $i < \dim (lk_\Delta F)$. But either $H_0(lk_\Delta F) = 0$ or $lk_\Delta F = \{\emptyset\}$. In either case we must have $\dim (lk_\Delta F) \leq 0$.

(ii) This is a straightforward application of the long exact sequence for relative homology, together with some standard computations concerning the homology of the simplicial complexes $lk_\Delta F$ and $lk_\Gamma F$. $\square$

Most of the conjectures of the previous section have obvious extensions to the relative case. Let us just state two of these generalizations here. We define a relative poset (P, I) (where P has a $\hat{0}$) to be *Cohen-Macaulay* if $(\Delta_0(P), \Delta_0(I))$ is a relative Cohen-Macaulay complex, where $\Delta_0(Q)$ denotes the order complex of $Q-\{\hat{0}\}$ as defined in Section 4. Moreover, if $\gamma: P \to \boldsymbol{R}[x]$ is acceptable and P has rank d, then as in (21) define

$$\begin{aligned} \gamma^*(P, I, x) &= \sum_{t \in P-I} \gamma(t, x)(x-1)^{d-\rho(t)} \\ \gamma(P, I, x) &= x^{d+1}\gamma^*(P, I, 1/x). \end{aligned} \tag{30}$$

5.5. Conjecture. *Let P be a lower Eulerian meet-semilattice of rank d, and let I be an order ideal of P. Suppose (P, I) is a relative Cohen-Macaulay poset.*

(a) (*extends Conjecture* 4.2 (b)) *Each $h_i(P, I) \geq 0$.*

(b) (*extends Conjecture* 4.3) *Suppose $\gamma: P \to \boldsymbol{R}[x]$ is acceptable and $\gamma(t, x)$ has nonnegative coefficients for all $t \in P-I$. Then the polynomial $\gamma(P, I, x)$ has nonnegative coefficients.* □

Next we come to a relative version of Theorem 2.4 (the generalized Dehn-Sommerville equations). Let $\hat{P}$ be an Eulerian poset of rank d, and I any *subset* of $P=\hat{P}-\{\hat{1}\}$. Let $\mu_{\hat{P}-I}$ denote the Möbius function of the poset $\hat{P}-I$, and when $t \in \hat{P}-I$ write $\mu_{\hat{P}-I}(t)$ for $\mu_{\hat{P}-I}(t, \hat{1})$. For any acceptable $\gamma: P \to \boldsymbol{R}[x]$, define

$$\phi_I(\hat{P}, x) = \sum_{t \in \hat{P}-I} \gamma(t, x)\mu_{\hat{P}-I}(t)(1-x)^{d-\rho(t)}. \tag{31}$$

5.6. Proposition. *We have*

$$x^d\phi_I(\hat{P}, 1/x) = \phi_{P-I}(\hat{P}, x).$$

Proof. By the definitions (31) of $\phi_I(\hat{P}, x)$ and (20) of acceptability, we have

$$\begin{aligned} x^d\phi_I(\hat{P}, 1/x) &= x^d \sum_{t \in \hat{P}-I} \sum_{s \leq t \text{ in } \hat{P}} x^{-\rho(t)}\gamma(s, x)(x-1)^{\rho(t)-\rho(s)}\mu_{\hat{P}-I}(t)\left(1-\frac{1}{x}\right)^{d-\rho(t)} \\ &= \sum_s (-1)^{d-\rho(s)}(1-x)^{d-\rho(s)}\gamma(s, x) \sum_{\substack{t \geq s \\ t \in \hat{P}-I}} \mu_{\hat{P}-I}(t). \end{aligned} \tag{32}$$

Denote the inner sum in (32) by $\nu(s)$. If $s \in P-I$ then by the defining recurrence for μ (e.g., [S_9, Ch. 2.7, equation (14)]) we have $\nu(s)=0$. If $s \in \hat{I}=I \cup \{\hat{1}\}$, then it follows from [$S_6$, Proposition 2.2] [$S_9$, Proposition 3.14.5] that $\nu(s)=(-1)^{d-\rho(s)}\mu_{\hat{I}}(t)$. Substituting into (32) yields $\phi_{P-I}(\hat{P}, x)$, as desired. □

Note. When we put $\phi = g$ and $I = \emptyset$, then Proposition 5.6 is equivalent to Theorem 2.4. However, our proof of Proposition 5.6 in this case requires the fact that g is acceptable, and this fact is also equivalent to Theorem 2.4.

We conclude this section with a relative version of Lemma 4.6 and Theorem 4.4. The proofs are essentially the same and will be omitted.

5.7. Lemma. *Let Σ be a finite, rational, pointed fan in $\boldsymbol{R}^n$. Let Ψ be a subfan, and let $K[\Sigma/\Psi]$ be the ideal of $K[\Sigma]$ spanned by all monomials x^a satisfying $a \in (|\Sigma| - |\Psi|) \cap \boldsymbol{Z}^n$. If (Σ^*, Ψ^*) (where * denotes dual) is a Cohen-Macaulay relative poset, then $K[\Sigma/\Psi]$ is a Cohen-Macaulay $K[\Sigma]$-module.* □

5.8. Theorem. *Let Γ be an integral polyhedral complex, and let Λ be a subcomplex. Define*

$$\omega^*(\Gamma, \Lambda, x) = \sum_{\mathscr{P} \in \Gamma - \Lambda} \omega(\mathscr{P}, x)(x-1)^{d - \dim \mathscr{P}},$$

where $d = \dim(\Gamma, \Lambda) = \max\{\dim \mathscr{P} : \mathscr{P} \in \Gamma - \Lambda\}$. If $(P(\Gamma), P(\Lambda))$ is a Cohen-Macaulay relative poset, then $\omega^*(\Gamma, \Lambda, x)$ has nonnegative coefficients. □

§6. Relative posets and shellability

Let (P, I) be any relative lower Eulerian poset, with maximal elements $t_1, t_2, \cdots, t_r$. Without loss of generality we suppose that no $t_i \in I$. Let P_i denote the order ideal of P generated by t_i, i.e.,

$$P_i = \{t \in P : t \leq t_i\}.$$

Set $I_i = P_i \cap (P_1 \cup \cdots \cup P_{i-1} \cup I)$, $1 \leq i \leq r$. Then I_i is an order ideal of P_i, and $P - I$ is a disjoint union of the subsets $P_i - I_i$, $1 \leq i \leq r$. Hence, if P is graded of rank d then for any acceptable $\gamma : P \to \boldsymbol{R}[x]$ we have by (30),

$$\gamma(P, I, x) = \sum_{i=1}^{r} \gamma(P_i, I_i, x). \tag{33}$$

Thus we can deduce information about $\gamma(P, I, x)$ if we know enough about each $\gamma(P_i, I_i, x)$. For instance, if it is known that each $\gamma(P_i, I_i, x)$ has nonnegative coefficients, then the same is true for $\gamma(P, I, x)$.

Let us consider a special case of particular interest. Suppose $\hat{P}$ is the face lattice of a convex d-polytope $\mathscr{P}$. Following [B-M], we say that a linear ordering $F_1, \cdots, F_r$ of the facets ($(d-1)$-faces) of $\mathscr{P}$ is a *shelling* if $(F_1 \cup \cdots \cup F_i) \cap F_{i+1}$ is a $(d-2)$-ball for $1 \leq i \leq r-2$, while $(F_1 \cup \cdots \cup F_{r-1}) \cap F_r$ is a $(d-2)$-sphere. By [B-M], a shelling of $\mathscr{P}$ always exists.

Define P_i to be the order ideal of P generated by F_i, where $F_1, \cdots, F_r$ is a shelling, and let $I_i = P_i \cap (P_1 \cup \cdots \cup P_{i-1})$. Then $\Delta_0(P_i)$ triangulates a $(d-1)$-ball, while $\Delta_0(I_i)$ triangulates a $(d-2)$-ball contained in $\partial \Delta_0(P_i)$. Hence by Corollary 5.4 (ii), (P_i, I_i) is a Cohen-Macaulay relative poset. If we take $\gamma(x, t) = g(P_t, x)$, then equation (33) builds up the polynomial $\gamma(P, x) = f(P, x) = \Sigma h_{d-i} x^i = \Sigma h_i x^i$ one step at a time from the shelling. Moreover, Conjecture 5.5(a) implies that the amount $\gamma(P_i, I_i, x) = f(P_i, I_i, x)$ added at each step has nonnegative coefficients. (When $\mathscr{P}$ is rational, then Corollary 3.2 implies that $f(P_1, I_1, x)$ and $f(P_r, I_r, x)$ have nonnegative coefficients, since $f(P_1, I_1, x) = x^d g(P_1, 1/x)$ and $f(P_r, I_r, x) = g(P_r, x)$. However, even when $\mathscr{P}$ is rational we don't know whether the other $f(P_i, I_i, x)$ have nonnegative coefficients.)

6.1. Example. Let $\mathscr{P}$ be a 3-cube, with shelling $X, Y, Z, -Z, -Y, -X$ (using an obvious notation for the six facets of $\mathscr{P}$). Then the polynomials

$$q_i := f(P_i, I_i, x) = \sum_{t \in P_i - I_i} g(P_t, x)(x-1)^{d-\rho(t)}$$

are calculated to be:

$$\begin{aligned}
q_1 &= (x-1)^3 + 4(x-1)^2 + 4(x-1) + x + 1 = x^3 + x^2 \\
q_2 &= 2(x-1)^2 + 3(x-1) + x + 1 = 2x^2 \\
q_3 &= q_4 = (x-1)^2 + 2(x-1) + x + 1 = x^2 + x \\
q_5 &= (x-1) + x + 1 = 2x \\
q_6 &= x + 1.
\end{aligned}$$

Hence $f(P, x) = q_1 + \cdots + q_6 = x^3 + 5x^2 + 5x + 1$.

Next we wish to discuss the connection between the shellability of a polytope $\mathscr{P}$ and the generalized Dehn-Sommerville equations (Theorem 2.4). For this purpose we require the following lemma (which can be generalized but which is adequate for our purposes as it stands).

6.2. Lemma. *Let P be a regular CW-complex whose underlying space $|P|$ is a $(d-1)$-ball. Identify P with its face poset, so P is lower Eulerian of rank $d-1$. Let I be a subcomplex ($=$order ideal) of P such that $|I|$ is a $(d-2)$-ball contained in $|\partial P|$. Let $\bar{I}$ be the subcomplex of P generated by $\partial P - I$. (By the Alexander-Newman theorem* (e.g., [*R*, p. 48]), *$|\bar{I}|$ is also a $(d-2)$-ball if ∂P is a piecewise-linear sphere; but, as pointed out to me by R. Edwards, $|\bar{I}|$ need not be a $(d-2)$-ball in general.) Clearly $I \cup \bar{I} = \partial P$, $I \cap \bar{I} = \partial I = \partial \bar{I}$. Then for any acceptable $\gamma: P \to \boldsymbol{R}[x]$, we have*

$$x^d \gamma(P, I, 1/x) = \gamma(P, \bar{I}, x).$$

Proof. We have

$$x^d \gamma(P, I, 1/x) = x^d \sum_{t \in P-I} \left(\frac{1}{x}-1\right)^{d-\rho(t)} x^{-\rho(t)} \sum_{s \leq t} \gamma(s, x)(x-1)^{\rho(t)-\rho(s)}$$

$$= \sum_{s \in P} (x-1)^{d-\rho(s)} \gamma(s, x) \sum_{\substack{t \geq s \\ t \in P-I}} (-1)^{d-\rho(t)}. \tag{34}$$

Denote the inner sum in (34) by $\nu(s)$. Since P is lower Eulerian and I is an order ideal we have

$$\nu(s) = (-1)^{d+1-\rho(s)} [\mu_{\hat{P}-I}(s, \hat{1}) - \mu_I(s, \hat{1})].$$

By the assumptions on $|P|$ and $|I|$ we have (see [S_9, Propositions 3.8.9 and 3.14.5])

$$\mu_{\hat{P}-I}(s, \hat{1}) = \begin{cases} (-1)^{d+1-\rho(s)}, & s \notin \bar{I} \\ 0, & s \in \bar{I} \end{cases}$$

$$\mu_I(s, \hat{1}) = \begin{cases} (-1)^{d-\rho(s)}, & s \notin \bar{I} \\ 0, & s \in \bar{I}, \end{cases}$$

and the proof follows. □

Let us return to the situation where $F_1, \cdots, F_r$ is a shelling of the d-polytope $\mathscr{P}$. Suppose that the reverse order $F_r, \cdots, F_1$ is also a shelling of $\mathscr{P}$. (Such is the case for the shelling constructed in [B-M] and is possibly true for *every* shelling of $\mathscr{P}$; see [P, Theorem 5.4.8].) Thus for $2 \leq i \leq r-1$, $(F_1 \cup \cdots \cup F_{i-1}) \cap F_i$ is a $(d-2)$-ball C_i on the boundary of the $(d-1)$-ball $B_i = F_1 \cup \cdots \cup F_{i-1}$, and $\bar{C}_i = (F_r \cup \cdots \cup F_{i+1}) \cap F_i$ is the complementary homology $(d-2)$-ball (possibly always a $(d-2)$ ball?), i.e.,

$$\bar{C}_i = \mathrm{cl}\,(B_i - C_i),$$

where cl denotes closure. As done earlier in this section, let $\hat{P}$ denote the face lattice of $\mathscr{P}$, and define P_i to be the order ideal generated by F_i. Set $I_i = P_i \cap (P_1 \cup \cdots \cup P_{i-1})$ and $\bar{I}_i = P_i \cap (P_{i+1} \cup \cdots \cup P_r)$. Thus by (33),

$$f(P, x) = \sum_{i=1}^{r} f(P_i, I_i, x) = \sum_{i=1}^{r} f(P_i, \bar{I}_i, x). \tag{35}$$

It follows from Lemma 6.2 that $x^d f(P_i, I_i, 1/x) = f(P_i, \bar{I}_i, x)$ for $2 \leq i \leq r-1$, while for $i=1$ and $i=r$ this formula follows, e.g., from Theorem 2.4. Hence from (35) we have $x^d f(P, 1/x) = f(P, x)$. Thus we have given a "shelling" proof of Theorem 2.4 for the case of convex polytopes, generalizing the proof [Mc, p. 182] for simplicial polytopes. Note that this proof of $x^d f(P, 1/x) = f(P, x)$ uses the fact that $g\colon \hat{Q} \to \boldsymbol{R}[x]$ is acceptable when $\hat{Q}$ is the face lattice of a facet F of $\mathscr{P}$. But this fact is equivalent to $x^{d-1} f(Q, 1/x) = f(Q, x)$. Hence we can give a completely "geometric" proof that $x^d f(P, 1/x) = f(P, x)$ by induction on d.

Note. The deduction of $f(P, x) = x^d f(P, 1/x)$ from Lemma 6.2 only used Lemma 6.2 in the case where $\hat{P}$ is Eulerian. This special case can also be deduced from Proposition 5.6. It is possible to state a common generalization of Proposition 5.6 and Lemma 6.2, but this result seems rather contrived and will not be given here.

Added in proof. Conjecture 4.1 has been disproved by F. Brenti. All parts of Conjecture 4.2 remain open.

References

[B-B] M. M. Bayer and L. J. Billera, Counting faces and chains in polytopes and posets, in Combinatorics and Algebra (C. Greene, ed.), Contemp. Math., Vol. 34, Amer. Math. Soc., Providence, R.I., 1984, pp. 207–252.

[B-B-D] A. A. Beilinson, J. Bernstein and P. Deligne, Faisceaux pervers, Asterisque, **100** (1982), 5–171.

[B-L] L. J. Billera and C. W. Lee, A proof of the sufficiency of McMullen's conditions for f-vectors of simplicial convex polytopes, J. Combin. Theory (A), **31** (1981), 237–255.

[B-J] A. Björner, Posets, regular CW complexes, and Bruhat order, European J. Combin., **5** (1984), 7–16.

[B-F-S] A. Björner, P. Frankl and R. Stanley, The number of faces of balanced Cohen-Macaulay complexes and a generalized Macaulay theorem, Combinatorica, to appear.

[B-M] M. Bruggesser and P. Mani, Shellable decompositions of cells and spheres, Math. Scand., **29** (1971), 197–205.

[D] V. I. Danilov, The geometry of toric varieties, Russian Math. Surveys, **33** (1978), 97–154. Translated from Uspekhi Math. Nauk., **33** (1978), 85–134.

[G-M$_1$] M. Goresky and R. MacPherson, Intersection homology theory, Topology, **19** (1980), 135–162.

[G-M$_2$] ——, Intersection homology II, Invent. Math., **72** (1983), 77–130.

[Gr] B. Grünbaum, Convex Polytopes, John Wiley & Sons, New York, NY, 1967.

[H] M. Hochster, Rings of invariants of tori, Cohen-Macaulay rings generated by monomials, and polytopes, Ann. Math., **96** (1972), 318–337.

[Ka] G. Kalai, Rigidity and the lower bound theorem, preprint.

[K-L] D. Kazhdan and G. Lusztig, Representations of Coxeter groups and Hecke algebras, Invent. Math., **53** (1979), 165–184.

[Mc] P. McMullen, The maximum numbers of faces of a convex polytope, Mathematika, **17** (1970), 179–184.

[Mu] J. R. Munkres, Topological results in combinatorics, Michigan Math. J., **31** (1984), 113–128.

[P] J. S. Provan, Decompositions, shellings, and diameters of simplicial complexes and convex polyhedra, Technical Report No. 354, School of Operations Research and Industrial Engineering, Cornell University, August 1977.

[R] T. B. Rushing, Topological Embeddings, Academic Press, New York and London, 1973.

[S_1] R. P. Stanley, Combinatorial reciprocity theorems, Adv. in Math., **14** (1974), 194–253.

[S_2] ——, Hilbert functions of graded algebras, Adv. in Math., **28** (1978), 57–83.

[S_3] ——, Cohen-Macaulay complexes, in Higher Combinatorics (M. Aigner, ed.), Reidel, Dordrecht and Boston, 1977, pp. 51–62.

[S_4] ——, Decompositions of rational convex polytopes, Ann. Discrete Math., **6** (1980), 333–342.

[S_5] ——, The number of faces of a simplicial convex polytope, Adv. in Math., **35** (1980), 236–238.

[S_6] ——, Some aspects of groups acting on finite posets, J. Combn. Theory (A), **32** (1982), 132–161.

[S_7] ——, Combinatorics and Commutative Algebra, Progress in Mathematics, Vol. 41, Birkhäuser, Boston-Basel-Stuttgart, 1983.

[S_8] ——, The number of faces of simplicial polytopes and spheres, in Discrete Geometry and Convexity (J. E. Goodman, et al., eds.), Ann. N.Y. Acad. Sci., Vol. 440, New York Academy of Sciences, New York, NY, 1985, pp. 212–223.

[S_9] ——, Enumerative Combinatorics, Vol. I, Wadsworth and Brooks/Cole, Monterey, CA, 1986.

[W] J. W. Walker, Topology and combinatorics of ordered sets, Ph. D. thesis, M.I.T., 1981.

[Y] S. Yuzvinsky, Cohen-Macaulay rings of sections, to appear.

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