ORDERED STRUCTURES AND PARTITIONS*

by

Richard P. Stanley

Department of Mathematics
University of California
Berkeley, California 94720

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Abstract

A general theory is developed for the enumeration of order-reversing maps of finite ordered sets P into chains. This theory comprehends many apparently disparate topics in combinatorial theory, including (1) ordinary partitions, (2) ordered partitions (compositions), (3) plane and multidimensional partitions, with applications to Young tableaux, (4) the Eulerian numbers and their refinements, (5) the tangent and secant numbers (or Euler numbers) and their refinements, (6) the indices of permutations, (7) trees, (8) stacks, and (9) protruded partitions, with applications to the Fibonacci numbers. The main tool used is that of generating functions. In particular, we study how the structure of P influences the form of the generating functions under consideration. As an application, we derive new combinatorial relationships between a finite ordered set P and its distributive lattice of order ideals.

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generating function, reciprocity, extension to a total order, distributive lattice, order polynomial, (P,ω)-separator, index of a permutation, (P,ω)-Eulerian number, plane partition, tree, generalized hook length, α-symmetric ordered set, stack, protruded partition, Euler numbers, tangent and secant number.

I. $(P,\omega;m)$-PARTITIONS

1. Introduction. Our basic object is to study order-reversing maps $\sigma: P \to \underline{N}_0$ of a finite (partially) ordered set P into the non-negative integers $\underline{N}_0$, and their connection with partitions. A survey of the classical theory of partitions is given by Hardy and Wright [19, Ch. XIX].

The basic concept of a labeled ordered set (P,ω) is introduced in the next section. Here ω is a bijection $\omega: P \to \{1,2,\ldots,p\}$ (where $|P| = p$), and is called a labeling of P. A (P,ω)-partition is an order-reversing map $\sigma: P \to \underline{N}$, with various stipulations determined by ω as to when $\sigma(X)$ can equal $\sigma(Y)$, for $X,Y \in P$. The general concept of (P,ω)-partition appears to be new, though several special cases have been considered before.[1]

As in the ordinary theory of partitions, a basic tool which we will use is that of generating functions. In this chapter (§§1-13), we will investigate the properties of some generating functions associated with (P,ω)-partitions. Highlights include the theory of ω-separators (§7), which allows the generating functions to be expressed in a simple form, and the Reciprocity Theorem (§10), which connects a labeling ω with the complementary labeling $\overline{\omega}$ defined by $\overline{\omega}(X) = p + 1 - \omega(X)$.

In Chapter II (§§14-19) we consider the special case of

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[1]Although the "syzygetic theory" of MacMahon (pronounced mak'may-on) [23, Sect. VIII] implicitly includes (P,ω)-partitions, he is primarily interested in other applications.

natural labelings, i.e., labelings $\omega: P \to \{1,2,\ldots,p\}$ which are order-preserving. In this case the form of the generating functions is closely related to the counting of chains of order ideals of P. Using this relationship, we obtain new combinatorial information about the distributive lattice $J(P)$ of order ideals of P. We also show that the generating functions (when ω is natural) satisfy certain functional equations if P satisfies appropriate "chain conditions" (§18).

Chapter III is devoted to applications and shows how many apparently disparate combinatorial topics are unified by the theory of (P,ω)-partitions. These topics include plane partitions, trees, stacks, protruded partitions, and counting permutations by positions of "descents". This latter topic includes the theory of Eulerian numbers and the tangent and secant numbers.

2. *Basic Definitions*. The fundamental object of study in this paper is called a *$(P,\omega;m)$-partition*, where (P,ω) is a *labeled ordered set* and m is a non-negative integer. Before defining these concepts, we first consider the case of ordinary partitions. Throughout this paper, we will use the notation

$$\begin{aligned} \underline{m} &= \{1,2,\ldots,m\} \\ \underline{m}_0 &= \{0,1,\ldots,m\} \\ \underline{N} &= \{1,2,3,\ldots\} \\ \underline{N}_0 &= \{0,1,2,\ldots\} . \end{aligned}$$

These sets will be considered as partially ordered sets with the usual order relation $\leqslant$. Thus, for instance, $\underline{m}$ is an m-element *chain* (also called a *totally ordered set* or *linearly ordered set*).

We also use the notation

$$S = \{m_1, m_2, \ldots, m_s\}_<$$

to denote that $S = \{m_1, m_2, \ldots, m_s\}$ with $m_1 < m_2 < \ldots < m_s$.

If n is a non-negative integer, then a *partition of n into* $\leqslant p$ *parts* (or into p parts, allowing 0 as a part) can be regarded as a sequence of integers $\lambda_1, \lambda_2, \ldots, \lambda_p$ satisfying

$$\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_p \geqslant 0 \tag{1}$$

$$\lambda_1 + \lambda_2 + \ldots + \lambda_p = n . \tag{2}$$

A classical result, known to Euler, states that if a_n is the number of partitions of n into $\leqslant p$ parts, then

$$\sum_{n=0}^{\infty} a_n x^n = 1/(1-x)(1-x^2)\ldots(1-x^p)$$

(ignoring inessential questions of convergence). Similarly a partition of n into p *distinct* parts (allowing 0) is obtained by replacing (1) with $\lambda_1 > \lambda_2 > \ldots > \lambda_p \geqslant 0$. Such a partition is also called a *strict* partition. The corresponding generating function is

$$\sum_{n=0}^{\infty} b_n x^n = x^{\binom{p}{2}}/(1-x)(1-x^2)\ldots(1-x^p) .$$

More generally, we can replace (1) by

$$\lambda_1 \sim \lambda_2 \sim \ldots \sim \lambda_p \geqslant 0 \tag{3}$$

where each "$\sim$" can be either ">" or "$\geqslant$" .

Conditions (1) and (2) lead to the alternative definition of a partition of n into p parts $\geqslant 0$ as an order-reversing map σ of a p-element chain $\underline{p}$ into the non-negative integers $\underline{N}_0$, such that

$$\sum_{X\in\underline{p}} \sigma(X) = n .$$

Similarly a strict partition into p parts $\geqslant 0$ is a <u>strict</u> order-reversing map $\sigma\colon \underline{p} \to \underline{N}_0$, i.e., a map satisfying $X<Y \Rightarrow \sigma(X)>\sigma(Y)$.

Our object in this paper is to study the generating functions obtained when $\underline{p}$ is replaced by an arbitrary finite ordered set P and (3) is replaced by a suitable analogue. We also will briefly discuss the further extension to the case where P is infinite (§20). Until further notice, <u>P is a finite ordered set of cardinality p</u> .

2.1. <u>Definition</u>. A <u>labeling</u> of P is a bijection $\omega\colon P \to \underline{p}$. A labeling ω is called a <u>natural labeling</u> if it satisfies

$$X \leqslant Y \Rightarrow \omega(X) \leqslant \omega(Y) ,$$

while ω is called a *strict labeling* if

$$X \leqslant Y \Rightarrow \omega(X) \geqslant \omega(Y) .$$

An ordered set together with a labeling ω is called a *labeled ordered set*.

2.2. *Definition*. If ω is a labeling of P, then a *(P,ω)-partition of n* is a map $\sigma\colon P \to \underline{N}_0$ satisfying the conditions

(i) $X \leqslant Y$ in $P \Rightarrow \sigma(X) \geqslant \sigma(Y)$, i.e., σ is order-reversing,

(ii) $X < Y$ in P and $\omega(X) > \omega(Y) \Rightarrow \sigma(X) > \sigma(Y)$,

(iii) $\sum_{X \in P} \sigma(X) = n$.

If ω is a natural labeling, then σ is called for short a *P-partition*. If ω is a strict labeling, then σ is called a *strict P-partition*.

Note that a P-partition is simply an order-reversing map $P \to \underline{N}_0$, while a strict P-partition is a strict order-reversing map $P \to \underline{N}_0$.

If σ is a (P,ω)-partition, then the values $\sigma(X)$, $X \in P$, are called the *parts* of σ. We frequently will consider (P,ω)-partitions with largest part $\leqslant m$, where m is some non-negative integer. We therefore define a *$(P,\omega;m)$-partition* to be a (P,ω)-partition with largest part $\leqslant m$.

The class of all (P,ω)-partitions is denoted by $\mathcal{A}(P,\omega)$, and the class of all $(P,\omega;m)$-partitions by $\mathcal{A}(P,\omega;m)$, so

$$\mathcal{A}(P,\omega;0) \subseteq \mathcal{A}(P,\omega;1) \subseteq \cdots \subseteq \mathcal{A}(P,\omega) .$$

Define two labelings ω and ω' to be <u>equivalent</u> (denoted $\omega \sim \omega'$) if $\mathcal{A}(P,\omega) = \mathcal{A}(P,\omega')$. (So also $\mathcal{A}(P,\omega;m) = \mathcal{A}(P,\omega';m)$ for all m.) This defines an equivalence relation on the $p!$ labelings ω. It is easily seen that $\omega \sim \omega'$ means

$$\omega(X) < \omega(Y) \Leftrightarrow \omega'(X) < \omega'(Y) ,$$

whenever Y <u>covers</u> X, i.e., whenever $X < Y$ and no $Z \in P$ satisfies $X < Z < Y$. Let $\langle\omega\rangle$ denote the equivalence class (relative to $\sim$) containing ω. One equivalence class consists of all the natural labelings. The number of natural labelings is denoted $e(P)$ and will be considered in more detail later. Similarly, the strict labelings form an equivalence class which also contains $e(P)$ elements.

<u>Problems</u>. A number of interesting combinatorial problems are associated with the labeling of ordered sets, as follows:

(a) How many equivalence classes of labelings of a given ordered set P are there? If P is a p-element chain $\underline{p}$, then there are 2^{p-1} classes, corresponding to the 2^{p-1} ways of specifying the inequalities in (3). More generally, there are 2^{p-1} classes whenever the Hasse diagram of P (considered as a graph) is a tree.

On the other hand, if P consists of p disjoint points then there is only one class.[1] (b) Define a partial ordering Q on the classes of labelings of P by the condition $\langle\omega\rangle \leqslant \langle\omega'\rangle$ if $\mathcal{L}(P,\omega) \subseteq \mathcal{Q}(P,\omega')$. Then Q has a 0 and 1 , but it need not be a lattice (e.g., when $P = \underline{2} \times \underline{2}$). What is the structure of Q ? (c) What is the most number of classes of labelings any ordered set of cardinality p can have? (d) Given a labeled ordered set (P,ω) , how many labelings are equivalent to ω ?

Remark. The reader may be wondering why we define a (P,ω)-partition to be order-reversing, rather than order-preserving. It turns out that our definition conforms more closely with certain conventions in lattice theory and the theory of partitions, which we will not enter into here. Of course, one may obtain the order-preserving case simply by dualizing P , so the theories are equivalent.

3. Generating functions for $(P,\omega;m)$-partitions. Let the elements of the labeled ordered set (P,ω) be given by $X_1, X_2, \ldots, X_p$. If $\sigma \in \mathcal{A}(P,\omega)$, we write $\sigma(i)$ for $\sigma(X_i)$ when no confusion will result. More generally, if f is any function whose domain is P , we sometimes write $f(i)$ for $f(X_i)$.

3.1. Definition. Let (P,ω) be a labeled ordered set with elements $X_1,\ldots,X_p$. Define the generating function $F(P,\omega;x_1,\ldots,x_p)$ (denoted $F(P,\omega)$ for short) in the variables $x_1,\ldots,x_p$ by

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[1] It can be shown that the number of equivalence classes is $(-1)^p\chi(-1)$, where χ is the chromatic polynomial of the Hasse diagram of P (considered as a graph).

$$F(P,\omega) = \sum_{\sigma \in \mathcal{A}(P,\omega)} x_1^{\sigma(1)} x_2^{\sigma(2)} \cdots x_p^{\sigma(p)} .$$

Since the generating function $F(P,\omega)$ explicitly lists all (P,ω)-partitions, the ordered set P can be uniquely recovered from it. Namely, $X_i < X_j$ in P if and only if the exponent $\sigma(i)$ of X_i is always greater than or equal to the exponent $\sigma(j)$ of X_j in every term of $F(P,\omega)$. For reasons of simplicity and applicability, it is more fruitful to consider generating functions less discriminating than $F(P,\omega)$.

3.2. <u>Definition</u>. Let (P,ω) be a labeled ordered set with elements $X_1,\ldots,X_p$. Define the generating functions $U_m(P,\omega;x)$ and $U(P,\omega;x)$ (denoted $U_m(P,\omega)$ and $U(P,\omega)$ for short) by

$$U_m(P,\omega) = \sum_{\sigma \in \mathcal{A}(P,\omega;m)} x^{\sigma(1)+\sigma(2)+\cdots+\sigma(p)}$$

$$U(P,\omega) = \sum_{\sigma \in \mathcal{A}(P,\omega)} x^{\sigma(1)+\sigma(2)+\cdots+\sigma(p)} .$$

Hence $U(P,\omega) = \lim_{m\to\infty} U_m(P,\omega) = F(P,\omega;x,x,\ldots,x)$. The coefficient of x^n in $U_m(P,\omega)$ is equal to the number of $(P,\omega;m)$-partitions of n, while the coefficient of x^n in $U(P,\omega)$ is equal to the number of (P,ω)-partitions of n. Note that $U_m(P,\omega)$ is necessarily a polynomial in x.

If ω is a natural labeling, the symbol ω is omitted from the notation. Hence if ω is natural, we write $F(P)$ for $F(P,\omega)$, $U(P)$ for $U(P,\omega)$, etc. Similarly if ω is strict we write

$\bar{F}(P)$ for $F(P,\omega)$, $\bar{U}(P)$ for $U(P,\omega)$, etc.

4. *Distributive lattices*. We will investigate the effect of the structure of P on the generating functions defined above. It turns out that these generating functions reflect fundamental properties of the distributive lattice $J(P)$ of order ideals of P , so first we will review some properties of the lattice $J(P)$. Recall that an *order ideal* of an ordered set P is a subset I of P satisfying

$$X \in I \text{ and } Y \leqslant X \Rightarrow Y \in I .$$

The set of order ideals of P , ordered by inclusion, forms a distributive lattice denoted $J(P)$. A fundamental structure theorem of Birkhoff [4, p. 59, Thm. 3] states that if L is any finite distributive lattice, then there is a unique finite ordered set P such that $L = J(P)$.

If P is finite of cardinality p , then $J(P)$ has length p . More generally, the *rank* (length from 0) $\nu(I)$ of any $I \in J(P)$ is equal to the cardinality of the order ideal I . An alternative formulation of this statement is the following: The number of chains $0 \leqslant I \leqslant 1$ in $J(P)$ such that $\nu(I) = n$ is equal to the number of $(P;2)$-partitions of n . For $\sigma\colon P \to \{0,1\}$ is a $(P;2)$-partition of n if and only if $\sigma^{-1}(1)$ is an order ideal of P of cardinality n .

The above paragraph provides the reader with a first glimpse of a phenomenon which pervades our approach to $(P,\omega;m)$-partitions, viz., the connection between $(P,\omega;m)$-partitions and the counting of chains in $J(P)$. We require one further connection for the present.

4.1. <u>Proposition</u>. The number $e(P)$ of order-preserving bijections (or natural labelings) $P \to \underline{p}$ is equal to the number of maximal chains in $J(P)$.

<u>Proof</u>. Let $\sigma\colon P \to \underline{p}$ be a natural labeling. Define $I_i = \sigma^{-1}(\{1,2,\ldots,i\})$, $i=0,1,\ldots,p$. Then $\phi = I_0 \subset I_1 \subset \ldots \subset I_p = P$ is a maximal chain of order ideals of P. Conversely, given such a maximal chain, the map $\sigma\colon P \to \underline{p}$ defined by $\sigma(i) \in I_i - I_{i-1}$ is a natural labeling. □

If $\sigma(X_{i_j}) = j$, then we sometimes denote an order-preserving bijection $\sigma\colon P \to \underline{p}$ by the permutation $X_{i_1}, X_{i_2}, \ldots, X_{i_p}$ of the elements of P. We also say that the permuation $X_{i_1}, X_{i_2}, \ldots, X_{i_p}$ <u>extends</u> <u>P</u> <u>to</u> <u>a</u> <u>total</u> <u>order</u>.

The number $e(P)$ goes under such names as "the number of extensions of P to a total order," "the number of ways of sorting P topologically," "the number of order-compatible linearizations of P," etc. Many different combinatorial problems are equivalent to finding $e(P)$ for an appropriate P; for some examples see [33] or [36].

5. <u>The form of the generating functions</u>. We will now establish some fundamental properties of the generating functions $F(P,\omega)$ and $U(P,\omega)$. Deeper properties will be considered later.

Let the elements of P be denoted $X_1, X_2, \ldots, X_p$, and let σ be <u>any</u> map $P \to \underline{N}_0$. Then there is a uniquely defined chain of subsets of P ,

$$\phi = P_0 \subset P_1 \subset \ldots \subset P_k = P \qquad (4)$$

such that

(i) If $X, Y \in P_i - P_{i-1}$ for some $i = 1, 2, \ldots, k$, then $\sigma(X) = \sigma(Y)$.

(ii) If $X \in P_i$, $Y \notin P_i$ for some $i = 1, 2, \ldots, k$, then $\sigma(Y) < \sigma(X)$.

If $\sigma \in \mathcal{A}(P, \omega)$ gives rise to the chain (4), then each P_i is an order ideal of P (since σ is order-reversing), and any map $\tau: P \to \underline{N}_0$ giving rise to (4) lies in $\mathcal{A}(P, \omega)$. We will then call (4) an <u>ω-compatible</u> chain of order ideals.

Observe that a chain $\phi = I_0 \subset I_1 \subset \ldots \subset I_k = P$ of order ideals is ω-compatible if and only if the restriction of ω to each $I_{i+1} - I_i$ (considered as a sub-ordered set of P) is order-preserving. Thus, for instance, (a) if ω is natural, then any chain of order ideals is ω-compatible, (b) if ω is strict, only those chains such that each $I_{i+1} - I_i$ is an anti-chain are ω-compatible, (c) the chain $\phi = I_0 \subset I_1 = P$ is ω-compatible if and only if ω is natural, and (d) every <u>maximal</u> chain of order ideals (i.e., each $|I_{i+1} - I_i| = 1$) is ω-compatible for any ω .

It is easily seen (by summing geometric series) that the terms of the generating function $F(P,\omega)$ arising from those $\sigma \in \mathcal{A}(P,\omega)$ which give the ω-compatible chain (4) are given by

$$\rho(P_1)\rho(P_2)\dots\rho(P_{k-1})/(1-\rho(P_1))(1-\rho(P_2))\dots(1-\rho(P_k)) \tag{5}$$

where

$$\rho(P_i) = \prod_{x_j \in P_i} x_j .$$

There follows:

5.1. Proposition. We have

$$F(P,\omega) = \sum \frac{\rho(P_1)\rho(P_2)\dots\rho(P_{k-1})}{(1-\rho(P_1))\dots(1-\rho(P_k))} , \tag{6}$$

where the sum is over all ω-compatible chains of order ideals of P . □

From the preceding proposition there follows:

5.2. Corollary. $F(P,\omega)$ is a rational function of $x_1,\dots,x_p$ whose denominator can be taken as

$$(1-\rho(I_1))(1-\rho(I_2))\dots(1-\rho(I_r)) ,$$

where the I_j's are the non-void order ideals of P . □

5.3. Corollary. We have

$$U(P,\omega;x) = \frac{W(P,\omega;x)}{(1-x)(1-x^2)\dots(1-x^p)} ,$$

where $W(P,\omega;x)$ is a polynomial in x.

Proof of 5.3. If each x_i is set equal to x in (6), then the denominator of (6) becomes $(1-x^{p_1})(1-x^{p_2})\dots(1-x^{p_k})$, where $p_i = |P_i|$. This denominator divides $(1-x)(1-x^2)\dots(1-x^p)$. Hence, when each x_i is set equal to x in $F(P,\omega)$, giving $U(P,\omega)$, the sum in Proposition 5.1 can be put over a common denominator $(1-x)(1-x^2)\dots(1-x^p)$. □

5.4. Corollary. With $W(P,\omega;x)$ defined by Corollary 5.3, we have

$$W(P,\omega;1) = e(P).$$

In particular, if a_n denotes the number of (P,ω)-partitions of n, then

$$a_n = \frac{e(P)n^{p-1}}{p!(p-1)!}\left(1+O\left(\frac{1}{n}\right)\right) \quad \text{as} \quad n \to \infty .$$

Proof. When each $x_i = x$ in (6) and each term of (6) is put over the denominator $(1-x)(1-x^2)\dots(1-x^p)$, the numerator will have a factor of the form $1-x^i$ and therefore vanish at $x = 1$, except for those terms where $k = p$. These terms with $k = p$ will arise if and only if (4) is a maximal chain of order ideals of P, and conversely every maximal chain of order ideals yields a term (5) with $k = p$. Each such term contributes 1 to $W(P,\omega;1)$ so $W(P,\omega;1)$ equals the number of maximal chains of order ideals of P . By Proposition 4.1, $W(P,\omega;1) = e(P)$.

The asymptotic formula for a_n now follows from standard techniques for estimating coefficients in the expansion of rational functions. All the poles of $U(P,\omega;x)$ lie on the unit circle, and the pole with largest multiplicity is at $x = 1$, with multiplicity p . The Laurent expansion of $U(P,\omega,x)$ about $x = 1$ begins

$$U(P,\omega;x) = \frac{e(P)}{(1-x)^p p!} + O((x-1)^{1-p}) ,$$

so $a_n = \frac{(-1)^n e(P)}{p!} \binom{-p}{n}(1+O(1/n)) = \frac{e(P)n^{p-1}}{p!(p-1)!}(1+O(1/n))$. □

6. The theory of ω-separators. Proposition 5.1 may be used to find some additional properties of the generating functions $F(P,\omega)$, $U(P,\omega)$, and even $U_m(P,\omega)$. It is advantageous, however, to introduce a new tool, which we call the *ω-separator* of (P,ω) . The ω-separator will enable us to analyze in considerable detail properties of our generating functions which seem difficult, if not impossible, to determine directly from Proposition 5.1.

Recall the expression for $F(P,\omega)$ in Proposition 5.1 was obtained by summing over chains (4). Each such chain represents a condition of the form

$$\sigma(i_1) \sim \sigma(i_2) \sim \dots \sim \sigma(i_p) ,$$

where "$\sim$" can be either ">" or "=" . (Recall $\sigma(i)$ stands for $\sigma(X_i)$.) We now consider the possibility of finding $F(P,\omega)$ by summing over conditions of the form

$$\sigma(i_1) \sim \sigma(i_2) \sim \ldots \sim \sigma(i_p) \text{ , "}\sim\text{" either "}>\text{" or "}\geq\text{" .} \quad (7)$$

It is no longer apparent that the set $\mathcal{A}(P,\omega)$ of all (P,ω)-partitions can be partitioned into disjoint classes each satisfying a different condition of the form (7). We shall show, however, that such a construction is always possible, and in fact can be explicitly described. The key result is the next rather technical lemma.

6.1. Lemma. For each $\sigma \in \mathcal{A}(P,\omega)$, there is exactly one extension $X_{i_1}, X_{i_2}, \ldots, X_{i_p}$ of P to a total order satisfying

(i) $\sigma(i_j) < \sigma(i_k) \Rightarrow j > k$, and

(ii) $\omega(i_j) > \omega(i_{j+1}) \Rightarrow \sigma(i_j) > \sigma(i_{j+1})$.

Proof. Suppose $\sigma \in \mathcal{A}(P,\omega)$. Define $i_1, i_2, \ldots, i_p$ as follows: For some $j_1, j_2, \ldots, j_r$ we have

$$\begin{aligned}(a) \quad \sigma(i_1) = \sigma(i_2) = \ldots = \sigma(i_{j_1}) \\ > \sigma(i_{j_1}+1) = \ldots = \sigma(i_{j_2}) > \ldots > \sigma(i_{j_{r-1}+1}) \\ = \ldots = \sigma(i_{j_r}) ,\end{aligned}$$

where $j_r = p$, and

$$\text{(b)} \qquad \begin{array}{c} \omega(i_1) < \omega(i_2) < \cdots < \omega(i_{j_1}) \, , \\ \omega(i_{j_1+1}) < \omega(i_{j_1+2}) < \cdots < \omega(i_{j_2}) \, , \\ \vdots \\ \omega(i_{j_{r-1}+1}) < \omega(i_{j_{r-1}+2}) < \cdots < \omega(i_{j_r}) \, . \end{array}$$

We first show $X_{i_1}, \ldots, X_{i_p}$ extends P to a total order. If i_j and i_k, with $j < k$, are in the same row of (b), then since $\omega(i_k) > \omega(i_j)$ and $\sigma(i_j) = \sigma(i_k)$, we cannot have $X_{i_k} < X_{i_j}$ by definition of a (P,ω)-partition. If i_j and i_k, with $j < k$, belong to distinct rows of (b), then by (a), $\sigma(i_j) > \sigma(i_k)$. Since σ is order-reversing, again we cannot have $X_{i_k} < X_{i_j}$. Thus $X_{i_1}, \ldots, X_{i_p}$ extends P to a total order.

We now show that the extension $X_{i_1}, \ldots, X_{i_p}$ satisfies (i) and (ii). First (i) follows immediately from (a). To prove (ii), assume $\omega(i_j) > \omega(i_{j+1})$. Then $\omega(i_j)$ must be the last entry in some row of (b) and $\omega(i_{j+1})$ the first entry of the next row. Thus by (a), $\sigma(i_j) > \sigma(i_{j+1})$, as desired.

It remains to prove the uniqueness of the extension $X_{i_1}, \ldots, X_{i_p}$, given σ. Let $X_{k_1}, \ldots, X_{k_p}$ be another extension satisfying (i) and (ii). By (i), for each $t = 1,2,\ldots,r$, we have that $k_{t-1}+1$, $k_{t-1}+2, \ldots, k_t$ $(k_0=1)$ is a permutation of $j_{t-1}+1$, $j_{t-1}+2, \ldots, j_t$. Thus σ takes the same value on these elements, so by (ii) the subscripts $k_{t-1}+1$, $k_{t-1}+2,\ldots,k_t$ must be arranged so that $\omega(k_{t-1}+1) < \omega(k_{t-1}+2) < \ldots < \omega(k_t)$. Hence $k_i = j_i$ for all i . □

Remark. A result equivalent to Lemma 6.1 was proved by MacMahon [23, §§189-197] in the case when P is a disjoint union of points, and extended by him to finite order ideals of $\underline{N} \times \underline{N}$, naturally labeled (the case of interest in the theory of *plane partitions*; see §21). Knuth [22] extended MacMahon's result to arbitrary finite ordered sets P , naturally labeled, and used a special case to construct an algorithm for enumerating *solid partitions*.

We now define the *ω-separator* $\mathcal{L}(P,\omega)$ (denoted $\mathcal{L}$ for short) of a labeled ordered set (P,ω) . $\mathcal{L}$ is a set of permutations of $1,2,\ldots,p$, namely, those permutations of the form

$$\omega(X_{i_1}) \ , \ \omega(X_{i_2}),\ldots,\omega(X_{i_p}) \ ,$$

where $X_{i_1},X_{i_2},\ldots,X_{i_p}$, extends P to a total order. Hence $\mathcal{L}$ has cardinality $e(P)$.

If $\omega(X_{i_1}),\omega(X_{i_2}),\ldots,\omega(X_{i_p})$ is a permutation π in an ω-separator $\mathcal{L}$, we say that a (P,ω)-partition σ is *compatible* with the permutation π if:

(a) $\sigma(X_{i_1}) \geqslant \sigma(X_{i_2}) \geqslant \ldots \geqslant \sigma(X_{i_p})$, and

(b) $\sigma(X_{i_j}) > \sigma(X_{i_{j+1}})$ if $\omega(X_{i_j}) > \omega(X_{i_{j+1}})$.

The following theorem is the basic result of this paper.

6.2. <u>Theorem</u>. Every (P,ω)-partition σ is compatible with precisely one permutation in the ω-separator $\mathcal{L}(P,\omega)$. Conversely, any map $\sigma\colon P \to \underline{N}_0$ compatible with some permutation in $\mathcal{L}(P,\omega)$ is a (P,ω)-partition.

<u>Proof</u>. The first statement is merely a rephrasing of Lemma 6.1. To prove the second statement, suppose we have a map $\sigma\colon P \to \underline{N}_0$ compatible with some permutation $\pi = (\omega(X_{i_1}),\omega(X_{i_2}),\ldots)$ in $\mathcal{L}(P,\omega)$. Since $X_{i_1}, X_{i_2},\ldots$ extends P to a total order, σ is order-reversing. If $X_i < X_j$ in P, then X_i appears to the left of X_j in π. If $\omega(X_i) > \omega(X_j)$, then somewhere between $\omega(X_i)$ and $\omega(X_j)$ there appears a descent in π. Hence $\sigma(X_i) > \sigma(X_j)$, so σ is a (P,ω)-partition. □

In writing specific examples of ω-separators $\mathcal{L}$, we will insert (for reasons of emphasis) a ">" sign between two consecutive terms $\omega(X_{i_j})$ and $\omega(X_{i_{j+1}})$ of a permutation $\pi \in \mathcal{L}$ if $\omega(X_{i_j}) > \omega(X_{i_{j+1}})$. Thus we write $4 > 2 > 1\ 3$ instead of $4\ 2\ 1\ 3$.

<u>Example</u>. Let (P,ω) be the labeled ordered set of Figure 1.

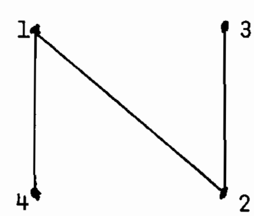

Figure 1.

Then $e(P) = 5$ and the ω-separator $\mathcal{L}(P,\omega)$ is given by

$$\begin{array}{l} 2 \quad 4 > 1 \quad 3 \\ 2 \quad 4 > 3 > 1 \\ 4 > 2 > 1 \quad 3 \\ 4 > 2 \quad 3 > 1 \\ 2 \quad 3 \quad 4 > 1 \quad . \end{array} \tag{8}$$

It is convenient to introduce the following notation dealing with permutations. If $\pi = (i_1,i_2,\ldots,i_p)$ is any permutation of $(1,2,\ldots,p)$, define

$$\mathcal{D}(\pi) = \{j \mid i_j > i_{j+1}\} .$$

Thus for the ω-separator of (8), the sets $\mathcal{D}(\pi)$ are given by $\{2\}$, $\{2,3\}$, $\{1,2\}$, $\{1,3\}$, $\{3\}$. (In general, however, the $e(P)$ sets $\mathcal{D}(\pi)$ need not be distinct.)

Also define the <u>index</u> $\operatorname{ind}(\pi)$ by

$$\operatorname{ind}(\pi) = \sum_{j \in \mathcal{D}(\pi)} j .$$

The index of a permutation was first considered by MacMahon [23, §104], who called it the "greater index."

Finally, define a <u>descent</u> (relative to $\pi = (i_1,i_2,\ldots,i_p)$) to be a pair i_j, i_{j+1} with $i_j > i_{j+1}$. We say that π has d descents to the right of i_k if

$$|\{j \mid k \leqslant j < p\ ,\ i_j > i_{j+1}\}| = d\ .$$

7. Generating functions in terms of ω-separators. Theorem 6.7 allows us to find simpler expressions for the generating functions we have been considering.

7.1. Proposition. Let (P,ω) be a labeled ordered set of cardinality p . Then

$$F(P,\omega) = \sum \frac{x_{i_1}^{c_1} x_{i_2}^{c_2} \cdots x_{i_p}^{c_p}}{(1-x_{i_1}x_{i_2}\cdots x_{i_p})(1-x_{i_1}x_{i_2}\cdots x_{i_{p-1}})\cdots(1-x_{i_1})}\ ,$$

where the sum is over all permutations

$$\pi:\ \omega(X_{i_1}),\omega(X_{i_2}),\ldots,\omega(X_{i_p})$$

in the ω-separator $\mathcal{L}(P,\omega)$, and where c_j is the number of descents occurring to the right of $\omega(X_{i_j})$ in π .

Proof. Let π be a permutation in $\mathcal{L}(P,\omega)$, with $\mathcal{D}(\pi) = \{m_1,\ldots,m_s\}_<$. The generating function for those $\sigma \in \mathcal{A}(P$ compatible with this permutation is given by

$$\sum_{k_p=0}^{\infty} \sum_{k_{p-1}=k_p}^{\infty} \cdots \sum_{k_{m_s+1}=k_{m_s+2}}^{\infty} \sum_{k_{m_s}=k_{m_s+1}+1}^{\infty} \cdots \sum_{k_{m_1+1}=k_{m_1+2}}^{\infty}$$

$$\sum_{k_{m_1}=k_{m_1+1}+1}^{\infty} \cdots \sum_{k_2=k_3}^{\infty} \sum_{k_1=k_2}^{\infty} x_{i_1}^{k_1} x_{i_2}^{k_2} \cdots x_{i_p}^{k_p}\ .$$

This is just a nested sum of geometric series, summing to $(x_{i_1}^{c_1} x_{i_2}^{c_2} \cdots x_{i_p}^{c_p})/(1-x_{i_1})(1-x_{i_1}x_{i_2}) \cdots (1-x_{i_1}x_{i_2} \cdots x_{i_p})$, with c_j as in the statement of the proposition. By Theorem 6.2, the sum of these generating functions over all permutations in $\mathcal{L}(P,\omega)$ includes each $\sigma \in \mathcal{A}(P,\omega)$ exactly once, and thus equals $F(P,\omega)$. □

By setting each $x_i = x$ in Proposition 7.1 and observing that $c_1+c_2+\ldots+c_p = m_1+m_2+\ldots+m_s = \operatorname{ind}(\pi)$, where $\mathcal{S}(\pi) = \{m_1,m_2,\ldots,m_s\}$, we obtain:

7.2. <u>Corollary</u>. Let (P,ω) be a labeled ordered set of cardinality p. Then

$$W(P,\omega) = \sum_{\pi} x^{\operatorname{ind}(\pi)}$$

where the sum is over all permutations π in $\mathcal{L}(P,\omega)$. □

For instance, if (P,ω) is the labeled ordered set of Figure 1, then we immediately read off from (8) that

$$W(P,\omega) = x^2 + 2x^3 + x^4 + x^5 .$$

Note that Corollary 7.2 provides an alternative (and considerably more transparent) proof of the relation $W(P,\omega;1) = e(P)$ (Corollary 5.4). This follows from Corollary 7.2 since $\mathcal{L}(P,\omega)$ contains $e(P)$ permutations π.

As an immediate consequence of Corollary 7.2, we obtain an interesting fact which seems difficult to prove by other means.

7.3. Corollary. The coefficients of $W(P,\omega)$ are non-negative.[2] □

8. The polynomials $W_s(P,\omega)$. In this section we consider the problem of obtaining expressions for the polynomials $U_m(P,\omega)$. First consider the classical case of ordinary partitions. Letting a_{mn} denote the number of partitions of n into p non-negative parts with largest part $\leqslant m$, a well-known result of Euler (cf. [19, Thm. 349]) states that

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{mn} q^m x^n = \sum_{m=0}^{\infty} \begin{bmatrix} m+p \\ p \end{bmatrix} q^m$$

$$= 1/(1-q)(1-qx)(1-qx^2)\dots(1-qx^p) , \qquad (9)$$

where $\begin{bmatrix} m+p \\ p \end{bmatrix}$ denotes the Gaussian coefficient (or generalized binomial coefficient, since $\lim_{x\to 1} \begin{bmatrix} n \\ m \end{bmatrix} = \binom{n}{m})$,

$$\begin{bmatrix} m+p \\ p \end{bmatrix} = \frac{(1-x^{m+p})(1-x^{m+p-1})\dots(1-x^{m+1})}{(1-x^p)(1-x^{p-1})\dots(1-x)} .$$

Unless otherwise stated, Gaussian coefficients are always understood here to be in the variable x. If another variable y is used, we denote this by $\begin{bmatrix} n \\ m \end{bmatrix}_y$.

---

[2] A similar device was used by Gordon [15] to prove the non-negativity of certain coefficients arising in the theory of multipartite partitions.

A variation of (9) is obtained by letting b_{mn} denote the number of strict partitions of n into p non-negative parts $\leqslant m$. Here

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} b_{mn} q^m x^n = x^{\binom{p}{2}} \sum_{m=0}^{\infty} \begin{bmatrix} m+1 \\ p \end{bmatrix} q^m$$

$$= q^{p-1} x^{\binom{p}{2}} / (1-q)(1-qx)\dots(1-qx^p) . \qquad (10)$$

It is formulas such as (9) and (10) which we now generalize to arbitrary (P,ω)-partitions.

Let $\omega(X_{i_1}), \omega(X_{i_2}), \dots, \omega(X_{i_p})$ be a permutation π in the ω-separator $\mathcal{L}(P,\omega)$. If $\mathcal{D}(\pi) = \{m_1, m_2, \dots, m_s\}_<$ and if σ is a (P,ω) partition of n with largest part $\leqslant m$ compatible with π, then define the map $\tau\colon P \to \underline{N}_0$ by

$$\tau(i_j) = \sigma(i_j) - c_j , \qquad (11)$$

where c_j is the number of descents appearing to the right of $\omega(X_{i_j})$ in π. Then τ is a P-partition of $n + \sum c_j = n + \sum m_i$ with largest part $\leqslant m - s$. Conversely, given such a τ, we can get σ by (11). It follows from (9) that the contribution from the permutation π to the generating function $U_m(P,\omega)$ is

$$x^{\mathrm{ind}(\pi)} \begin{bmatrix} m+p-s \\ p \end{bmatrix} \qquad (12)$$

This leads us to define for each $s \in \underline{N}_0$ a polynomial

$$W_s(P,\omega) = W_s(P,\omega;x) = \sum_{\pi} x^{\operatorname{ind}(\pi)}, \tag{13}$$

where the sum is over all permutations π in $\mathcal{L}(P,\omega)$ satisfying $|\mathcal{S}(\pi)| = s$.

Note the following elementary properties of $W_s(P,\omega)$.

8.1. <u>Proposition</u>. $W_s(P,\omega)$ is a polynomial in x with non-negative integer coefficients, satisfying

(i) $W_s(P,\omega) = 0$ if $s \geqslant p$, and

(ii) $\displaystyle\sum_{s=0}^{p-1} W_s(P,\omega) = W(P,\omega)$. $\square$

We have immediately from (12):

8.2. <u>Proposition</u>. For all $m \geqslant 0$,

$$U_m(P,\omega) = \sum_{s=0}^{p-1} \binom{p+m-s}{p} W_s(P,\omega). \quad \square \tag{14}$$

By reversing the order of summation in (14), we also have

$$U_m(P,\omega) = \sum_{s=0}^{m} \binom{p+s}{p} W_{m-s}(P,\omega). \tag{15}$$

Using the identity given by the last equality of (9), together with (15), there follows:

8.3. <u>Proposition</u>. We have

$$\sum_{m=0}^{\infty} U_m(P,\omega)q^m = \frac{\displaystyle\sum_{s=0}^{p-1} W_s(P,\omega)q^s}{(1-q)(1-qx)\dots(1-qx^p)}. \quad \square$$

It is a simple matter to invert the relation (15) and express the W_s's in terms of the U_m's .

8.4. Proposition. We have

$$W_s(P,\omega) = \sum_{m=0}^{s} (-1)^m x^{\binom{m}{2}} \begin{bmatrix} p+1 \\ m \end{bmatrix} U_{s-m}(P,\omega) .$$

Proof. By Proposition 8.3,

$$(1-q)(1-qx)\dots(1-qx^p) \sum_{m=0}^{\infty} U_m(P,\omega)q^m = \sum_{s=0}^{p-1} W_s(P,\omega)q^s \qquad (16)$$

By a well-known identity of Euler [15, Thm. 348],

$$(1-q)(1-qx)\dots(1-qx^p) = \sum_{k=0}^{\infty} (-1)^k x^{\binom{k}{2}} \begin{bmatrix} p+1 \\ k \end{bmatrix} q^k$$

Expanding the left-hand side of (16) as a Cauchy product and equating coefficients of q^s gives the result. □

9. The numbers $\alpha(P,\omega;S)$ and $\beta(P,\omega;S)$. Let (P,ω) be a labeled ordered set of cardinality p , and let π be a permutation in the ω-separator $\mathcal{L}(P,\omega)$. We seek a combinatorial interpretation of the set $\mathcal{S}(\pi)$ which does not involve $\mathcal{L}(P,\omega)$. To this end, let $S \subseteq \underline{p-1}$ and define $\alpha(P,\omega;S)$ to be the number of ω-compatible chains

$$\phi = I_0 \subset I_1 \subset \dots \subset I_{s+1} = P \qquad (17)$$

of order ideals of P satisfying $|I_i| = m_i$ $(i=1,2,\ldots,s)$, where $S = \{m_1,m_2,\ldots,m_s\}_<$.

For instance,

$$\alpha(P,\omega;\phi) = \begin{cases} 1, & \text{if } \omega \text{ is natural} \\ 0, & \text{otherwise} \end{cases} .$$

Moreover, since every maximal chain of order ideals is ω-compatible, we have

$$\alpha(P,\omega;\underline{p-1}) = e(P) .$$

Recall that if ω is natural (in which case we adhere to our previous convention by writing $\alpha(P;S)$ for $\alpha(P,\omega;S)$) , then every chain of order ideals (17) is ω-compatible. Thus for instance $\alpha(P;m)$ (short for $\alpha(P;\{m\})$) is equal to the number of order ideals of P of cardinality m .

Define numbers $\beta(P,\omega;S)$ by

$$\beta(P,\omega;S) = \sum_{T \subseteq S} (-1)^{|S-T|}\alpha(P,\omega;T)$$

Equivalently, by the Principle of Inclusion-Exclusion,

$$\alpha(P,\omega;S) = \sum_{T \subseteq S} \beta(P,\omega;T) .$$

9.1. Theorem. Let (P,ω) be a labeled ordered set of cardinality p , and let $S \subseteq \underline{p-1}$. Then $\beta(P,\omega;S)$ is equal to the number of permutations π in $\mathcal{L}(P,\omega)$ satisfying $\mathcal{D}(\pi) = S$

Proof. We introduce a new generating function $G(P,\omega;x_1\ldots,x_p)$ defined as follows: the coefficient of $x_1^{a_1} x_2^{a_2} \ldots x_p^{a_p}$ in $G(P,\omega)$ is 0 unless $a_1 \geqslant a_2 \geqslant \ldots \geqslant a_p$. When $a_1 \geqslant \ldots \geqslant a_p$, then this coefficient is equal to the number of (P,ω)-partitions with parts $a_1,a_2,\ldots,a_p$. Thus $G(P,\omega)$ can be obtained from $F(P,\omega)$ by relabeling the subscripts of each term of $F(P,\omega)$ so that the exponents appear in descending order. If $\pi = (\omega(X_{i_1}),\omega(X_{i_2}),\ldots,\omega(X_{i_p}))$ is a permutation in $\mathcal{L}(P,\omega)$ with $\mathcal{S}(\pi) = \{m_1,\ldots,m_s\}_<$, then it is easily seen that the contribution to $G(P,\omega)$ from π is

$$\frac{(x_1x_2\ldots x_{m_1})(x_1x_2\ldots x_{m_2})\ldots(x_1x_2\ldots x_{m_s})}{(1-x_1)(1-x_1x_2)\ldots(1-x_1x_2\ldots x_p)} .$$

Hence

$$G(P,\omega) = \frac{\sum_{S\subseteq \underline{p-1}} \gamma(P,\omega;S)(x_1x_2\ldots x_{m_1})\ldots(x_1x_2\ldots x_{m_s})}{(1-x_1)(1-x_1x_2)\ldots(1-x_1x_2\ldots x_p)} , \qquad (18)$$

where $\gamma(P,\omega;S)$ is the number of permutations π in $\mathcal{L}(P,\omega)$ satisfying $\mathcal{S}(\pi) = S$. On the other hand, we also have (e.g., from Proposition 5.1)

$$G(P,\omega) = \sum \frac{(x_1x_2\ldots x_{\nu_1})(x_1x_2\ldots x_{\nu_2})\ldots(x_1x_2\ldots x_{\nu_{k-1}})}{(1-x_1x_2\ldots x_{\nu_1})\ldots(1-x_1x_2\ldots x_{\nu_k})} , \qquad (19)$$

where the sum is over all ω-compatible chains $\phi = I_0 \subset I_1 \subset \ldots \subset I_k = P$ of order ideals, with $\nu_i = |I_i|$.

When the expression (19) for $G(P,\omega)$ is given a common denominator $(1-x_1)(1-x_1x_2)\dots(1-x_1x_2\dots x_p)$, the typical term in the numerator has the form

$$(-1)^j(x_1x_2\dots x_{\nu_1})(x_1x_2\dots x_{\nu_2})\dots(x_1x_2\dots x_{\nu_{k-1}})$$
$$\cdot\ (x_1x_2\dots x_{\mu_1})(x_1x_2\dots x_{\mu_2})\dots(x_1x_2\dots x_{\mu_j}) \quad ,$$

where $\{\mu_1,\mu_2,\dots,\mu_j\} \subseteq \underline{p-1} - \{\nu_1,\nu_2,\dots,\nu_{k-1}\}$. Relabeling the numbers $\nu_1,\nu_2,\dots,\nu_{k-1}$, $\mu_1,\mu_2,\dots,\mu_j$ as $m_1,m_2,\dots m_s$ $(s=k+j-1)$, it follows that the coefficient of $(x_1x_2\dots x_{m_1})(x_1x_2\dots x_{m_2})\ \dots\ (x_1x_2\dots x_{m_s})$ in the numerator of $G(P,\omega)$ is given by

$$\sum_{T\subseteq S} (-1)^{|S-T|}\alpha(P,\omega;T) \;=\; \beta(P,\omega;S) \quad .$$

Comparing with (18), we get $\gamma(P,\omega;S) = \beta(P,\omega;S)$. ☐

Theorem 9.1 can also be given a purely lattice-theoretic proof (not involving generating functions). Such a proof appears in Stanley [35], in a more general lattice-theoretic context.

Theorem 9.1 allows us to deduce many interesting properties of the numerical invariants $\beta(P,\omega;S)$ of (P,ω) . For instance,

$$\beta(P,\omega;S) \geqslant 0 \qquad (20)$$

for all $S \subseteq \underline{p-1}$. In Chapter II, we will investigate how properties of the numbers $\beta(P;S)$ are related to the structure of P (where ω is natural). Note also that the polynomials $W_s(P,\omega)$ may be expressed in terms of the $\beta(P,\omega;S)$'s by

$$W_s(P,\omega) = \sum_S \beta(P,\omega;S)x^{m_1+\cdots+m_s},$$

where the sum is over all subsets $S = \{m_1,\ldots,m_s\}$ of $\underline{p-1}$ of cardinality s.

10. <u>The reciprocity theorem</u>. If ω is a labeling of an ordered set P of cardinality p, define the <u>complementary labeling</u> $\bar{\omega}$ by

$$\bar{\omega}(X) = p + 1 - \omega(X).$$

For instance, if ω is natural then $\bar{\omega}$ is strict. The next result shows the relationship between the generating functions $F(P,\omega)$ and $F(P,\bar{\omega})$.

10.1. <u>The Reciprocity Theorem</u>. We have

$$(x_1x_2\ldots x_p)F(P,\bar{\omega};x_1,x_2,\ldots,x_p)$$
$$= (-1)^pF(P,\omega;\frac{1}{x_1},\frac{1}{x_2},\ldots,\frac{1}{x_p}).$$

<u>Proof</u>. Let $X_{i_1},X_{i_2},\ldots,X_{i_p}$ be an extension of P to a total order, thus inducing a permutation π in $\mathcal{L}(P,\omega)$ and $\bar{\pi}$

in $\mathcal{L}(P,\overline{\omega})$. Now $\omega(X_{i_j}) > \omega(X_{i_{j+1}})$ if and only if $\overline{\omega}(X_{i_j}) < \overline{\omega}(X_{i_{j+1}})$. Hence if c_j denotes the number of descents appearing after $\omega(X_{i_j})$ in π , and if $\overline{c}_j$ denotes the number appearing after $\overline{\omega}(X_{i_j})$ in $\overline{\pi}$, we have $c_j + \overline{c}_j = p - j$. Hence, by Proposition 7.1,

$$\begin{aligned}
&(x_1x_2\ldots x_p)F(P,\overline{\omega};x_1,x_2,\ldots,x_p)\\
&= \sum \frac{x_{i_1}^{\overline{c}_1+1}\, x_{i_2}^{\overline{c}_2+1}\cdots x_{i_p}^{\overline{c}_p+1}}{(1-x_{i_1}x_{i_2}\ldots x_{i_p})(1-x_{i_1}x_{i_2}\ldots x_{i_{p-1}})\ldots(1-x_{i_1})}\\
&= \sum \frac{x_{i_1}^{p-c_1}\, x_{i_2}^{p-1-c_2}\cdots x_{i_p}^{1-c_p}}{(1-x_{i_1}x_{i_2}\ldots x_{i_p})\ldots(1-x_{i_1})}\\
&= (-1)^pF(P,\omega;\tfrac{1}{x_1},\tfrac{1}{x_2},\ldots,\tfrac{1}{x_p}). \qquad \square
\end{aligned}$$

By setting each $x_i = x$ in Theorem 10.1, we get a reciprocity theorem for $U(P,\omega)$ and $W(P,\omega)$.

10.2. Corollary. We have

$$x^pU(P,\overline{\omega};x) = (-1)^pU(P,\omega;\tfrac{1}{x})$$

and

$$W(P,\overline{\omega};x) = x^{\binom{p}{2}}W(P,\omega;\tfrac{1}{x}) . \qquad \square$$

As an immediate application of Corollary 10.2, we have:

10.3. Corollary. The degree of $W(P,\omega)$ is equal to $\binom{p}{2}-n_0$, where n_0 is the smallest integer n for which there exists a $(P,\overline{\omega})$-partition of n. Moreover, $W(P,\omega)$ is always a monic polynomial. □

Corollary 10.3 provides a simple method for determining $\deg W(P,\omega)$, since n_0 is easy to compute. Further results along these lines, in the case of natural labelings, will be discussed in Chapter II.

To obtain a reciprocity theorem for the W_s's, it is simplest to use the definition (13) of $W_s(P,\omega)$.

10.3. Proposition. We have

$$W_s(P,\overline{\omega};x) = x^{\binom{p}{2}} W_{p-1-s}(P,\omega;\tfrac{1}{x}) .$$

Proof. Let π be a permutation in $\mathcal{L}(P,\omega)$ and $\overline{\pi}$ the corresponding permutation in $\mathcal{L}(P,\overline{\omega})$. Then $\mathcal{D}(\pi) \cap \mathcal{D}(\overline{\pi}) = \phi$ and $\mathcal{D}(\pi) \cup \mathcal{D}(\overline{\pi}) = \underline{p-1}$. Hence if π contributes a term x^k to $W_s(P,\omega)$ (via (13)), then $\overline{\pi}$ contributes $x^{\binom{p}{2}-k}$ to $W_{p-1-s}(P,\overline{\omega})$, and the proof follows. □

For instance, if $P = \underline{p}$ and ω is natural, then

$$W_s(P,\omega) = \begin{cases} 1, & \text{if } s = 0 \\ 0, & \text{otherwise} . \end{cases} \qquad (21)$$

Here Propositions 8.2 and 8.3 reduce to (9). By Proposition 10.3,

$$W_s(P,\overline{\omega}) = \begin{cases} x^{\binom{p}{2}} , & \text{if } s = p-1 \\ 0 , & \text{otherwise} \end{cases} .$$

So in this case Propositions 8.2 and 8.3 reduce to (10).

Note that the expression (14) for $U_m(P,\omega)$ is defined when m is negative. We can ask whether there is some interpretation of $U_{-m}(P,\omega)$, $m>0$. The key to this result is Proposition 10.3.

10.4. Proposition. $U_{-1}(P,\omega) = 0$, and for any integer m ,

$$x^p U_m(P,\omega;x) = (-1)^p U_{-(m+2)}(P,\overline{\omega};\tfrac{1}{x}) .$$

Proof. The result $U_{-1}(P,\omega) = 0$ follows immediately from setting $m = -1$ in (14), since then each Gaussian coefficient vanishes.

By Proposition 10.3, $W_s(P,\omega;x) = x^{\binom{p}{2}} W_{p-1-s}(P,\overline{\omega};\frac{1}{x})$. Substituting this into (14), we get

$$\begin{aligned} U_m(P,\omega;x) &= \sum_{s=0}^{p-1} \begin{bmatrix} p+m-s \\ p \end{bmatrix} x^{\binom{p}{2}} W_{p-1-s}(P,\overline{\omega};\tfrac{1}{x}) \\ &= x^{\binom{p}{2}} \sum_{s=0}^{p-1} \begin{bmatrix} m+s+1 \\ p \end{bmatrix} W_s(P,\overline{\omega};\tfrac{1}{x}) \end{aligned}$$

The following identity is easily verified:

$$\begin{bmatrix} a \\ b \end{bmatrix}_x = (-1)^b x^{-\binom{b+1}{2}} \begin{bmatrix} b-a-1 \\ b \end{bmatrix}_{1/x} .$$

Hence

$$U_m(P,\omega;x) = x^{\binom{p}{2}} \sum_{s=0}^{p-1} (-1)^p x^{-\binom{p+1}{2}} \begin{bmatrix} p-(m+2)-s \\ p \end{bmatrix}_{1/x} W_s(P,\overline{\omega};\tfrac{1}{x})$$

$$= (-1)^p x^{-p} U_{-(m+2)}(P,\overline{\omega};\tfrac{1}{x})$$

and the proof follows. □

We conclude this section with a reciprocity result between $\beta(P,\omega;S)$ and $\beta(P,\overline{\omega};S)$; its proof is immediate from Theorem 9.1.

10.5. <u>Proposition</u>. Let (P,ω) be a labeled ordered set of cardinality p . Then

$$\beta(P,\overline{\omega};S) = \beta(P,\omega;\underline{p-1} - S) . \qquad \square$$

11. <u>An application to r-dimensional partitions</u>. Let P be a finite order ideal of $\underline{N}_0^r = \underline{N}_0 \times \underline{N}_0 \times \dots \times \underline{N}_0$ (r times) . We denote elements of $\underline{N}_0^r$ as

$$\underline{N}_0^r = \{(m_1,\dots,m_r) \mid m_i \geqslant 0 \ \text{ for } \ i=1,\dots,r\} .$$

Let $S = \{i_1,i_2,\dots,i_t\} \subseteq \underline{r}$, and for each integer $n \geqslant 0$, consider maps τ of P into the <u>positive</u> integers $\underline{N}$ with the following properties:

(i) $\tau(m_1,\ldots,m_r) \geqslant \tau(m_1,m_2,\ldots,m_i+1,\ldots,m_r)$, for all $i \in \underline{r}$

(ii) $\tau(m_1,\ldots,m_r) > \tau(m_1,m_2,\ldots,m_i+1,\ldots,m_r)$, if $i \in S$,

(iii) $\sum_{X\in P} \tau(X) = n$.

Then τ is called an <u>r-dimensional partition of n of shape P strict in directions</u> $i_1,i_2,\ldots,i_t$. Let a_n be the number of such τ's (as a function of n , with r , P , and S fixed) and let

$$A(x) = \sum_{n=0}^{\infty} a_n x^n .$$

Also, let b_n be the number of r-dimensional partitions of n of shape P strict in directions $j_1,\ldots,j_{p-t}$, where $\{j_1,\ldots,j_{p-t}\} = \underline{r} - S$. Define

$$B(x) = \sum_{n=0}^{\infty} b_n x^n .$$

11.1 <u>Proposition</u>. $A(x)$ and $B(x)$ are rational functions of x related by

$$B(x) = (-1)^p x^p A(\tfrac{1}{x}) ,$$

where $|P| = p$.

<u>Proof</u>. One can easily choose a labeling ω of P which is decreasing in directions $i_1,i_2,\ldots,i_t$ and increasing in the other directions $(j_1,\ldots,j_{p-t})$. Then

$$A(x) = x^p U(P,\omega;x)$$

$$B(x) = x^p U(P,\overline{\omega};x) .$$

By the Reciprocity Theorem for $U(P,\omega)$ (Corollary 10.2), we have $x^p U(P,\overline{\omega};x) = (-1)^p U(P,\omega;\frac{1}{x})$, from which the proof follows. □

We remark that Proposition 11.1 was proved in the case $r = 3$ by B. Gordon [16] using _ad hoc_ techniques. There is considerable interest in the problem of determining $A(x)$ explicitly. The case $r = 2$ has a well-developed theory--here 2-dimensional partitions are known as _plane partitions_. See §21 and the survey article by Stanley [34] for results on plane partitions. For $r \geqslant 3$ almost nothing is known, and Proposition 11.1 casts only a faint glimmer of light on a vast darkness.

12. _Operations on ordered sets_. We will consider various operations which can be formed on ordered sets, and their effect on the generating functions $U(P,\omega)$ (or $W(P,\omega)$) and $U_m(P,\omega)$ (or $W_s(P,\omega)$) .

We will discuss three operations:

(i) the dual P^* of P

(ii) the disjoint union $P + Q$

(iii) the ordinal sum $P \oplus Q$.

The definitions and basic properties of these operations can be found, e.g., in Birkhoff [4].

One operation is conspicuously absent from the above three--the direct (or cartesian) product $P \times Q$. It would be very desirable to have a formula (call it $\widetilde{\mathscr{F}}$) expressing $W_s(P\times Q)$ in terms of the $W_r(P)$'s and $W_r(Q)$'s (valid for all finite P and Q) (recall from §3 that $W_s(P)$ is short for $W_s(P,\omega)$ when ω is natural). Unfortunately, no such formula $\mathscr{F}$ exists. To see this, let P and Q be the ordered sets of Figure 2 (naturally labeled). Then $W_s(P) = W_s(Q)$ for all s , viz.,

$$\begin{aligned} W_0(P) &= W_0(Q) = 1 \\ W_1(P) &= W_1(Q) = x+2x^2+2x^3+x^4 \\ W_2(P) &= W_2(Q) = x^4+x^5+x^6 \\ W_s(P) &= W_s(Q) = 0 \;, \quad s \geqslant 3 \;. \end{aligned}$$

Hence the expression for $W_2(P\times\underline{2})$ obtained from $\mathscr{F}$ would have to agree with the expression for $W_2(Q\times\underline{2})$. In particular, we would have

$$\begin{aligned} e(P\times\underline{2}) &= \sum_s W_s(P\times\underline{2};1) \\ &= \sum_s W_s(Q\times\underline{2};1) \\ &= e(Q\times\underline{2}) \;. \end{aligned}$$

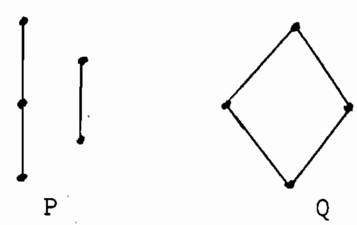

Figure 2.

This contradicts the fact that $e(P\times\underline{2}) = 2100$ and $e(Q\times\underline{2}) = 2160$.

We first consider the formula for $W_s(P^*,\omega)$.

12.1. Proposition. Let (P,ω) be a finite labeled ordered set of cardinality p, and let ω^* be the labeling of P^* defined by

$$\omega^*(X) = p+1-\omega(X) .$$

Then for all $s \geqslant 0$, we have

$$W_s(P^*,\omega^*;x) = x^{ps}W_s(P,\omega;\tfrac{1}{x}) .$$

In particular, since ω^* is a natural labeling of P^* if and only if ω is a natural labeling of P,

$$W_s(P^*;x) = x^{ps}W_s(P;\tfrac{1}{x}) .$$

Proof. Let $\pi = (\omega(X_{i_1}),\omega(X_{i_2}),\ldots,\omega(X_{i_p}))$ be a permutation in $\mathcal{L}(P,\omega)$, so there is a corresponding permutation $\pi^* = (\omega(X_{i_p}),\ldots,\omega(X_{i_2}),\omega(X_{i_1}))$ in $\mathcal{L}(P^*,\omega^*)$. Now $\omega(X_{i_j}) > \omega(X_{i_{j+1}})$ if and only if $\omega^*(X_{i_{j+1}}) > \omega^*(X_{i_j})$. It follows that if $\mathcal{D}(\pi) = \{m_1,\ldots,m_s\}$, then $\mathcal{D}(\pi^*) = \{p-m_1,\ldots,p-m_s\}$. The proof now follows from the definition (13) of $W_s(P,\omega)$. □

We next consider the problem of computing $W_s(P \oplus Q,\omega)$.

12.2. Proposition. Let (P,ω_1) and (Q,ω_2) be finite labeled ordered sets of cardinalities p and q, respectively. Let $\omega = \omega_1 \oplus \omega_2$ be the labeling of $P \oplus Q$ defined by

$$\omega(X) = \begin{cases} \omega_1(X) & , \text{ if } X \in P \\ \omega_2(X) + p , & \text{ if } X \in Q . \end{cases}$$

Then for all $s \geq 0$, we have

$$W_s(P \oplus Q, \omega) = \sum_{k=0}^{s} x^{kp} W_{s-k}(P; \omega_1) W_k(Q, \omega_2) .$$

In particular, since ω is natural if and only if ω_1 and ω_2 are natural,

$$W_s(P \oplus Q) = \sum_{k=0}^{s} x^{kp} W_{s-k}(P) W_k(Q) .$$

Proof. Every permutation π in $\mathcal{L}(P \oplus Q, \omega_1 \oplus \omega_2)$ is of the form

$$\pi = (i_1, i_2, \ldots, i_p, j_1, j_2, \ldots, j_q)$$

where $\pi_1 = (i_1, i_2, \ldots, i_p)$ is in $\mathcal{L}(P, \omega_1)$ and $\pi_2 = (j_1, j_2, \ldots, j_q)$ is in $\mathcal{L}(P, \omega_2)$. Thus if $\mathcal{S}(\pi_1) = \{m_1, \ldots, m_s\}$ and $\mathcal{S}(\pi_2) = \{n_1, \ldots, n_t\}$, then $\mathcal{S}(\pi) = \{m_1, \ldots, m_s, n_1+p, \ldots, n_t+p\}$. The proof follows from the definitions (13) of $W_s(P, \omega)$. ☐

12.3. Corollary. In the notation of the previous Proposition we have

$$W(P \oplus Q, \omega) = W(P, \omega_1) \sum_s x^{sp} W_s(Q, \omega_2) .$$

Proof. Sum on s the expression for $W_s(P \oplus Q, \omega)$ of the previous proposition, and interchange the order of summation. The proof now follows from $\sum W_s(P,\omega) = W(P,\omega)$ (Proposition 8.1(ii)). □

Two simple special cases of Proposition 12.2 occur when either P or Q is a chain. Here $W_s(\underline{r},\omega) = x^d \delta_{st}$ for the appropriate values of d and t (δ_{st} is the Kronecker delta). In particular, $W_s(\underline{r}) = \delta_{0s}$. We explicitly state the resulting formulas when ω is a natural labeling.

12.4. Corollary. We have

(i) $W_s(P \oplus \underline{r}) = W_s(P)$

(ii) $W_s(\underline{r} \oplus P) = x^{rs} W_s(P)$. □

We now turn to consideration of the disjoint union $P + Q$. If (P,ω_1) and (Q,ω_2) are labeled ordered sets, then let $\omega = \omega_1 + \omega_2$ be any labeling of $P + Q$ satisfying

$$\begin{aligned} \omega_1(X) < \omega_1(Y) &\Rightarrow \omega(X) < \omega(Y) , \text{ and} \\ \omega_2(X) < \omega_2(Y) &\Rightarrow \omega(X) < \omega(Y) . \end{aligned} \tag{22}$$

There are many such labelings, e.g., the labeling ω of Proposition 12.1, and all are equivalent.

The formula relating $W_s(P+Q,\omega)$ to the $W_i(P,\omega_1)$'s and $W_j(Q,\omega_2)$'s has a fairly simple form due to what appears to be a "fluke," viz., the existence of a combinatorial identity given by the next lemma. It would be interesting to find a proof of Proposition 12.6 which bypasses Lemma 12.5. Among other things, such a proof would give a new proof of Lemma 12.5.

12.5. <u>Lemma</u>. The following identities are valid for non-negative integers α,β :

(i) $$\begin{bmatrix} p+\alpha \\ p \end{bmatrix}\begin{bmatrix} q+\beta \\ q \end{bmatrix}$$

$$= \sum_{i\geq 0} \begin{bmatrix} p+q+i \\ i \end{bmatrix} x^{(\alpha-i)(\beta-i)} \begin{bmatrix} q+\beta-\alpha \\ \beta-i \end{bmatrix}\begin{bmatrix} p+\alpha-\beta \\ \alpha-i \end{bmatrix}$$

(ii) $$x^{\alpha\beta}\begin{bmatrix} p+\alpha-\beta \\ \alpha \end{bmatrix}\begin{bmatrix} q+\beta-\alpha \\ \beta \end{bmatrix}$$

$$\sum_{i\geq 0} (-1)^i x^{\binom{i}{2}} \begin{bmatrix} p+q+1 \\ i \end{bmatrix}\begin{bmatrix} p+\alpha-i \\ p \end{bmatrix}\begin{bmatrix} q+\beta-i \\ q \end{bmatrix}$$

<u>Proof</u>. (i) is an identity kindly proved by H. Gould [17] upon request from this writer. Subsequently I was informed by George Andrews that (i) is equivalent to an identity of E.M. Wright [40].

(ii) is simply the inverse form of (i), as may be seen by introducing a new variable k and putting $k-\alpha$ for α and $k-\beta$ for β in (i). This puts (i) in the form

$$A_k = \sum_{i\geq 0} \begin{bmatrix} p+q+1 \\ i \end{bmatrix} B_{k-i} \, .$$

Arguing as in the proof of Proposition 8.4,

$$B_k = \sum_{i\geq 0} (-1)^i x^{\binom{i}{2}} \begin{bmatrix} p+q+1 \\ i \end{bmatrix} A_i .$$

Restoring $k-\alpha$ to α and $k-\beta$ to β gives (ii). □

12.6. <u>Proposition</u>. Let (P,ω_1) , (Q,ω_2) , and $(P+Q,\omega)$ satisfy (22). Then

(i) $U_m(P+Q,\omega) = U_m(P,\omega_1)U_m(Q,\omega_2)$,

(ii)

$$W_s(P+Q,\omega)$$

$$= \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} x^{(s-i)(s-j)} \begin{bmatrix} p+j-i \\ s-i \end{bmatrix} \begin{bmatrix} q+i-j \\ s-j \end{bmatrix} W_i(P,\omega_1)W_j(Q,\omega_2) .$$

<u>Proof</u>. (i) Obvious.

(ii) Substituting (i) into the expression for $W_s(P+Q,\omega)$ given by Proposition 8.4, we get

$$W_s(P+Q,\omega) = \sum_{m=0}^{s} (-1)^m x^{\binom{m}{2}} \begin{bmatrix} p+q+1 \\ m \end{bmatrix} U_{s-m}(P,\omega_1)U_{s-m}(Q,\omega_2) \qquad (23)$$

Hence, by (15),

$$W_s(P+Q,\omega) = \sum_{m=0}^{s} (-1)^m x^{\binom{m}{2}} \begin{bmatrix} p+q+1 \\ m \end{bmatrix} \left(\sum_{k=0}^{s-m} \begin{bmatrix} p+k \\ p \end{bmatrix} W_{s-m-k}(P,\omega_1) \right)$$
$$\cdot \left(\sum_{k=0}^{s-m} \begin{bmatrix} q+k \\ q \end{bmatrix} W_{s-m-k}(Q,\omega_2) \right) .$$

The coefficient of $W_i(P,\omega_1)W_j(Q,\omega_2)$ in this expansion is

$$\sum_{m\geq 0} (-1)^m x^{\binom{m}{2}} \begin{bmatrix} p+q+1 \\ m \end{bmatrix} \begin{bmatrix} p+s-m-i \\ p \end{bmatrix} \begin{bmatrix} q+s-m-j \\ q \end{bmatrix}$$

By Lemma 12.5, this latter sum is equal to

$$x^{(s-i)(s-j)} \begin{bmatrix} p+j-i \\ s-i \end{bmatrix} \begin{bmatrix} q+i-j \\ s-j \end{bmatrix} ,$$

and the proof follows. □

A particularly simple case occurs when ω is natural and Q is a chain.

12.7. <u>Corollary</u>. We have

$$W_s(P+\underline{q}) = \sum_{i=0}^{p-1} x^{s(s-i)} \begin{bmatrix} p-i \\ s-i \end{bmatrix} \begin{bmatrix} q+i \\ s \end{bmatrix} W_i(P) . \qquad \square$$

When in addition P is a chain, we obtain a formula for $W_s(\underline{p}+\underline{q})$ which will be used in §23 in the discussion of "stacks."

12.8. <u>Corollary</u>. $W_s(\underline{p}+\underline{q}) = x^{s^2} \begin{bmatrix} p \\ s \end{bmatrix} \begin{bmatrix} q \\ s \end{bmatrix}$. □

A result equivalent to Corollary 12.8 was obtained by MacMahon [23, vol. 2, §460], using the notation $PF_s(pq;\infty)$ for our $W_s(\underline{p}+\underline{q})$. It is not difficult to see that Corollary 12.8, together with (23), is equivalent to the special case $\alpha = \beta$ of Lemma 12.5(ii).

Finally, we note that by letting $m \to \infty$ in Proposition 12.6(i

we get

$$U(P+Q,\omega) = U(P,\omega_1)U(P,\omega_2) .$$

From this it follows that

$$W(P+Q,\omega) = \begin{bmatrix} p+q \\ p \end{bmatrix} W(P,\omega_1)W(Q,\omega_2) . \tag{24}$$

In particular, $e(P+Q) = \binom{p+q}{p} e(P)e(Q)$, a fact which can easily be proved directly.

13. The order polynomial and (P,ω)-Eulerian numbers. Setting $x = 1$ in the polynomials $U_m(P,\omega)$ and $W_s(P,\omega)$ gives certain basic numerical invariants of the ordered set P . An inkling of this fact is Corollary 5.4, viz., $W(P,\omega;1) = e(P)$. We will obtain a (P,ω)-generalization of the classical Eulerian numbers and an interesting polynomial $\Omega(P,\omega;m)$ associated with P .

First consider the numbers $W_s(P,\omega;1)$, $s=0,1,\ldots,p-1$. These are denoted by

$$w_s(P,\omega) = W_s(P,\omega;1) ,$$

or more simply by w_s when no confusion will result. Observe that

$$w_0 + w_1 + \ldots + w_{p-1} = e(P) . \tag{25}$$

It follows from the definition (13) of $W_s(P,\omega)$ that w_s is equal to the number of permutations $i_1,i_2,\ldots,i_p$ in $\mathcal{L}(P,\omega)$ with exactly s descents. The classical Eulerian numbers $A_{p,s+1}$ are defined combinatorially to be the total number of permutations

$i_1, i_2, \ldots, i_p$ of the integers $1, 2, \ldots, p$ with exactly s descents (see, e.g., [27, pages 38 and 215]). Hence $w_s = A_{p,s+1}$ when P is a disjoint union $\underline{1} + \underline{1} + \ldots + \underline{1} = p\underline{1}$ of p points. This justifies calling w_s a *(P,ω)-Eulerian number*.

By setting $P = p\underline{1}$ in many of the following formulas, we obtain known results about ordinary Eulerian numbers and Eulerian polynomials (cf. [5], [6], [11], [27]). In particular, in [5] Carlitz defines "q-Eulerian numbers" $A_{ms}(q)$, which in our notation are given by

$$A_{ms}(q) = W_{m-s}(m\underline{1}, q) .$$

Carlitz gives their combinatorial interpretation (due to Riordan) in Section 13 of [5]. This interpretation is immediate from our definition (13) of $W_s(P,\omega)$.

As a further example, suppose P is a disjoint union of chains, naturally labeled. Then the (P,ω)-Eulerian numbers arising in this case have been considered by MacMahon [23, §§178-181], though of course from a different point of view. Specifically, let λ be a partition of p with parts $\lambda_1, \lambda_2, \ldots, \lambda_t$. Let P be the ordered set $\underline{\lambda_1} + \underline{\lambda_2} + \ldots + \underline{\lambda_t}$, naturally labeled. Then MacMahon's invariant N_s, is equal in our terminology to w_{s-1}. These numbers have also been investigated by Carlitz, Roselle, and Scoville [8].

We now introduce some new numbers closely related to the

(P,ω)-Eulerian numbers. Define $\Omega(P,\omega;m)$ to be the number of $(P,\omega;m-1)$ partitions, i.e.,

$$\Omega(P,\omega;m) = U_{m-1}(P,\omega;1) = |\mathcal{A}(P,\omega;m-1)| .$$

We call $\Omega(P,\omega;m)$ the $\underline{(P,\omega)}$-order polynomial. Define moreover $e_s(P,\omega)$ to be the number of surjective (P,ω)-partitions $P \to \underline{s}$ (denoted e_s for short). Also define $\bar{e}_s(P,\omega) = e_s(P,\bar{\omega})$ (denoted $\bar{e}_s$ for short).

13.1. Proposition. We have $e_s = 0$ unless $1 \leqslant s \leqslant p$ (except that $e_0 = 1$ if P is void), $e_p = e(P)$, and $\Omega(P,\omega;m)$ is a polynomial in m of degree p and leading coefficient $e(P)/p!$, viz.,

$$\Omega(P,\omega;m) = \sum_{i=1}^{p} e_i \binom{m}{i} . \tag{26}$$

Proof. The statements about e_s are clear from its definition. Now $\Omega(P,\omega;m)$ is the number of (P,ω)-partitions $\sigma: P \to \underline{m-1}_0$. If the range of σ has i elements, then there are precisely e_i (P,ω)-partitions $\tau: P \to \underline{m-1}_0$ with the same range. Summing over all i gives (26). From (26) the assertions about $\Omega(P,\omega;m)$ are immediate. □

In the language of the calculus of finite differences, e_i is the i-th difference at 0 of $\Omega(P,\omega;m)$, i.e.,

$$e_i = \Delta^i \Omega(P,\omega;0) .$$

The reciprocity result of Proposition 10.4 allows us to connect $\Omega(P,\omega)$ with $\Omega(P,\overline{\omega})$ and the e_i's with the $\overline{e}_i$'s .

13.2. Proposition. Let (P,ω) be a labeled ordered set of cardinality p . Then

(i) $\Omega(P,\overline{\omega};m) = (-1)^p \Omega(P,\omega;-m)$

(ii) $\overline{e}_s = (-1)^p \sum_{i=1}^{p} (-1)^i e_i \binom{i-1}{s-1}$.

Proof. (i) follows from putting $x = 1$ in Proposition 10.4. Using Proposition 13.1, (i) takes the form

$$\sum_{i=1}^{p} e_i \binom{m}{i} = (-1)^p \sum_{i=1}^{p} \overline{e}_i \binom{-m}{i} . \tag{27}$$

(ii) is now a simple consequence of (27) and the identity (cf. [27, Ex. 16, p. 43])

$$\binom{m}{k} = (-1)^k \sum_{i=1}^{k} \binom{k-1}{i-1}\binom{-m}{i} .$$

We omit the details. □

We now consider the relation of the (P,ω)-Eulerian numbers w_s to the order polynomial $\Omega(P,\omega)$.

13.3. Proposition. We have

(i) $\Omega(P,\omega;m) = \sum_{s=0}^{p-1} \binom{p+m-1-s}{p} w_s$.

(ii) $\sum_{m=0}^{\infty} \Omega(P,\omega;m)q^m = \left[\sum_{s=0}^{p-1} w_s q^{s+1}\right]/(1-q)^{p+1}$.

(iii) $e_s = \sum_{k=0}^{p-1} w_k \binom{p-k-1}{p-s}$.

(iv) $w_s = (-1)^s \sum_{k=1}^{p} (-1)^{k+1} e_k \binom{p-k}{p-1-s}$.

Proof. (i) Substitute $x = 1$ in Proposition 8.2 and use $U_{m-1}(P,\omega;1) = \Omega(P,\omega;m)$.

(ii) Substitute $x = 1$ in Proposition 8.3.

(iii) Although an algebraic proof of (iii) can be given using (i) or (ii), a combinatorial proof seems more appropriate. We count how many surjective (P,ω)-partitions $\sigma\colon P \to \underline{s-1}_0$ are compatible with a given permutation π in the ω-separator $\mathcal{L}(P,\omega)$, and sum over all π to get e_s . If $|\mathcal{D}(\pi)| \geq s$, then there are no such σ . If $|\mathcal{D}(\pi)| = k \leq s-1$, then π has k descents and $p-1-k$ ascents. There are $\binom{p-1-k}{s-1-k} = \binom{p-1-k}{p-s}$ ways of choosing $s-1-k$ of the ascents giving a total of $s-1$ descents plus chosen ascents. Once these $s-1-k$ ascents have been chosen, there is precisely one $\sigma\colon P \to \underline{s-1}_0$ compatible with π such that $\sigma(X_{i_j}) > \sigma(X_{i_{j+1}})$ whenever $\omega(X_{i_j}) < \omega(X_{i_{j+1}})$ is a chosen ascent. Thus there are $\binom{p-1-k}{p-s}$ σ's compatible with π . Since w_k is the number of permutations π in $\mathcal{L}(P,\omega)$ satisfying $|\mathcal{D}(\pi)| = k$, the result follows.

(iv) This follows from (ii) using standard inverse relations (e.g., see [28, Table 2.1, nos. 2 and 3]). □

The polynomial $\sum_{s=0}^{p-1} w_s q^s$ which appears in Proposition 13.1(ii) (multiplied by q) can be regarded as a *(P,ω)-Eulerian polynomial*. When $P = p\underline{1}$ it reduces to the classical Eulerian polynomial $E_p(q)$. Since $\Omega(p\underline{1},\omega;m) = m^p$ (every labeling ω of $p\underline{1}$ is natural), Proposition 13.3(ii) becomes

$$\sum_{m=0}^{\infty} m^p q^m = qE_p(q)/(1-q)^{p+1} ,$$

a standard result on the Eulerian polynomials (see [27, pp. 38-39]

II. NATURAL LABELINGS

In this chapter we will consider more closely the various generating functions and numerical invariants previously discussed, in the case where ω is a <u>natural labeling</u>. Though most of our results on natural labelings can be generalized to arbitrary labelings, they are not so simple nor elegant in their full generality.

Now when ω is natural, $\alpha(P;S)$ counts the <u>total</u> number of chains $\phi = I_0 \subset I_1 \subset \ldots \subset I_{s+1} = P$ of order ideals of P such that $|I_i| = m_i$, where $S = \{m_1,\ldots,m_s\}_<$. Hence by studying the numbers $\beta(P;S) = \sum_{T \subseteq S} (-1)^{|S-T|}\alpha(P;T)$ we gain considerable insight into the relationship between a finite ordered set P and its distributive lattice $J(P)$ of order ideals. An investigation of the properties of the numbers $\beta(P;S)$ will occupy a major portion of this chapter.

We assume throughout this chapter that all labelings are natural. We will call an ω-separator simply a <u>separator</u>, and we write $U(P)$ for $U(P,\omega)$, etc. We will also adopt the convention of denoting the elements of P by $X_1,X_2,\ldots,X_p$, where $\omega(X_i) = i$.

14. <u>A Möbius-theoretic interpretation of $\beta(S)$</u> . Recall that every finite ordered set P has associated with it a <u>Möbius function</u> μ (see Rota [30]). The theory of Möbius functions has been extensively developed, but we require only a fundamental theorem of Philip Hall [18]. Namely, if P is a finite ordered set with 0 and 1 and Möbius function μ , then

$$\mu(0,1) \;=\; \sum_k (-1)^k c_k \,, \tag{28}$$

where c_k is the number of chains $0 = X_0 < X_1 < \ldots < X_k = 1$ of P of length k (so $c_0 = 0$ unless $|P| = 1$.

Let P be a finite ordered set of cardinality p, so $J(P)$ has length p (see §4). If $S \subseteq \underline{p-1}$, define the ordered set $J(P,S)$ to be the sub-ordered set of $J(P)$ consisting of the 0 and 1 of $J(P)$, together with all elements I whose ranks $|I|$ are in S. Thus for instance $J(P,\phi) = \underline{2}$ and $J(P,\underline{p-1}) = J(P)$. We now conclude from the definition of $\alpha(P;S)$ (§9) and Hall's theorem (28) the following result.

14.1. Proposition. Let μ_S denote the Möbius function of $J(P,S)$, and let $s = |S|$. Then

$$\mu_S(0,1) \;=\; (-1)^s \beta(P;S) \,. \qquad \square$$

Since every segment of $J(P,S)$ is of the form $J(P',S')$ for an appropriate P' and S', the fact that $\beta(P;S) \geqslant 0$ (eqn. (20)) is translated into the following result.

14.2. Proposition. The Möbius function of $J(P,S)$ alternates in sign; i.e., if $[X,Y]$ is a segment of $J(P,S)$ of length k, then

$$(-1)^k \mu_S(X,Y) \;\geqslant\; 0 \,. \qquad \square$$

Thus we have the first known example of a general class of ordered sets which are not lattices but whose Möbius functions alternate in sign. A more extensive class of such ordered sets, obtained by generalizing Proposition 14.2, appears in [35].

Recall that the Möbius function μ of a distributive lattice $L = J(P)$ is given by [30]

$$\mu(X,Y) = \begin{cases} (-1)^r, & \text{if } [X,Y] \text{ is a boolean algebra of rank } r \\ 0, & \text{otherwise} \end{cases} .$$

Thus by Proposition 14.1,

$$\beta(P;\underline{p-1}) = \begin{cases} 1, & \text{if } P \text{ has no chains of length} \geqslant 1 \\ 0, & \text{otherwise} \end{cases} .$$

This illustrates how the length of chains in P influences the behavior of the function $\beta(P;S)$. A much more general result of this nature will be given in §16.

15. Some properties of $\beta(P;S)$. We know from (20) that

$$\beta(P;S) \geqslant 0 \tag{29}$$

for all $S \subseteq \underline{p-1}$. We will determine various conditions as to when $\beta(P;S) = 0$ or $\beta(P;S) > 0$. First we give a necessary and sufficient condition for strict inequality to hold in (29).

15.1. Theorem. Let P be an ordered set of cardinality p, and let $S \subseteq \underline{p-1}$. Suppose

$$\underline{p-1} - S = \{n_1, n_2, \ldots, n_t\}_< .$$

Then $\beta(P;S) > 0$ if and only if there exists a chain

$$\phi = I_0 \subset I_1 \subset \ldots \subset I_t \subset I_{t+1} = P \qquad (30)$$

of order ideals of P such that

(i) $|I_i| = n_i$, for $i=1,2,\ldots,t$, and

(ii) Each subset $I_{i+1} - I_i$ of P $(0 \leq i \leq t)$ is an antich

of P.

(In other words, $\beta(P,S) > 0$ if and only if $\alpha(P,\omega;\underline{p-1} - S) >$ where ω is a <u>strict</u> labeling of P.)

<u>Proof</u>. Suppose $\beta(P;S) > 0$. Let $\pi = (i_1,i_2,\ldots,i_p)$ be a permutation in the separator $\mathcal{L}(P)$ with $\mathcal{S}(\pi) = S$. Such a perm tion π exists by Theorem 9.1. Then for $0 \leq j \leq t$, we have $i_{n_j+1} > i_{n_j+2} > \ldots > i_{n_{j+1}}$ (with the convention $n_0 = 0, n_{t+1} = p$ It is easily seen that this means that the set T_j $= \{X_{i_k} | n_j+1 \leq k \leq n_{j+1}\}$ is an antichain of P. (Recall our conv tion $\omega(X_i) = i$.) Hence the required chain of order ideals is gi by defining

$$I_{j+1} = T_0 \cup T_1 \cup \ldots \cup T_j .$$

Conversely, suppose we have a chain (30) of order ideals. L ω be any labeling of P with the property that $\omega(I_i)$ $= \{1,2,\ldots,n_i\}$, for $0 \leq i \leq t+1$. Any such labeling ω is natural since each $I_{i+1} - I_i$ is an antichain. Then one of the permutations π in $\mathcal{L}(P)$ is the following:

$$
\begin{aligned}
\pi:\ & n_1 > n_1-1 > n_1-2 > \cdots > 1 \\
& n_2 > n_2-1 > \cdots > n_1+1 \\
& \vdots \\
& n_{j+1} > n_{j+1}-1 > \cdots > n_j + 1 \\
& \vdots \\
& n_{t+1} > n_{t+1}-1 > \cdots > n_t+1 .
\end{aligned}
$$

For this permutation π we have $\mathscr{D}(\pi) = \underline{p-1} - \{n_1, n_2, \ldots, n_t\}$, and the proof follows. □

15.2. Corollary. If $\beta(P;S) > 0$ and $T \subseteq S$, then $\beta(P;T) > 0$.

Proof. Any refinement

$$\phi = I'_0 \subset I'_1 \subset \cdots \subset I'_r \subset I'_{r+1} = P$$

of the chain (30) retains the property that each subset $I'_{i+1} - I'_i$ is an antichain of P (since any subset of an antichain is an antichain). The proof now follows immediately from Theorem 15.1. □

Observe that Corollary 15.2 is *false* whenever ω is not natural, for then $\beta(P,\omega;\phi) = 0$.

A special case of Theorem 15.1 is of some interest. We omit the relatively straightforward proof.

15.3. Corollary. Let P be an ordered set of cardinality p , and let $m \in \underline{p-1}$. The following conditions are equivalent:

(i) $\beta(P;m) = 0$.

(ii) $\beta(P;S) = 0$ whenever $m \in S$.

(iii) P has a unique order ideal of cardinality m .

(iv) P can be written as an ordinal sum $P_1 \oplus P_2$, where $|P_1| = m$. □

A further property of the numbers $\beta(P;S)$, the proof of which becomes trivial when separators are invoked, is the following.

15.4. Proposition. Let P and Q be partial orders on the same set of cardinality p , such that Q is an extension of the ordering of P ; i.e., if $X \leqslant Y$ in P , then $X \leqslant Y$ in Q . Then $\beta(P;S) \geqslant \beta(Q;S)$ for all $S \subseteq \underline{p-1}$.

Proof. Let ω be a natural labeling of Q . Since Q extends P , ω is also a natural labeling of P , so every permutation in $\mathcal{L}(Q)$ is also in $\mathcal{L}(P)$. The proof follows from Theorem 9.1. □

Finally, we mention a "reconstruction" problem.

Problem. If P and Q are ordered sets of the same cardinality p such that $\beta(P;S) = \beta(Q;S)$ for all $S \subseteq \underline{p-1}$, then are P and Q isomorphic? We conjecture that the answer is no, although we have verified that the answer is yes if $p \leqslant 6$. An equivalent formulation of the problem is the following: If for every sequence $a_0, a_1, \cdots$ of non-negative integers summing to p , we have that the number of P-partitions with a_i parts equal to i is the same as the number of Q-partitions with a_i parts equal to i , then are P and Q isomorphic?

16. The extreme-value theorem. In this section, we investigate more closely the relationship between the combinatorial properties of

p and the behavior of the function $\beta(P,\cdot)$. We require a fair amount of new notation. In what follows, we assume P is a finite ordered set of cardinality p , with longest chain of length ℓ (or cardinality $\ell+1$) .

(a) If $X \in P$, define

$\nu(X)$ = length of longest chain of P with top X .

$\delta(X)$ = length of longest chain of P with bottom X .

(b) Define ν_k(resp. δ_k) to be the number of elements $X \in P$ satisfying $\nu(X) = k$(resp. $\delta(X) = k$) . Thus $\nu_0 + \nu_1 + \dots + \nu_\ell = \delta_0 + \delta_1 + \dots + \delta_\ell = p$. Also define

$$\Delta_r = \delta_r + \delta_{r+1} + \dots + \delta_\ell \,, \quad 0 \leqslant r \leqslant \ell + 1$$

$$\Gamma_r = \nu_0 + \nu_1 + \dots + \nu_{r-1} \,, \quad 0 \leqslant r \leqslant \ell + 1 \,,$$

so

$$p = \Delta_0 > \Delta_1 > \dots > \Delta_{\ell+1} = 0$$

$$0 = \Gamma_0 < \Gamma_1 < \dots < \Gamma_{\ell+1} = p \,.$$

(c) Define finite sequences $a_0, a_1, \dots, a_{p-\ell-1}$ and $b_0, b_1, \dots, b_{p-\ell-1}$ by

$$\begin{aligned}[a_0, a_1, \dots, a_{p-\ell-1}] = [1, &\tbinom{\delta_0-1}{1}, \tbinom{\delta_0-1}{2}, \dots, \tbinom{\delta_0-1}{\delta_0-1}, \tbinom{\delta_1-1}{1}, \\ &\tbinom{\delta_1-1}{2}, \dots, \tbinom{\delta_1-1}{\delta_1-1}, \tbinom{\delta_2-1}{1}, \tbinom{\delta_2-1}{2}, \\ &\dots, \tbinom{\delta_2-1}{\delta_2-1}, \dots, \tbinom{\delta_\ell-1}{\delta_\ell-1}] \,.\end{aligned}$$

$$[b_0,b_1,\ldots,b_{p-\ell-1}] = [1,\binom{\nu_0-1}{1},\binom{\nu_0-1}{2},\ldots,\binom{\nu_0-1}{\nu_0-1},\binom{\nu_1-1}{1},$$
$$\binom{\nu_1-1}{2},\ldots,\binom{\nu_1-1}{\nu_1-1},\binom{\nu_2-1}{1},\binom{\nu_2-1}{2},$$
$$\ldots,\binom{\nu_2-1}{\nu_2-1},\ldots,\binom{\nu_\ell-1}{\nu_\ell-1}].$$

(d) If n is a non-negative integer and $0 \leqslant k \leqslant n$, define $L(n,k)$ to be the lattice of all k-subsets of $\underline{n}$, ordered as follows: If $T_1 = \{m_1,\ldots,m_k\}_<$ and $T_2 = \{n_1,\ldots,n_k\}_<$ are subsets of $\underline{n}$, then define $T_1 \leqslant T_2$ in $L(n,k)$ if and only if $m_i \leqslant n_i$ for $i=1,2,\ldots,k$.

$L(n,k)$ is easily seen to be a distributive lattice of cardinality $\binom{n}{k}$ and height $k(n-k)$. In fact, $L(n,k) \cong J(\underline{k} \times \underline{n-k})$, a fact for which we have no need.

16.1. Theorem. Let P be a finite ordered set of cardinality p, with longest chain of length ℓ. Let $S \subseteq \underline{p-1}$. If $|S| > p-\ell-1$, then $\beta(P;S) = 0$.

Proof. Suppose

$$\phi = I_0 \subset I_1 \subset \ldots \subset I_t \subset I_{t+1} = P$$

is a chain of order ideals such that each subset $I_{i+1} - I_i$ $(0 \leqslant i$ is an antichain of P. Now P contains a chain of length ℓ, each of the $\ell+1$ elements of this chain must belong to a different antichain $I_{i+1} - I_i$. Hence $t \geqslant \ell$. It follows from Theorem 15 that if $\beta(P;S) \neq 0$, then $|S| \leqslant p-\ell-1$. $\square$

It is not hard to see that the following converse to Theorem 16.1 is true: There is some $S \subseteq \underline{p-1}$ of cardinality $p-\ell-1$ such that $\beta(P;S) > 0$. We can prove, however, a much stronger result.

16.2. <u>The extreme value theorem</u>. Let P be a finite ordered set of cardinality p, with longest chain of length ℓ. Let $0 \leqslant s \leqslant p-\ell-1$. Then using the notation (a)-(d) above, we have:

(i) The set Λ of elements S of $L(p-1,s)$ for which $\beta(P;S) > 0$ has a unique maximal element $M(P,s)$, and a unique minimal element $m(P,s)$ (where Λ is considered as a sub-ordered set of the lattice $L(p-1,s)$).

We denote $M(P,p-\ell-1)$ and $m(P,p-\ell-1)$ by $M(P)$ and $m(P)$, respectively.

(ii) $M(P,s)$ consists of the largest s elements of $M(P)$, and $m(P,s)$ consists of the smallest s elements of $m(P)$.

(iii) $M(P) = \underline{p-1} - \{\Delta_1,\Delta_2,\ldots,\Delta_\ell\}$.

$m(P) = \underline{p-1} - \{\Gamma_1,\Gamma_2,\ldots,\Gamma_\ell\}$.

(iv) $\beta(P;M(P,s)) = a_s$,

$\beta(P;m(P,s)) = b_s$.

<u>Proof</u>. We only give proofs of the statements involving $M(P,s)$; those involving $m(P,s)$ are done by a dual argument. Define $M'(P,s)$ to be the set whose elements are the largest elements of the set

$$M'(P) = \underline{p-1} - \{\Delta_1,\Delta_2,\ldots,\Delta_\ell\}$$

Suppose $\beta(P;S) > 0$. Then by Theorem 15.1, there exists a chain

$$\phi = I_0 \subset I_1 \subset \ldots \subset I_t \subset I_{t+1} = P \qquad (31)$$

of order ideals of P such that each $I_{i+1} - I_i$ is an antichain of P, and such that $|I_i| = n_i$, where $\underline{p-1} - S = \{n_1, n_2, \ldots, n_t\}_<$. Thus for each $k = 0,1,\ldots,t+1$ the subset $P - I_k$ contains no cha[in] of P of length greater than $t-k$. Hence $P - I_k$ contains only elements $X \in P$ satisfying $\delta(X) \leqslant t-k$, so $|P-I_k| \leqslant p-\Delta_{t-k+1}$ a[nd] $n_k = |I_k| \geqslant \Delta_{t-k+1}$. It follows that if $S = \{m_1, m_2, \ldots, m_s\}_<$, th[en] m_k is less than or equal to the k-th largest element of $\underline{p-1} - \{\Delta_1, \Delta_2, \ldots, \Delta_\ell\}$, i.e., $S \leqslant M'(P,s)$ in the lattice $L(p-1, \ldots$

To complete the proof of the theorem, it remains only to prove (iv). Let us call a permutation π in the separator $\mathcal{L}(P)$ an *extreme permutation* if $\mathcal{S}(\pi) = M'(P,s)$. Thus (iv) is equivalent to the statement that there are exactly a_s extreme permutations π satisfying $|\mathcal{S}(\pi)| = s$.

Suppose a_s has the form $\binom{\delta_r - 1}{k}$ in accordance with the def[ini]tion (c) of a_s. Thus

$$(\delta_0 - 1) + (\delta_1 - 1) + \ldots + (\delta_{r-1}) < s$$

$$\leqslant (\delta_0 - 1) + (\delta_1 - 1) + \ldots + (\delta_r - 1),$$

and

$$(\delta_0 - 1) + (\delta_1 - 1) + \ldots + (\delta_{r-1} - 1) = s - k.$$

Let π be an extreme permutation in $\mathcal{L}(P)$ satisfying $|\mathcal{S}(\pi)| = s$. Since $\mathcal{S}(\pi) = M'(P,s)$, the "tail" of π has the form

$$\ldots < i_{\Delta_r+1} > i_{\Delta_r+2} > \ldots > i_{\Delta_{r-1}}$$

$$< i_{\Delta_{r-1}+1} > i_{\Delta_{r-1}+2} > \ldots > i_{\Delta_{r-2}}$$

$$\vdots$$

$$< i_{\Delta_1+1} > i_{\Delta_1+2} > \ldots > i_{\Delta_0} .$$

Hence for $0 \leqslant j \leqslant r-1$, the subset $T_j = \{X_{i_a} | \Delta_{j+1}+1 \leqslant a \leqslant \Delta_j\}$ is an antichain of P. Thus T_j consists of those elements X of P satisfying $\delta(X) = j$, $0 \leqslant j \leqslant r-1$. Hence the section of π displayed above is uniquely determined.

We have accounted for

$$(\delta_0-1) + (\delta_1-1) + \ldots + (\delta_{r-1}-1) = s - k$$

of the s descents in π. The remaining k descents must occur between i_{Δ_r-k} and i_{Δ_r}, in order that $\mathcal{S}(\pi) = M'(P,s)$. Thus the set $T = \{X_{i_a} | \Delta_r-k \leqslant a \leqslant \Delta_r\}$ is a subset of the antichain $A = \{X | \delta(X)=r\}$. Moreover, the labels of the elements of T must be arranged in π in descending order, while the labels of all the unaccounted for elements of P must be arranged in ascending order (otherwise $\mathcal{S}(\pi) \neq M'(P,s)$). Thus in order for there not to be a descent preceding $i_{\Delta_r - k}$, we must have that $i_{\Delta_r - k}$ is the largest label of any element in the antichain A. The remaining k elements of T may be chosen arbitrarily from the remaining $\delta_r - 1$ elements of A. Hence there are $\binom{\delta_r - 1}{k} = a_s$ possible permutations π and the proof is complete. □

An alternative method for proving the above theorem appears in [35, Thm. 10.3] in a more general lattice-theoretic context.

Example. This example illustrates the ease in applying Theorem 16.2 to a specific ordered set. Let P be the labeled ordered set of Figure 3. This labeling is natural. The appropriate numerology for P is

$$p = 8,\quad \ell = 3$$

$$\delta_0 = 2,\quad \delta_1 = 4,\quad \delta_2 = 1,\quad \delta_3 = 1$$

$$\Delta_0 = 8,\quad \Delta_1 = 6,\quad \Delta_2 = 2,\quad \Delta_3 = 1$$

$$[a_0, a_1, a_2, a_3, a_4] = [1,1,3,3,1] .$$

Thus

$$\begin{aligned} M(P) = M(P,4) &= \{3,4,5,7\},\quad \beta(3,4,5,7) = 1 \\ M(P,3) &= \{4,5,7\},\quad \beta(4,5,7) = 3 \\ M(P,2) &= \{5,7\},\quad \beta(5,7) = 3 \\ M(P,1) &= \{7\},\quad \beta(7) = 1 \\ M(P,0) &= \phi,\quad \beta(\phi) = 1 . \end{aligned}$$

The extreme rows are given as follows:

$$
\begin{array}{lllllllll}
s = 0: & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
s = 1: & 1 & 2 & 3 & 4 & 5 & 6 & 8 > 7 & \\[1em]
 & 1 & 2 & 3 & 4 & 6 > 5 & 8 > 7 & & \\
s = 2: & 1 & 2 & 3 & 5 & 6 > 4 & 8 > 7 & & \\
 & 1 & 2 & 4 & 5 & 6 > 3 & 8 > 7 & & \\[1em]
 & 1 & 2 & 3 & 6 > 5 > 4 & 8 > 7 & & & \\
s = 3: & 1 & 2 & 4 & 6 > 5 > 3 & 8 > 7 & & & \\
 & 1 & 2 & 5 & 6 > 4 > 3 & 8 > 7 & & & \\[1em]
s = 4: & 1 & 2 & 6 > 5 > 4 > 3 & 8 > 7 & & & &
\end{array}
$$

Dually we have $\nu_0 = 2$, $\nu_1 = 2$, $\nu_2 = 3$, $\nu_3 = 1$,

$[b_0, b_1, b_2, b_3, b_4] = [1,1,1,2,1]$, so $m(P) = \{1,3,5,6\}$ etc.

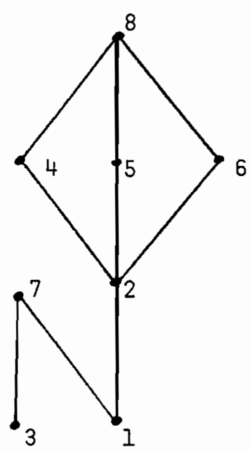

Figure 3

We give an interesting simple corollary to Theorem 16.2(iii). Further results of this nature will be given in §19.

16.3. <u>Corollary</u>. Let P be a finite ordered set with longest chain of length ℓ. Then $m(P) = M(P)$ if and only if every element of P is contained in a chain of length ℓ.

<u>Proof</u>. By Theorem 16.2(iii), $m(P) = M(P)$ if and only if $\{\Delta_1, \Delta_2, \ldots, \Delta_\ell\} = \{\Gamma_1, \Gamma_2, \ldots, \Gamma_\ell\}$. This holds if and only if $\nu_0 = \delta_\ell$, $\nu_1 = \delta_{\ell-1}, \ldots, \nu_\ell = \delta_0$. Since $\nu(X)+\delta(X)$ is equal to the length of the longest chain containing X, the result follows.

17. <u>Numerology of $W_s(P)$ and $U_m(P)$</u>. We now discuss some special properties of the polynomials $W_s(P,\omega)$ and $U_m(P,\omega)$ when ω is a natural labeling. A $(P;m)$-partition of n is simply an order-reversing map $\sigma\colon P \to \underline{m}_0$ satisfying

$$\sum_{X \in P} \sigma(X) = n .$$

The set of all such σ, ordered by defining $\sigma \leqslant \tau$ if and only if $\sigma(X) \leqslant \tau(X)$ for all $X \in P$, is simply the distributive lattice $\underline{m+1}^P$ (using the notation of [4,Ch.III,§1]). The height of σ in $\underline{m+1}^P$ is n. Now by the laws of cardinal arithmetic for ordered sets [<u>ibid</u>.], we have

$$\underline{m+1}^P = (\underline{2}^{\underline{m}})^P = \underline{2}^{\underline{m}\times\underline{P}} = J(\underline{m}\times P) .$$

There follows:

17.1. Proposition. Let P be a finite ordered set. The coefficient of x^n in $U_m(P)$ is equal to the number of order ideals of $\underline{m}\times P$ of cardinality n. □

As a special case of Proposition 17.1, the coefficient of x^n in $U_1(P) = W_1(P) + (1+x+\ldots+x^p)$ is the number of order ideals of P of cardinality n. In other words, $U_1(P)$ is the "rank generating function" for $J(P)$. This fact can be easily seen in many different ways.

17.2. Corollary. Let P be a finite ordered set, and let k and m be non-negative integers. Then

$$U_m(P\times\underline{k}) = U_k(P\times\underline{m}) .$$

Proof. Follows from Proposition 17.1, since the coefficient of x^n in both $U_m(P\times\underline{k})$ and $U_k(P\times\underline{m})$ is equal to the number of order ideals of $P\times\underline{k}\times\underline{m}$ of cardinality n. □

In particular, if $B_n = \underline{2}^n$ is the boolean algebra with n atoms, then $U_1(B_{n+1}) = U_2(B_n)$. The identity $\underline{2}^{B_{n+1}} = \underline{3}^{B_n}$ is given by N.M. Riviere (in a different form) in his paper [29] on free distributive lattices with n generators.

We now give some "numerology" of the polynomials $W_s(P)$. The notation is from §16, definitions (a)-(d).

17.3. Proposition. Let P be a finite ordered set of cardinality p and longest chain of length ℓ. Then:

(i) $W_s(P) = 0$ if $s > p - \ell - 1$.

(ii) If $0 \leqslant s \leqslant p - \ell - 1$, then

$$\deg W_s(P) = \sum i ,$$

where the sum is over all $i \in M(P,s)$.

(iii) The leading coefficient of $W_s(P)$ is a_s .

(iv) The exponent of the largest power of x dividing $W_s(P$
is $\sum i$, where the sum is over all $i \in m(P,s)$.

(v) The coefficient of the non-zero term of $W_s(P)$ of leas
degree is b_s .

Proof. (i) is an immediate corollary of Theorem 16.1, while (ii)-(v) follow from Theorem 16.2. □

In particular, we have, as a restatement of Corollary 10.3 fo the case of natural labelings, that

$$\deg W(P) = \binom{p}{2} - \sum_{X \in P} \delta(X) ,$$

where $\delta(X)$ is the length of the longest chain in P with botto

18. Chain conditions. The Reciprocity Theorem of §10 gives a connection between generating functions for P-partitions and th for strict P-partitions. When P satisfies certain "chain condi we can moreover construct direct combinatorial correspondences be these two types of partitions. These two kinds of connections wi lead to certain functional identities satisfied by the generating functions. There are four types of chain conditions which will b

relevant to us.

(a) A finite ordered set P is said to satisfy the strong chain condition if every maximal chain of P has the same length.

(b) P is said to satisfy the ν-chain condition if every subset of P of the form $\{X \mid X \leqslant X_0$ for some fixed $X_0 \in P\}$ satisfies the strong chain condition.

(c) Dually to (b), P is said to satisfy the δ-chain condition if every subset of P of the form $\{X \mid X \geqslant X_0$ for some fixed $X_0 \in P\}$ satisfies the strong chain condition.

(d) P is said to satisfy the λ-chain condition if every element of P is contained in a chain of length ℓ, where ℓ is the length of the longest chain of P.

It is easily seen that the following relations hold among the above four chain conditions, and that no other relations hold not a consequence of these:

$$\text{strong} \Rightarrow \delta\ ,\ \nu\ ,\ \text{and}\ \lambda$$

$$\delta \text{ and } \lambda \Rightarrow \text{strong}$$

$$\nu \text{ and } \lambda \Rightarrow \text{strong}\ .$$

For instance, there are five non-isomorphic ordered sets of cardinality six satisfying λ but neither ν nor δ, and none of smaller cardinality. If P has a 0 and 1, then the ν, δ, and strong chain conditions are equivalent. If P has a 0, then $\delta \Rightarrow$ strong; while if P has a 1, then $\nu \Rightarrow$ strong.

We remind the reader of the notation (from §16),

$$\delta(X) = \text{length of longest chain of } P \text{ with bottom } X$$

$$\nu(X) = \text{length of longest chain of } P \text{ with top } X .$$

It is easy for us to see the effect of the λ-chain condition on the $W_s(P)$'s .

18.1. <u>Proposition</u>. Let P be a finite ordered set of cardinality p , with longest chain of length ℓ . Then $W_{p-\ell-1}(P;1) = 1$ if and only if P satisfies the λ-chain condition.

<u>Proof</u>. Follows from Corollary 16.3. □

To study the remaining three chain conditions, we need to analyze a certain correspondence $\sigma \to \sigma'$ between P-partitions $\sigma \in \mathcal{A}(P)$ and strict P-partitions $\sigma' \in \overline{\mathcal{A}}(P)$. This correspondence is defined by

$$\sigma'(X) = \sigma(X) + \delta(X) , \quad X \in P .$$

It is easily seen that if σ is a P-partition of n , then σ' is a strict P-partition of $n + \Delta(P) = n + \sum_{X\in P} \delta(X)$. If the largest part of σ is m , then the largest part of σ' is $\leqslant m + \ell$, where ℓ is the length of the longest chain of P . Clearly the correspondence $\sigma \to \sigma'$ is injective.

18.2. <u>Lemma</u>. The injection $\sigma \to \sigma'$ is a bijection from $\mathcal{A}(P)$ to $\overline{\mathcal{A}}(P)$ if and only if P satisfies the δ-chain condition.

Thus if $a(k)$ is the number of P-partitions of k and $b(k)$

the number of strict P-partitions of k, then $a(n) \leq b(n+\Delta(P))$ for all n, with equality holding for all n if and only if P satisfies the δ-chain condition.

Proof. The only statement which is not obvious is that if P does not satisfy the δ-chain condition, then there is a $\tau \in \overline{\mathcal{A}}(P)$ such that $\tau-\delta \notin \mathcal{A}(P)$. Assume that P does not satisfy the δ-chain condition. Then there exist two elements X_0, Y_0 of P such that Y_0 covers X_0 and $\delta(X_0) > \delta(Y_0) + 1$. Define Y by

$$\tau(X) = \begin{cases} \delta(X), & \text{if } X \geq X_0 \text{ and } X \neq Y_0 \\ \delta(X)+1, & \text{if } X \not\geq X_0 \text{ or } X = Y_0 . \end{cases}$$

It is easily seen that $\tau \in \overline{\mathcal{A}}(P)$, but

$$\tau(X_0) - \delta(X_0) = 0 < 1 = \tau(Y_0) - \delta(Y_0) .$$

Since $X_0 < Y_0$, $\tau - \delta \notin \mathcal{A}(P)$. □

18.3. Lemma. Let P be a finite ordered set with longest chain of length ℓ. The injections $\sigma \to \sigma'$ between $\mathcal{A}(P;0)$ and $\overline{\mathcal{A}}(P;\ell)$ and between $\mathcal{A}(P;1)$ and $\overline{\mathcal{A}}(P;\ell+1)$ are both bijections if and only if P satisfies the strong chain condition.

Proof. The "if" part is clear. To prove the "only if" part, assume that P fails to satisfy the strong chain condition. If in addition P fails to satisfy the λ-chain condition, then define the map τ of P into $\underline{N}_0$ by

$$\tau(X) = \ell - \nu(X) .$$

Then τ and δ are easily seen to be distinct elements of $\overline{\mathcal{A}}(P;\ell)$. Since $|\mathcal{A}(P;0)| = 1$, the correspondence $\sigma \to \sigma'$ is not a bijecti[on] between $\mathcal{A}(P;0)$ and $\overline{\mathcal{A}}(P;\ell)$.

Hence we may assume P satisfies the λ-chain condition. Let $X_0 < X_1 < \ldots < X_m$ be a maximal chain of P with $m < \ell$. Let k be the greatest integer, $0 \leqslant k \leqslant m$, such that $\delta(X_k) > m - k$. Since P satisfies the λ-chain condition and X_0 is a minimal ele[ment] of P, $\delta(X_0) = \ell > m$, so k always exists. Furthermore $k \neq m$ since X_k is a maximal element of P. Define a map $\tau: P \to \underline{\ell+1}_0$ as follows:

$$\tau(X) = \begin{cases} \delta(X) & , \text{ if } X \not\leqslant X_{k+1} \\ \max(\delta(X), \ \delta(X_{k+1}) + \lambda(X,X_{k+1}) + 1) & , \text{ if } X \leqslant X_{k+1}, \end{cases}$$

where $\lambda(Y,Z)$ denotes the length of the longest chain in the segment $[Y,Z]$. It is not hard to see that $\tau \in \overline{\mathcal{A}}(P;\ell+1)$. More-over,

$$\tau(X_k) - \delta(X_k) = 0 , \quad \tau(X_{k+1}) - \delta(X_{k+1}) = 1 ,$$

so $\tau - \delta \notin \mathcal{A}(P)$. Hence the correspondence $\sigma \to \sigma'$ between $\mathcal{A}$[(P)] and $\overline{\mathcal{A}}(P;\ell+1)$ is not a bijection, and the proof is complete. $\square$

18.4. Proposition. Let P be a finite ordered set of cardi[nal]ity p, with elements $X_1,\ldots,X_p$. The following conditions are equivalent:

(i) $F(P;\frac{1}{x_1},\ldots,\frac{1}{x_p}) = \pm(x_1^{a_1}\ldots x_p^{a_p})F(P;x_1,\ldots,x_p)$,

for some integers $a_1,a_2,\ldots,a_p$.

(ii) $F(P;\frac{1}{x_1},\ldots,\frac{1}{x_p}) = (-1)^p\ x_1^{1+\delta(1)}\ldots x_p^{1+\delta(p)}\ F(P;x_1,\ldots,x_p)$

(where as usual $\delta(i)$ means $\delta(X_i)$) .

(iii) $x^{\binom{p}{2}-\Delta(P)}\ W(P;\frac{1}{x}) = W(P;x)$,

where $\Delta(P) = \sum_{X\in P} \delta(X)$.

(iv) P satisfies the δ-chain condition.

Proof. Suppose (i) holds. By the Reciprocity Theorem (§10), there follows

$$(-1)^p x_1x_2\ldots x_p\overline{F}(P) = \pm x_1^{a_1}x_2^{a_2}\ldots x_p^{a_p}F(P) . \tag{32}$$

Clearly the signs on both sides agree. Then (32) says that the correspondence $\sigma \to \sigma''$ defined by $\sigma''(i) = \sigma(i) + a_i - 1$ is a bijection between P-partitions σ of n and strict P-partitions σ' of $n + \sum(a_i-1)$. Thus we must have $a_i - 1 = \delta(i)$, so (ii) holds.

Set each $x_i = x$ in (ii) to obtain $U(P;\frac{1}{x}) = (-1)^p x^{p+\Delta}U(P;x)$. By definition,

$$W(P;x) = U(P;x)(1-x)(1-x^2)\ldots(1-x^p) ,$$

and (iii) follows immediately.

Assume (iii) holds. Then by Corollary 10.2,

$$\overline{U}(P) = x^{\Delta}U(P) .$$

Hence for all n, the number of P-partitions of n equals the num-ber of strict P-partitions of $n+\Delta$. By Lemma 18.2, P satisfies the δ-chain condition.

Finally, if (iv) holds, then (ii) (so *a fortiori* (i)) is an immediate consequence of Lemma 18.2. □

If we dualize Proposition 18.4, we get a result on the ν-chain condition. There does not appear to be a simple way to express $F(P^*)$ in terms of $F(P)$. However, we at least know from Propositi[on] 12.1 that

$$W(P^*;x) = \sum_s x^{ps} W_s(P;\tfrac{1}{x}) .$$

There follows:

18.5. Proposition. Let P be a finite ordered set of cardina[l]ity p. The following conditions are equivalent:

(i) $$x^{\binom{p}{2}-\Gamma(P)} \sum_s x^{-ps}\, W_s(P;x) = \sum_s x^{ps}\, W_s(P;\tfrac{1}{x})$$

where $\Gamma(P) = \sum_{X\in P} \nu(X)$.

(ii) P satisfies the ν-chain condition. □

Finally, we come to the strong chain condition.

18.6. Proposition. Let P be a finite ordered set of cardin[al]ity p, with longest chain of length ℓ. The following conditio[ns] are equivalent:

(i) P satisfies the strong chain condition.

(ii) For all $s = 0,1,\ldots,p-\ell-1$, we have

$$W_s(P;x) = x^{\binom{p}{2}-\Delta(P)} W_{p-\ell-s-1}(P;\tfrac{1}{x}) ,$$

where $\Delta(P) = \sum_{X\in P} \delta(X)$.

(iii) $W_{p-\ell-1}(P;1) = 1$ and $W_1(P;1) = W_{p-\ell-2}(P;1)$.

Proof. (i) $\Rightarrow$ (ii). Suppose P satisfies the strong chain condition. Then σ is a P-partition of n with largest part $\leqslant m$ if and only if $\sigma+\delta$ is a strict P-partition of $n+\Delta(P)$ with largest part $\leqslant m+\ell$. Hence $x^{\Delta(P)}U_m(P) = \overline{U}_{m+\ell}(P)$, so by Proposition 8.3,

$$x^{\Delta(P)}q^{\ell} \sum_{s=0}^{p-1} W_s(P)q^s = \sum_{s=0}^{p-1} \overline{W}_s(P)q^s .$$

But by Proposition 10.3, $\overline{W}_s(P;x) = x^{\binom{p}{2}} W_{p-1-s}(P;\frac{1}{x})$, so

$$x^{\Delta(P)}q^{\ell} \sum_{s=0}^{p-1} W_s(P;x)q^s = x^{\binom{p}{2}} \sum_{s=0}^{p-1} W_{p-1-s}(P;\tfrac{1}{x})q^s$$

Equating coefficients of q^k gives (ii).

(ii) $\Rightarrow$ (iii) Trivial: set $x=1$ and $s=0,1$ in (ii).

(iii) $\Rightarrow$ (i). Combining Propositions 8.2 and 10.3, we get

$$\overline{U}_m(P;x) = \sum_{s=0}^{p-1} \begin{bmatrix} p+m-s \\ p \end{bmatrix} x^{\binom{p}{2}} W_{p-1-s}(P;\tfrac{1}{x}) .$$

In the special cases $x=1$, $m=\ell$ or $\ell+1$, we get

$$\overline{U}_\ell(P;1) = W_{p-\ell-1}(P;1)$$

$$\overline{U}_{\ell+1}(P;1) = (p+1)W_{p-\ell-1}(P;1) + W_{p-\ell-2}(P;1) \ ,$$

since by Proposition 17.3(i), $W_{p-1-s}(P) = 0$ if $0 \leqslant s \leqslant \ell$. By assumption, $W_{p-\ell-1}(P;1) = 1$, so

$$\overline{U}_\ell(P;1) = 1 \tag{33}$$

$$\overline{U}_{\ell+1}(P;1) = p + 1 + W_{p-\ell-2}(P;1) \ .$$

But by Proposition 8.2,

$$U_1(P;1) = p + 1 + W_1(P;1)$$

Hence, by the assumption $W_1(P;1) = W_{p-\ell-2}(P;1)$, we get

$$\overline{U}_{\ell+1}(P;1) = U_1(P;1) \tag{34}$$

Thus by (33) and (34), $|\overline{\mathcal{A}}(P;\ell)| = |\mathcal{A}(P;0)|$ and $|\overline{\mathcal{A}}(P;\ell+1)| = |\mathcal{A}(P;1)|$. It follows from Lemma 18.3 that P satisfies the strong chain condition. □

Propositions 18.1, 18.4, 18.5, and 18.6 establish the rather remarkable fact that from the polynomials $W_s(P)$ it can be determined which of the strong, δ , ν , or λ-chain conditions are satisfied by P .

19. Properties of $\Omega(m)$. When ω is a natural labeling, the order polynomial $\Omega(m) = \Omega(P,\omega;m)$ has a number of interesting properties. Here $\Omega(m)$ is simply the number of order-reversing (or order-preserving) maps $P \to \underline{m}$. Some of the properties of $\Omega(m)$ are summarized in [32] and [37]. A polynomial related to $\Omega(m)$ has been studied by K. Johnson [20]. In general, our proofs are obtained by setting $x=1$ in the appropriate result concerning $U_m(P)$, since $\Omega(P;m) = U_{m-1}(P;1)$.

19.1. Proposition. Let P be a finite ordered set of cardinality p, with longest chain of length ℓ. Then we have:

(i) $\Omega(P;m+1)$ is the number of order ideals of $P\times\underline{m}$, $m \geqslant 0$. In particular, $\Omega(P;2)$ is the number of order ideals of P.

(ii) $\Omega(P;1) = 1$.

(iii) $\Omega(P;0) = \Omega(P;-1) = \ldots = \Omega(p;-\ell) = 0$.

(iv) $(-1)^p\Omega(-\ell-m) \geqslant \Omega(m) > 0$, $m \geqslant 1$.

(v) $\Omega(P\times\underline{m};n+1) = \Omega(P\times\underline{n};m+1)$.

Proof. (i) follows from Proposition 17.1. (ii) follows, e.g., from (i). (iii) follows from Proposition 13.2(i). (iv) follows from Proposition 13.2(i) and the fact that the map $\sigma \to \sigma'$ of §18 is an injection from $\mathcal{A}(P;m)$ to $\overline{\mathcal{A}}(P;m+\ell)$. (v) follows from Corollary 17.2. □

19.2. Proposition. P satisfies the λ-chain condition if and only if $\Omega(P;-\ell-1) = (-1)^p$.

Proof. Follows from Proposition 18.1. □

19.3. Proposition. The following conditions are equivalent:

(i) P satisfies the strong chain condition.

(ii) $(-1)^p\Omega(P;-\ell-2) \leqslant \Omega(P;2)$.

(iii) $\Omega(P;m) = (-1)^p\Omega(P;-\ell-m)$ for all m .

(iv) $e_s = \sum_{i=0}^{\ell} \bar{e}_{i+s} \binom{\ell}{i}$, $0 \leqslant s \leqslant p$.

Proof. The equivalence of (i), (ii), (iii) is a consequence of Proposition 18.6. The equivalence of (iii) and (iv) involves some elementary manipulations of binomial coefficients, which we omit.

The above Proposition leads to some curious results which seem difficult to prove by purely combinatorial reasoning.

19.4. Corollary. Let P be a finite ordered set of cardinality p , with every maximal chain of length ℓ . Then

(i) $2e_{p-1} = (p+\ell-1)e(P)$.

(ii) $2\bar{e}_{p-1} = (p-\ell-1)e(P)$.

(iii) $\sum_{s=1}^{p} e_s = 2^{\ell} \sum_{s=1}^{p} \bar{e}_s$.

(iv) The coefficient of m^{p-1} in $\Omega(P;m)$ is $\ell e(P)/2(p-1)!$

Proof. (i) Since $\Omega(P;m) = \sum_{s=1}^{p} e_s\binom{m}{s}$, there follows

$$\Omega(P;m) = \frac{e_p}{p!} m^p - \frac{(p-1)e_p-2e_{p-1}}{2(p-1)!} m^{p-1} + \dots \qquad (35)$$

On the other hand, by Proposition 19.3 $\Omega(P;m) = (-1)^p\Omega(P;-\ell-m)$. Equating coefficients of m^{p-1} in these two expressions for $\Omega(P$ yields (i).

(ii) Putting $s = p-1$ in Proposition 19.3(iv) gives

$$e_{p-1} = \bar{e}_{p-1} + \ell\bar{e}_p = \bar{e}_{p-1} + \ell e(P) ,$$

so (ii) follows by substituting this into (i).

(iii) Sum Proposition 19.3(iv) on s .

(iv) Follows from (i) and (35). □

19.5. Corollary. Let P be a finite ordered set of cardinality p , with every maximal chain of length ℓ . Then either $p+\ell-1$ is even, or $e(P)$ is even.

Proof. Follows from Corollary 19.4(i). □

A result similar to Corollary 19.5 follows from Proposition 18.4. It can be shown that this result implies Corollary 19.5 directly. We simply state it without proof.

19.6. Corollary. Suppose P satisfies the δ-chain condition and $\binom{p}{2} - \sum_{X\in P} \delta(X)$ is odd. Then $W(P;-1) = 0$ and $e(P)$ is even. □

19.7. Corollary. Let P be a finite ordered set of cardinality p , such that every maximal chain has length $p-4$. Let $j(P)$ denote the number of order ideals of P . Then

$$e(P) = 2(j(P)-p) .$$

Proof. Since $\Omega(P;m)$ is of degree p , we have

$$e(P) = \sum_{m=0}^{p} \binom{p}{m}(-1)^{p-m}\,\Omega(P;-p+2+m) . \tag{36}$$

By Propositions 19.1(i)(ii) and 19.3, $j(P) = (-1)^p\Omega(-p+2) = \Omega(2)$ and $1 = (-1)^p\Omega(-p+3) = \Omega(1)$. By Proposition 19.1(iii) ,

$0 = \Omega(-p+3) = \Omega(-p+4) = \ldots = \Omega(0)$. Substituting into (36) proves the corollary. □

An interesting problem (though probably intractable) is to determine for which P and Q we have $\Omega(P) = \Omega(Q)$. A number of such P and Q can be obtained from the easily verified relations $\Omega(P+Q) = \Omega(P)\Omega(Q)$, $\Omega(P) = \Omega(P^*)$, and $\Omega(P \oplus Q) = \Omega(Q \oplus P)$. There can, however, be other pairs, such as $P = 2\underline{2}$ and $Q = 1 \oplus 3\underline{1}$. Of the 63 ordered sets P of cardinality 5 , there are exactly 30 distinct order polynomials $\Omega(P)$.

III. APPLICATIONS

20. <u>Some remarks on infinite P</u>. For some of the applications which follow, we will need to consider infinite ordered sets P. Most of the preceding theory can be directly carried over to a class of infinite ordered sets satisfying appropriate finiteness conditions. For the sake of simplicity, we will consider here only the case of natural labelings, though certain other labelings are permissible. We would like to extend the notion of a P-partition $\sigma: P \to \underline{N}_0$ in such a way that the following finiteness conditions hold:

(A) For every element X of P, there is some P-partition σ such that $\sigma(X) > 0$.

(B) There exist only finitely many P-partitions of any given integer n.

Consideration of these properties leads us to defining a <u>W-ordered set</u> P to be an ordered set P satisfying:

(i) Every $X \in P$ is contained in a finite order ideal, and

(ii) For every integer $n \geq 0$, there exist only finitely many order ideals of P of cardinality n.

We then define a P-partition σ of a W-ordered set P to be an order-reversing map $\sigma: P \to \underline{N}_0$ such that all but finitely many values of σ are 0. The separator $\mathcal{L}(P)$ is defined as before, except that each permutation

$$\pi = (\omega(X_{i_1}), \omega(X_{i_2}), \ldots)$$

must come from a locally finite extension $X_{i_1}, X_{i_2}, \ldots$ of P to a total order, with only finitely many descents in π . This latte condition is equivalent to requiring $\omega(X_{i_j}) = j$ for all but finitely many j . For any finite $S \subseteq \underline{N}$, we define $\alpha(P;S)$ and $\beta(P;S)$ as before, with respect to the lattice $J_f(P)$ of <u>finite</u> order ideals of P .

We then have that the various generating functions defined for finite ordered sets can be extended to W-ordered sets. These gene ating functions can be computed from $\mathcal{L}(P)$ or from the numbers $\beta(P;S)$ as usual. Of course, the functions $W_s(P)$, $U_m(P)$, etc., will no longer be polynomials; in general, they are power series with finite coefficients.

Two previous results that no longer make sense are the Reciprocity Theorem (§10) (since the concept of a complementary labeli is no longer defined), and that half of the Extreme Value Theorem (§16) dealing with $M(P)$, a_i , Δ_k , etc. (since the numbers δ are no longer defined). However, the half of the Extreme Value Theorem dealing with $m(P)$, b_i , Γ_k , etc. is still valid. In particular, Proposition 17.3(iv-v) still holds.

In a certain trivial sense, the generating functions connect with W-ordered sets are just limits of those connected with finit ordered sets. If P is a W-ordered set with a natural labeling define for each $p \geqslant 0$ the subset (actually an order ideal) P_p by $P_p = \{X \mid \omega(X) \leqslant p\}$. It is then easy to see that

$$\lim_{p\to\infty} F(P_p) = F(P)$$

(and similarly for the other generating functions), in the sense that the coefficient of a given term of $F(P_p)$ equals the coefficient of the corresponding term of $F(P)$ for all sufficiently large p.

For instance, if $P = \underline{N}$, then a P-partition of n is just an an ordinary unrestricted partition of n, and $P_p = \underline{p}$. Since $U(\underline{p}) = 1/(1-x)(1-x^2)\ldots(1-x^p)$, letting $p \to \infty$ we get

$$U(\underline{N}) = 1/(1-x)(1-x^2)(1-x^3)\ldots ,$$

which of course is a classical result of Euler.

21. Plane partitions. Let λ be a partition of p, with parts $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_r > 0$. Define the order ideal $P(\lambda)$ of $\underline{N}^2 = \{(i,j) | i,j \in \underline{N}\}$ by the condition $(i,j) \in P(\lambda)$ if $1 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant \lambda_i$. Thus $|P(\lambda)| = p$. A $P(\lambda)$-partition with positive parts is known as a plane partition of shape λ. More generally, suppose μ is another partition, with parts $\mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant \mu_s \geqslant 0$, such that $s \leqslant r$ and $\mu_i \leqslant \lambda_i$ for $i=1,2,\ldots,s$. Hence $P(\mu) \subseteq P(\lambda)$. We denote the difference $P(\lambda) - P(\mu)$ by $P(\lambda/\mu)$. A $P(\lambda/\mu)$-partition with positive parts is called a skew plane partition of shape λ/μ.

Given $P(\lambda/\mu)$, let ω be a column-strict labeling, i.e., a labeling satisfying

$$\omega(i+1,j) > \omega(i,j)\ , \quad \text{if } (i+1,j) \text{ and } (i,j) \text{ are in } P(\lambda/\mu)$$

$$\omega(i,j+1) < \omega(i,j)\ , \quad \text{if } (i,j+1) \text{ and } (i,j) \text{ are in } P(\lambda/\mu)\ .$$

Let λ be a partition with parts $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r > 0$. Let the parts of the conjugate partition λ' [19, p. 274] be $\lambda_1' \geq \lambda_2' \geq \ldots \geq \lambda_c' > 0$. Define the <u>hook lengths</u> h_{ij} of λ by

$$h_{ij} = \lambda_i + \lambda_j' - i - j + 1 \quad (1 \leq i \leq r, 1 \leq j \leq \lambda_i)$$

We denote the hook lengths more simply by $h_1, h_2, \ldots, h_p$ (in some order). Similarly define the <u>contents</u> c_{ij} of λ by

$$c_{ij} = j - i \quad (1 \leq i \leq r, 1 \leq j \leq \lambda_i) .$$

We denote the contents by $c_1, c_2, \ldots, c_p$ (in some order).

The labeled ordered sets $(P(\lambda/u),\omega)$ have a number of remarkable properties associated with the concepts we have been consideri[ng.] We will merely state two such results here. For further aspects of the theory, together with proofs, see [33] and [34]. We shall use the notation

$$\underline{(k)} = 1-x^k$$

$$\underline{(k)}! = \underline{(1)}\,\underline{(2)} \cdots \underline{(k)} ,$$

so that

$$\underline{\begin{bmatrix} n \\ m \end{bmatrix}} = \frac{\underline{(n)}!}{\underline{(m)}!\,\underline{(n-m)}!}$$

Define a finite labeled ordered set (P,ω) of cardinality p

to be <u>α-symmetric</u> if for all $S = \{m_1, \ldots, m_s\}_< \subseteq \underline{p-1}$, the number $\alpha(P,\omega;S)$ depends only on the set of differences (including multiplicities) m_1 , m_2-m_1 , m_3-m_2 , ... , $p-m_s$. Thus, e.g., if (P,ω) is α-symmetric, then $\alpha(P,\omega;S) = \alpha(P,\omega;T)$ if $|S| = |T| = p-2$. It is easily seen that α-symmetry is equivalent to the following condition: If $\sigma \in \mathcal{A}(P,\omega)$, define the (formal) monomial

$$M(\sigma) = x_1^{r_1} x_2^{r_2} \ldots ,$$

where r_i parts of σ are equal to i . Define also the (formal) power series

$$\{P,\omega\} = \sum_{\sigma \in \mathcal{A}(P,\omega)} M(\sigma) .$$

Then (P,ω) is α-symmetric if and only if $\{P,\omega\}$ is a symmetric function of the x_i's .

21.1. <u>Proposition</u>. Let ω be a column-strict labeling of $P(\lambda/\mu)$. Then $(P(\lambda/\mu),\omega)$ is α-symmetric. □

When $P(\mu) = \phi$, the resulting symmetric function $\{P(\lambda),\omega\}$ is known as a <u>Schur function</u> [34], and is usually denoted $\{\lambda\}$ or e_λ . Similarly $\{P(\lambda/\mu),\omega\}$ is denoted $\{\lambda/\mu\}$ or $e_{\lambda/\mu}$ [34, §12].

<u>Conjecture</u>. Every finite α-symmetric labeled ordered set (P,ω) is isomorphic to a labeled ordered set of the form $(P(\lambda/\mu),\omega)$, where ω is a column-strict labeling.

The second remarkable property of plane partitions which we will state gives a simple determinant for the generating function

$U_m(P(\lambda/\mu),\omega)$. This determinant can be obtained, e.g., from a gener result of Aitken (see [34, Prop. 12.2]). When $\mu = \phi$, the determi ant can be explicitly evaluated [34, Thm. 15.3].

21.2. Proposition. Let ω be a column-strict labeling of $P(\lambda/\mu)$. Then

$$U_m(P(\lambda/\mu),\omega) = \left| \underline{\begin{bmatrix} m+\lambda_s-\mu_t+s-t \\ m \end{bmatrix}} \right|_1^r$$

where the parts of λ are $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_r > 0$ and the parts of μ are $\mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant \mu_r \geqslant 0$. □

21.3. Proposition. Let ω be a column-strict labeling of $P(\lambda)$. Then

$$U_m(P(\lambda),\omega) = x^k \frac{\underline{(m+c_1+1)}\,\underline{(m+c_2+1)}\cdots\underline{(m+c_p+1)}}{\underline{(h_1)}\,\underline{(h_2)}\cdots\underline{(h_p)}} ,$$

where $k = \sum \binom{\lambda_i'}{2} = \sum(i-1)\lambda_i = \sum h_i - \sum \binom{\lambda_i}{2} - p$, and where the c_i's are the contents and the h_i's the hook lengths of λ . □

As immediate corollaries, we see that

$$U(P(\lambda),\omega) = x^k / \underline{(h_1)}\,\underline{(h_2)}\cdots\underline{(h_p)} \tag{37}$$

and

$$e(P(\lambda)) = p!/h_1h_2\ldots h_p . \tag{38}$$

Equation (38) is a well-known result of Frame, Robinson, and Thrall [13].

Proposition 21.3 leads to several interesting consequences, discussed in [33] and [34]. One of these is that the number of order ideals of $\underline{r} \times \underline{s} \times \underline{t}$ is

$$\frac{\binom{t+r}{r}\binom{t+r+1}{r}\cdots\binom{t+r+s-1}{r}}{\binom{r}{r}\binom{r+1}{r}\cdots\binom{r+s-1}{r}} .$$

Though this result is implicit in the literature, it doesn't seem to have been stated explicitly before. The much easier determination of $|J(\underline{r}\times\underline{s})|$ appears, e.g., in [4, p. 66, ex. 8]. Nothing significant seems to be known in general about $|J(\underline{r} \times \underline{s} \times \underline{t} \times \underline{u})|$. It follows from Proposition 19.1(i) that the number of order ideals of $\underline{n_1} \times \underline{n_2} \times \ldots \times \underline{n_k}$ is a polynomial in each n_i when all the other n_j's are held fixed, but this observation does not appear to be of much value.

22. Trees. A tree is a finite ordered set with 0 whose Hasse diagram contains no cycles. A dual tree is the dual of a tree.

22.1. Proposition. Let P be a dual tree (naturally labeled). Then

$$U(P) = \prod_{X\in P} \left(\underline{h}(X)\underline{)}^{-1} , \qquad (39)$$

where $h(X)$ is the number of elements $Y \in P$ satisfying $Y \leqslant X$.

Proof. Perhaps the most straightforward proof is by induction on $|P|$. The proposition is immediate if $|P| = 1$; clearly then

$U(P) = [\underline{1}]^{-1}$. If P is a dual tree of cardinality $p > 1$, then P has the form $(P_1+P_2+\ldots P_k) \oplus \underline{1}$, where P_1, P_2, ..., P_k are smaller dual trees. We have from Corollary 12.4(i) that for any finite ordered set Q of cardinality q, $U(Q\oplus\underline{1}) = [\underline{q+1}]^{-1}U(Q)$. Since $U(Q_1+Q_2) = U(Q_1)U(Q_2)$, there follows

$$U(P) \quad = \quad [\underline{p}]^{-1}U(P_1)U(P_2)\ldots U(P_k) \; .$$

But $p = h(1)$, where 1 denotes the top element of P, and the proof follows by induction. □

As a corollary, we see that if P is a tree or dual tree of cardinality p, then

$$e(P) \quad = \quad p!/\prod_{X\in P} h(X) \; ,$$

as was observed by Knuth [21, 5.2.4].

Consideration of (37) and (39) leads us to ask what finite labeled ordered sets (P,ω) have the property that for each $X \in P$ there is a number $h(X)$ "naturally associated" with X such that

$$U(P,\omega) \quad = \quad x^k \prod_{X\in P} [\underline{h(X)}]^{-1} \; ,$$

for an appropriate k. We call the numbers $h(X)$ the <u>generalized hook lengths</u> of (P,ω). Of course this concept is somewhat vague since we have not defined precisely the meaning of a "natural association" between X and $h(X)$. Note that if (P,ω) possesses generalized hook lengths $h(X)$, then by Corollary 10.2 so does $(P,\bar{\omega})$.

With a few "sporadic" exceptions, we know basically of two (and probably a third) classes of connected ordered sets possessing generalized hook lengths:

(a) dual trees, naturally or strictly labeled,

(b) $(P(\lambda),\omega)$, for appropriate labelings ω .

The third possible class is:

(c) duals of finite order ideals of $J(\underline{2} \times \underline{N})$, appropriately labeled.

There are strong reasons for believing that the ordered sets of (c) possess generalized hook lengths (cf. [21, 5.2.4, Ex. 21]), but we do not elaborate on this here.

23. Stacks and V-partitions. An n-stack, as defined by E.M. Wright [38], [39], and considered previously by F.C. Auluck [3], is a finite sequence of positive integers whose sum is n , which first increases to its largest term and then decreases to the last term. For instance, (1122) , (2211) , (1221) are 6-stacks while (1212) , (2121) , (2112) are not. We consider a slight modification of stacks which, from the standpoint of P-partitions, appears to be a more natural concept. A V-partition of n is an n-stack which is "rooted" at one of its terms of largest size. For instance, the stack (1221) can be rooted in one of two ways, viz., $(1\underline{2}21)$ or $(12\underline{2}1)$.

A V-partition of n is immediately seen to be equivalent to a P-partition of n , where $P = \underline{1} \oplus 2\underline{N}$ (whence the terminology "V-partition"). Thus a V-partition may also be regarded as a finite order ideal of the ordered set $(\underline{1} \oplus 2\underline{N}) \times \underline{N}$.

We first establish a relationship between stacks and V-partitions which enables all our results on V-partitions to be interpreted in terms of stacks.

23.1. <u>Proposition</u>. Let s_n be the number of n-stacks and v_n the number of V-partitions of n (with $s_0 = 0$, $v_0 = 1$) . Then

$$\sum_{n=0}^{\infty} (s_n+v_n)x^n = \prod_{i=1}^{\infty} (\underline{i})^{-2} . \tag{40}$$

<u>Proof</u>. The coefficient of x^n in the right-hand side of (40) is the number of $2\underline{N}$-partitions of n , i.e., the number of pairs of sequences $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_j > 0$ and $\mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant \mu_k > 0$ satisfying $\sum\lambda_i + \sum\mu_i = n$. For each such $2\underline{N}$-partition, we associate either an n-stack or a V-partition of n , as follows:

(a) Associate the n-stack

$$(\lambda_j,\lambda_{j-1},\ldots,\lambda_1,\mu_1,\mu_2,\ldots,\mu_k)$$

if $\lambda_1 > \mu_1$.

(b) Associate the V-partition of n

$$(\lambda_j,\lambda_{j-1},\ldots,\lambda_1,\underline{\mu_1},\mu_2,\ldots,\mu_k)$$

(rooted at μ_1) if $\lambda_1 \leqslant \mu_1$.

It is easily seen that in this correspondence, each n-stack and each V-partition of n will occur exactly once as we range over all $2\underline{N}$-partitions of n . From this the proof follows. □

We begin our study of the generating function for V-partitions by first obtaining expressions for the generating functions arising from a more general situation. Though direct combinatorial proofs can be given of many of the results below, we can save considerable effort by appealing to some of our previous results.

23.2. <u>Proposition</u>. Let

$$v_{sn}(x) = \sum_{i=0}^{s} (-1)^i x^{\binom{i}{2}} \underline{\begin{bmatrix} s \\ i \end{bmatrix}} \underline{(s-i+1)}^{n-1} \underline{(s-i+2)}^{n-1} \cdots \underline{(s)}^{n-1} .$$

Then

$$W_s(n\underline{N}) = v_{sn}(x)/\underline{(s)}!^n$$

<u>Proof</u>. Since $U_m(\underline{N}) = 1/\underline{(m)}!$, we have by Proposition 12.6(i) that $U_m(n\underline{N}) = 1/\underline{(m)}!^n$. The proof now follows from Proposition 8.4 (letting $p \to \infty$) . □

The next corollary gives some information about the functions $v_{sn}(x)$.

23.3. <u>Corollary</u>. If $n \geqslant 2$, then $v_{sn}(x)$ is an integral polynomial in x with the following properties:

(i) $\deg v_{sn}(x) = n\binom{s+1}{2}-s$,

(ii) the leading coefficient of $v_{sn}(x)$ is $(-1)^{sn}$,

(iii) the largest power of x dividing $v_{sn}(x)$ is

$$\sum_{i=1}^{s} \left(i+\left[\frac{i-1}{n-1}\right]\right) = \binom{s+1}{2} + \frac{1}{2}\left((2s+1-n)\left[\frac{s}{n-1}\right]-(n-1)\left[\frac{s}{n-1}\right]^2\right)$$

(brackets denote the integer part),

(iv) the coefficient of the power of x in (iii) is

$$\begin{bmatrix} n-1 \\ (n-1)\{\frac{s}{n-1}\} \end{bmatrix}$$

(braces denote the fractional part),

(v) $v_{sn}(1) = 1$.

<u>Proof</u>. That $v_{sn}(x)$ is an integral polynomial is immediate from Proposition 23.2.

(i) The degree of the i-th term of the sum in Proposition 23.2 is

$$n(\binom{s+1}{2} - \binom{s-i+1}{2}) - i .$$

When $n \geqslant 2$, this is a strictly increasing function of i , $0 \leqslant i \leqslant s$, so its maximum in this range occurs at $i = s$.

(ii) By Proposition 23.2, the coefficient of the leading power of x in the $i = s$ term is $(-1)^s(-1)^{s(n-1)} = (-1)^{sn}$.

(iii) By Proposition 17.3(iv), the largest power of x divid[ing] $W_s(n\underline{N})$ is $\sum i$, where the sum is over all $i \in m(n\underline{N},s)$. Now for the ordered set $n\underline{N}$, $\nu_j = n$ for $j \geqslant 0$. Thus, by the Extreme Value Theorem (Theorem 16.2), $m(n\underline{N},s)$ consists of the smallest elements of the set $\underline{N}-\{n,2n,3n,\ldots\}$. A straightforward computat[ion] now proves (iii).

(iv) Again by the Extreme Value Theorem, the coefficient in question is b_s (as defined in §16). This is easily seen to equa[l] the expression given in the statement of (iv).

(v) Immediate from Proposition 23.2. □

Corollary 23.3(iii) and (iv) illustrates the power of the Extreme Value Theorem. It appears to be difficult to prove these results directly from the definition of $v_{sn}(x)$ given in Proposition 23.2 (but see the remark following the proof of Proposition 23.7).

23.4. Corollary. We have

(i) $$\sum_{s=0}^{\infty} v_{sn}(x)q^s/[s]!^n = \prod_{i=0}^{\infty}(1-qx^i)\Big(\sum_{j=0}^{\infty} q^j/[j]!^n\Big) .$$

(ii) $$U(\underline{r} \oplus n\underline{N}) = \frac{1}{[r-1]!}\sum_{s=0}^{\infty} x^{rs}/[s]!^n$$

$$= \frac{1}{[1][2]\cdots}\sum_{s=0}^{\infty} v_{sn}(x)x^{rs}/[s]!^n .$$

Proof. (i) is a straightforward consequence of the definition of $v_{sn}(x)$ and the identity

$$\prod_{i=0}^{\infty}(1-qx^i) = \sum_{j=0}^{\infty}(-1)^j x^{\binom{j}{2}} q^j/[j]!$$

(ii) By Corollary 12.4(ii),

$$U(\underline{r} \oplus n\underline{N}) = \frac{\sum_{s=0}^{\infty} x^{rs}W_s(n\underline{N})}{[1][2]\cdots}$$

The proof now follows from (i) and Proposition 23.2. □

When $n = 2$, the expression for $v_{sn}(x)$ becomes particularly simple and leads to further consequences.

23.5. Proposition. We have $v_{s2}(x) = x^{s^2}$.

First Proof. Substitute $n = 2$ into Corollary 23.3(i) and (iii). We find $\deg v_{s2}(x)$ = largest power of x dividing $v_{s2}(x) = s^2$. By Corollary 23.3(v), $v_{s2}(1) = 1$, so $v_{s2}(x) = x^{s^2}$.

Second Proof. Let $p \to \infty$ and $q \to \infty$ in Corollary 12.8. $\square$

It is surprising that not only can $v_{s2}(x)$ be given explicitly, but also $\beta(2\underline{N};S)$ for any finite set $S \subset \underline{N}$. This can be used to give a third proof of Proposition 3.4, but we will omit this proof here. We first give the easy computation of $\alpha(n\underline{N};S)$ for any n (whose proof is left to the reader).

23.6. Lemma. Let $S = \{m_1,m_2,\ldots,m_s\}_< \subset \underline{N}$. Then

$$\alpha(n\underline{N};S) = \binom{m_1+n-1}{n-1}\binom{m_2-m_1+n-1}{n-1}\cdots\binom{m_s-m_{s-1}+n-1}{n-1}. \qquad \square$$

To evaluate $\beta(2\underline{N};S)$, consider the following general situation. Let $f(m,n)$ be any function of two variables, and for any finite sequence $m_0,m_1,\ldots,m_s$ define

$$F(m_0,m_1,\ldots,m_s) = \sum f(m_0,m_{i_1})f(m_{i_1},m_{i_2})\cdots f(m_{i_{t-1}},m_{i_t}), \qquad (41)$$

where the sum is over all sequences $1 \leqslant i_1 < i_2 < \cdots < i_t \leqslant s$, with a term 1 included when $t = 0$. Thus $F(m_0) = 1$, $F(m_0,m_1) = 1 + f(m_0,m_1)$, etc.

Separating the different values of $i_1 = k$ from (41), we obtain

$$F(m_0,\ldots,m_s) = 1 + \sum_{k=1}^{s} f(m_0,m_k)F(m_k,m_{k+1},\ldots,m_s)$$

$$= f(m_0,m_1)F(m_1,m_2,\ldots,m_s) + F(m_0,m_2,\ldots,m_s) \ . \qquad (42)$$

23.7. Proposition. Let $S = \{m_1,m_2,\ldots,m_s\}_< \subset \underline{N}$. Then

$$\beta(2\underline{N};S) = m_1(m_2-m_1-1)(m_3-m_2-1)\ldots(m_s-m_{s-1}-1)$$

Proof. By definition of β (§9) and Lemma 23.6,

$$\beta(2\underline{N};S) = \sum_{T\subseteq S} (-1)^{s-t}\alpha(2\underline{N};T) \qquad (s=|S| \ , \ t=|T|)$$

$$= \sum (-1)^{s-t}(m_{i_1}+1)(m_{i_2}-m_{i_1}+1)\ldots(m_{i_t}-m_{i_{t-1}}+1) \ ,$$

where the latter sum is over all sequences $1 \leqslant i_1 < \ldots < i_t \leqslant s$. Thus $(-1)^s\beta(2\underline{N};S)$ has the form $F(m_0,m_1,\ldots,m_s)$ of (41), where $m_0 = 0$ and $f(m,n) = -(n-m+1)$. Now by (42),

$$F(m_0,m_1,\ldots,m_s) = -(m_1-m_0+1)F(m_1,m_2,\ldots,m_s) + F(m_0,m_2,\ldots,m_s) \ .$$

This recursion, together with the initial condition $F(m_0) = 1$, uniquely determines $F(m_0,m_1,\ldots,m_s)$. A simple calculation shows that the function

$$(-1)^s(m_1-m_0)(m_2-m_1-1)\ldots(m_s-m_{s-1}-1)$$

satisfies the same recursion, and the proof follows after setting

$m_0 = 0$. □

It appears unlikely that $\beta(n\underline{N};S)$ can be given as explicitly as in Proposition 23.7 when $n > 2$. Similarly, the expression for $v_{sn}(x)$ does not seem to simplify when $n > 2$. George Andrews has conjectured that there is an alternative expression for $v_{sn}(x)$ which reduces to Proposition 23.5 when $n = 2$, and which makes Corollary 23.3 evident for all n .

23.8. Corollary. We have

$$W(\underline{r} \oplus 2\underline{N}) = \sum_{s=0}^{\infty} \frac{x^{s(s+r)}}{(\underline{s})!^2}$$

Proof. Immediate from Corollary 23.4 and Proposition 23.5.

Auluck [3] shows that if s_n is the number of n-stacks, then

$$\sum_{n=0} s_n x^n = \Big(\sum_{i=1}^{\infty} (-1)^{i-1} x^{\binom{i+1}{2}}\Big) / (\underline{1})^2 (\underline{2})^2 \cdots .$$

Proposition 23.1 shows that an analogous result holds for the generating function $U(\underline{1} \oplus 2\underline{N})$, i.e., that there is a simple expression for $U(\underline{1} \oplus 2\underline{N})(\underline{1})^2(\underline{2})^2\cdots$. We in fact state such a result for $U(\underline{r} \oplus 2\underline{N})(\underline{1})^2(\underline{2})^2\cdots$ (for any $r \in \underline{N}$) . For a proof (incl a purely combinatorial argument in the case $r = 1$) , see [33, Ch. V, §3].

23.8. Proposition. Let $T(x) = \sum_{i=0}^{\infty} (-1)^i x^{\binom{i+1}{2}}$. Then, for $r > 0$,

$$U(\underline{r} \oplus \underline{N}) (\underline{1})^2 (\underline{2})^2 \ldots \quad = \quad \frac{p_r(x)T(x)-q_r(x)}{(\underline{r-1})!}$$

where $p_r(x)$, $q_r(x)$ are polynomials, both satisfying the recursion

$$s_r(x) \quad = \quad 2s_{r-1}(x) - (\underline{r-2})s_{r-2}(x) \qquad (r > 2)$$

($s=p$ or $s=q$) , with the initial conditions

$$\begin{aligned} p_1(x) &= 1 & \qquad q_1(x) &= 0 \\ p_2(x) &= 2 & \qquad q_2(x) &= 1 . \end{aligned} \qquad \square$$

The polynomial $p_r(x)$ has been studied by Goldman and Rota [14] in another context, viz., $p_r(q)$ is the total number of subspaces of a vector space of dimension $r-1$ over the field $GF(q)$.

24. Protruded Partitions. We define a protruded partition of n to be a decreasing sequence of positive integers

$$\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_r > 0 ,$$

together with a sequence μ_1 , μ_2 , ... , μ_r of non-negative integers satisfying

$$0 \leqslant \mu_i \leqslant \lambda_i , \quad i=1,2,\ldots,r , \qquad (43)$$

such that $\sum\lambda_i + \sum\mu_i = n$. (The μ_i are protrusions of the ordinary partition $\lambda_1 \geqslant \ldots \geqslant \lambda_r$.) Hence a protruded partition of n is simply a K-partition of n , where K is the ordered set of Figure 4.

Some of the combinatorial properties of the ordered set K and the distributive lattice $J_f(K)$ are considered in [36]. We shall content ourselves here with the determination of the generati functions $W_s(K)$ and $U_m(K)$.

24.1. Proposition. We have

$$W_s(K) = x^{s(s+1)}/\underline{(s)}! \prod_{i=1}^{s} (1-x-x^{i+1}) .$$

Proof. Clearly $K \cong \underline{1} \oplus (\underline{1}+K)$. By Corollary 12.4(ii),

$$W_s(\underline{1} \oplus (\underline{1}+K)) = x^s W_s(\underline{1}+K) .$$

By Corollary 12.7 (letting $p \to \infty$) ,

$$W_s(\underline{1}+K) = (\frac{x^s}{1-x}) W_{s-1}(K) + \underline{\begin{bmatrix} s+1 \\ 1 \end{bmatrix}} W_s(K) .$$

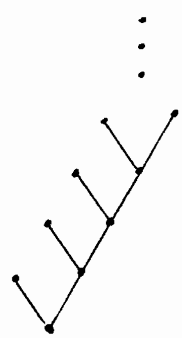

Figure 4

Hence

$$W_s(K) \;=\; \left(\frac{x^{2s}}{1-x}\right)W_{s-1}(K) \;+\; x^s \underline{\begin{bmatrix} s+1 \\ 1 \end{bmatrix}} W_s(K) \;,$$

so

$$W_s(K) \;=\; \frac{x^{2s}W_{s-1}(K)}{(1-x^s)(1-x-x^{s+1})} \;.$$

From this recursion and the initial condition $W_0(K) = 1$ we get the desired formula for $W_s(K)$. □

A virtually identical argument yields an expression for $W_s(K_n)$, where K_n is the ordered set satisfying $K_n \cong \underline{n} \oplus (\underline{1}+K_n)$. Similarly, one can obtain recursions for $W_s(P)$ when P satisfies such relations as $P \cong \underline{1} \oplus (\underline{2} \oplus P)$ or $P \cong \underline{1} \oplus 2P$. In these cases, however, it seems difficult to solve these recurrence relations for $W_s(P)$.

24.2. <u>Corollary</u>. We have

$$\sum_{m=0}^{\infty} U_m(K)q^m \;=\; P(q,x) \sum_{s=0}^{\infty} \frac{x^{s(s+1)}q^s}{\underline{(s)}!(1-x-x^2)(1-x-x^3)\ldots(1-x-x^{s+1})}$$

where $P(q,x) = 1/(1-q)(1-qx)(1-qx^2)\ldots$.

<u>Proof</u>. Immediate from Proposition 8.3. □

On the other hand, it is easy to give an explicit expression for $U_m(K)$.

24.3. <u>Proposition</u>. We have

$$U_m(K) \;=\; \prod_{j=1}^{m} (1-x^j-x^{j+1}-x^{j+2}-\ldots-x^{2j})^{-1} \;. \qquad (44)$$

Proof. A term x^{j+k} in the expansion of the product in (44) $(1 \leqslant j \leqslant m\ ,\ 0 \leqslant k \leqslant j\ )$ corresponds to the pair $j = \lambda_i\ ,\ k =$ in (43). □

Substituting the result of Proposition 24.3 into Proposition 24.2, we get an interesting combinatorial identity. It is possible to give a purely algebraic proof of this identity, but we do not do so here.

Note that as a special case of Proposition 24.3 we have $U_1(K) = 1/(1-x-x^2)$, the generating function for the Fibonacci numbers. It follows (from Proposition 17.1) that the number of order ideals of K of cardinality n is the n-th Fibonacci number. Some simpler proofs of this result are given in [36].

25. Permutations. Let M be a finite multiset of positive integers, say with λ_i copies of i , with $\sum\lambda_i = p$. A problem which has received considerable attention is that of enumerating permutations $(i_1,i_2,\ldots,i_p)$ of M according as to when $i_j >$ or $i_j \leqslant i_{j+1}$ (for each $j=1,2,\ldots,p-1)$. Of particular interest is the case when M is a set, i.e., when each $\lambda_i = 0$ or 1 . natural setting for this problem is that of (P,ω)-partitions where $P = \underline{\lambda_1} + \underline{\lambda_2} + \ldots$ and ω is natural. The permutations $\pi = (\omega(X_{i_1}),\omega(X_{i_2}),\ldots,\omega(X_{i_p}))$ in the separator $\mathcal{L}(P)$ correspond canonically to permutations $\pi' = (i_1',i_2',\ldots,i_p')$ of M by replacing the entry $\omega(X_{i_j})$ of $\mathcal{L}(P)$ by the symbol k , where $X_{i_j} \in \underline{\lambda_k}$

In this correspondence, $\omega(X_{i_j}) > \omega(X_{i_{j+1}})$ if and only if $i'_j > i'_{j+1}$. In particular, the index $\operatorname{ind}(\pi)$ (defined at the end of §6) is identical to the "greater index" (as defined by MacMahon [23, §104]) $\operatorname{ind}(\pi')$ of the permutation π' of M . Now by iterating the relation (24), we get

$$W(\underline{\lambda_1}+\underline{\lambda_2}+\cdots) = \frac{\underline{(p)}!}{\underline{(\lambda_1)}!\,\underline{(\lambda_2)}!\cdots}$$

Hence

$$\sum_{\pi'} x^{\operatorname{ind}(\pi')} = \frac{\underline{(p)}!}{\underline{(\lambda_1)}!\,\underline{(\lambda_2)}!\cdots}\,, \tag{45}$$

where the sum is over all permutations π' of the multiset M . This remarkable formula was first proved by MacMahon [24]. Similarly, when ω is a strict labeling, the number $\operatorname{ind}(\pi)$ equals the sum of MacMahon's greater and equal indices. Thus the formula of [23, §108] may be regarded as a special case of Corollary 10.2.

Later MacMahon [25] showed that the coefficient of x^n in (45) is also the number of permutations π' of M with n <u>inversions</u>, i.e., with n pairs (i,j) such that $i > j$ and i appears before j in π' . This result was rediscovered by Carlitz [7] and by Knuth [21, vol. III, §5.2.2, ex. 16]. A further proof was given by Abramson [1]. A combinatorial correspondence between permutations of M with n inversions and those with (greater) index n was given by Foata [10].

We have already mentioned in §13 how the P-Eulerian numbers

$w_s(\lambda_{\underline{1}} + \lambda_{\underline{2}} + \ldots)$ are equivalent to MacMahon's invariants $N_{s,\lambda}$ a reduce to the classical Eulerian numbers when each $\lambda_i = 1$. Combinatorially, the numbers $w_s(\lambda_{\underline{1}} + \lambda_{\underline{2}} + \ldots)$ count the number of permutations $(i_1, i_2, \ldots, i_p)$ of M with exactly s descents $i_j > i_{j+1}$. We can ask more specifically for the number of permutations of M with descents precisely after $i_{m_1}, i_{m_2}, \ldots, i_{m_s}$. Clea this number is $\beta(\lambda_{\underline{1}} + \lambda_{\underline{2}} + \ldots ; S)$, where $S = \{m_1, m_2, \ldots\}$.

MacMahon [23, §§178-181] was the first person to study these numbe His notation was as follows: Let λ be the partition of p into parts $\lambda_1, \lambda_2, \ldots,$ and let μ be an <u>ordered</u> partition (compositio $\mu_1 + \ldots + \mu_{s+1} = p$. Then MacMahon's invariant $N(\mu)_\lambda$ is our $\beta(\lambda_{\underline{1}} + \lambda_{\underline{2}} + \ldots ; S)$, where $S = \{\mu_1, \mu_1+\mu_2, \ldots, \mu_1+\mu_2+\ldots+\mu_s\}$.

The special case where each $\lambda_i = 1$ (so $P=p\underline{1}$) is related t the theory of <u>compositions</u> (or <u>ordered partitions</u>), since a $p\underline{1}$-partition is just a composition into p non-negative parts. Mac-Mahon [23, vol. I, p. 190] showed that

$$\beta(p\underline{1};S) = p!\,\left|1/(m_j - m_{i-1})!\right|_1^{s+1}, \tag{46}$$

where $S = \{m_1, \ldots, m_s\}_< \subseteq \underline{p-1}$, with the usual conventions $m_0 = 0$, $m_{s+1} = p$, $0! = 1$, $1/(-k)! = 0$ if $k > 0$. If this determinant for $\beta(p\underline{1};S)$ is expanded, it simply takes the form $\sum_{T \subseteq S} (-1)^{|S-T|} \alpha(p\underline{1};T)$, using the obvious fact that $\alpha(p\underline{1};T)$ is the multinomial coefficient $p!/n_1!(n_2-n_1)!(n_3-n_2)!\dots(p-n_t)!$, where $T = \{n_1,\dots,n_t\}_<$. MacMahon's result was rediscovered (in a slightly different form) by Niven [26] and further studied by De Bruijn [9]. In general, the numbers $\alpha(\lambda_{\underline{1}}+\lambda_{\underline{2}}+\dots;S)$ are the multiset analogues of the multinomial coefficients. No simple formula for them is known (though there are some recurrence relations and some "messy" formulas), and no formula for $\beta(\lambda_{\underline{1}}+\lambda_{\underline{2}}+\dots;S)$ is known analogous to MacMahon's formula (46) for $\beta(p\underline{1};S)$.

Let us now consider the numbers $\beta(p\underline{1};S)$ in more detail. If $S = \{2j+1 \mid 1 \leqslant 2j+1 \leqslant p-1\}$, then the number $t_p = \beta(p\underline{1};S)$ counts the number of "alternating permutations" of $\underline{p}$, i.e., permutations $i_1,i_2,\dots,i_p$ satisfying $i_1 > i_2 < i_3 > i_4 < \dots$. The numbers t_p are known as the *Euler numbers* (not to be confused with the Eulerian numbers of §13). André [2] derived the generating function

$$\sum_{p=0}^{\infty} \frac{t_p\, x^p}{p!} = \tan x + \sec x .$$

For this reason one also calls t_{2i+1} a *tangent number* and t_{2i} a *secant number*. For further properties of these numbers, see Foata-Schützenberger [11] and the references quoted there.

Foata and Schützenberger have elsewhere [12] considered the problem of "refining" the numbers t_p , i.e., of expressing t_p as the sum of combinatorially significant numbers. We will give a refinement of the numbers $\beta(p\underline{1};S)$ for any $S \subseteq \underline{p-1}$. This refinement differs from that of Foata-Schützenberger in the case $S = \{1,3,5,\ldots\}$.

Given any permutation $\pi = (i_1,i_2,\ldots,i_p)$ of $(1,2,\ldots,p)$, define $Z(\pi)$ to be ordered set with elements $X_1,X_2,\ldots,X_p$ generated by the relations

$$\left.\begin{array}{l} X_j < X_{j+1} \quad \text{if} \quad i_j < i_{j+1} \\ X_j > X_{j+1} \quad \text{if} \quad i_j > i_j{+}1 \end{array}\right. \qquad (j=1,2,\ldots,p-1)$$

Hence the Hasse diagram of $Z(\pi)$ is a "zig-zag," i.e., a tree wit only two end-points. Note that $\sigma\colon Z(\pi) \to \underline{p}$ is an order-preservi bijection if and only if the permutation $\pi' = (\sigma(X_1),\sigma(X_2),\ldots,\sigma($ has its descents in exactly the same places as does π , i.e., $\mathcal{D}(\pi') = \mathcal{D}(\pi)$. From this we get the following result.

25.1. <u>Proposition</u>. Let $T \subseteq \underline{p-1}$, and let π be any perm tion of $1,2,\ldots,p$ satisfying $\mathcal{D}(\pi) = T$. Let $N(T)$ be the to number of permutations π' of $1,2,\ldots,p$ satisfying $\mathcal{D}(\pi') = T$

Then $N(T) = \beta(p\underline{1};T) = e(Z(\pi))$. □

Hence, if ω is any labeling of $Z(\pi)$, then the $(Z(\pi),\omega)$ Eulerian numbers w_s (defined in §13) give a refinement of $N(T)$ i.e., $w_0 + w_1 + \ldots + w_{p-1} = N(T)$ (see (25)). A more discrimi refinement is given by the coefficients of the polynomials $W_s(Z($ or even the numbers $\beta(Z(\pi),\omega;S)$.

Two labelings ω of possible special interest are (a) natu labelings, and (b) the labeling $\omega(X_i) = i$. Let us call this la labeling ω the <u>linear labeling</u> of $Z(\pi)$. It is easy to see that the labeled ordered set $(Z(\pi),\omega)$ (where ω is linear) is a special case of the ordered sets $(P(\lambda/\mu),\ )$ considered in §21. From this follows several interesting properties of $(Z(\pi),\omega)$. For instance, from Proposition 21.1 we have that $(Z(\pi),\omega)$ is α-symmetric. Moreover, from Proposition 21.2 there follows

$$U_m(Z(\pi),\omega) \quad = \quad \left|\left|\begin{bmatrix} m+m_j-m_{i-1} \\ m \end{bmatrix}\right|\right|_1^{s+1} \qquad (47)$$

where $\mathcal{S}(\pi) = \{m_1,m_2,\ldots,m_s\}_<$ (with $m_0 = 0$, $m_{s+1} = p$) . Equation (47) is a generalization of MacMahon's formula (46) for $e(Z(\pi))$. Indeed, if in (47) we let $m \to \infty$, multiply by $\underline{[p]}!$, and set $x = 1$, we get (46).

Actually, the entire theory of $(Z(\pi),\omega)$-partitions (where ω is linear) was anticipated by MacMahon in his study of compositions. For instance, the theorem of Aitken from which Proposition 21.2 follows is given in the special case $P(\lambda/\mu) = Z(\pi)$ by MacMahon in [23, §168]. Moreover, MacMahon's "zig-zag graphs" [23, 129] are simply a way of representing $Z(\pi)$. For a connection between these considerations and the representation theory of the symmetric group, see Solomon [31, §6].

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