Smith Normal Form and Combinatorics

Richard P. Stanley

Smith normal form

A: $n \times n$ matrix over commutative ring ***R*** (with 1)

Suppose there exist ***P***, ***Q*** $\in \mathrm{GL}(n, R)$ such that

$$PAQ := B = \mathrm{diag}(d_1, d_1 d_2, \ldots, d_1 d_2 \cdots d_n),$$

where $d_i \in R$. We then call B a **Smith normal form (SNF)** of A.

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NOTE. (1) Can extend to $m \times n$.

$$(2)\ \mathrm{unit} \cdot \det(A) = \det(B) = d_1^n d_2^{n-1} \cdots d_n.$$

Thus SNF is a refinement of $\det$.

Row and column operations

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- Add a multiple of a row to another row.
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Over a field, SNF is **row reduced echelon form** (with all unit entries equal to 1).

Existence of SNF

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If R is a PIR then A has a unique SNF up to units.

Otherwise A "typically" does not have a SNF but may have one in special cases.

Algebraic note

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Example. ring of entire functions and ring of all algebraic integers (not PIR's)

Open: every matrix over a Bézout domain has an SNF.

Algebraic interpretation of SNF

R: a PID

A: an $n \times n$ matrix over R with rows $v_1, \dots, v_n \in R^n$

$\mathrm{diag}(e_1, e_2, \dots, e_n)$: SNF of A

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$R^n/(v_1, \dots, v_n)$: **(Kasteleyn) cokernel** of A

An explicit formula for SNF

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Theorem. $e_1 e_2 \cdots e_i$ *is the gcd of all* $i \times i$ *minors of* A*.*

minor: determinant of a square submatrix.

Special case: e_1 is the gcd of all entries of A.

An example

Reduced Laplacian matrix of K_4:

$$A = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$$

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What about SNF?

An example (continued)

$$\begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & -1 \\ -4 & 4 & -1 \\ 8 & -4 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & -1 \\ -4 & 4 & 0 \\ 8 & -4 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 0 & 0 & -1 \\ 0 & 4 & 0 \\ 4 & -4 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & -1 \\ 0 & 4 & 0 \\ 4 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Reduced Laplacian matrix of K_n

$$\begin{aligned} \boldsymbol{L_0(K_n)} &= nI_{n-1} - J_{n-1} \\ \det L_0(K_n) &= n^{n-2} \end{aligned}$$

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Trick: 2×2 submatrices (up to row and column permutations):

$$\begin{bmatrix} n-1 & -1 \\ -1 & n-1 \end{bmatrix}, \quad \begin{bmatrix} n-1 & -1 \\ -1 & -1 \end{bmatrix}, \quad \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix},$$

with determinants $n(n-2)$, $-n$, and 0. Hence $e_1 e_2 = n$. Since $\prod e_i = n^{n-2}$ and $e_i | e_{i+1}$, we get the SNF $\text{diag}(1, n, n, \ldots, n)$. ∎

Laplacian matrices of general graphs

SNF of the Laplacian matrix of a graph: very interesting

connections with sandpile models, chip firing, abelian avalanches, etc.

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SNF of random matrices

Huge literature on random matrices, mostly connected with eigenvalues.

Very little work on SNF of random matrices over a PID.

Is the question interesting?

$\mathbf{Mat}_k(n)$: all $n \times n$ $\mathbb{Z}$-matrices with entries in $[-k, k]$ (uniform distribution)

$p_k(n, d)$: probability that if $M \in \mathrm{Mat}_k(n)$ and SNF$(M) = (e_1, \ldots, e_n)$, then $e_1 = d$.

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Theorem. $\lim_{k\to\infty} p_k(n, d) = \dfrac{1}{d^{n^2}\zeta(n^2)}$

Specifying some e_i

with **Yinghui Wang**

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Two general results.

- Let $\alpha_1,\dots,\alpha_{n-1}\in\mathbb{P}$, $\alpha_i|\alpha_{i+1}$.

  $\boldsymbol{\mu_k(n)}$: probability that the SNF of a random $A\in\mathrm{Mat}_k(n)$ satisfies $e_i=\alpha_i$ for $1\leq\alpha_i\leq n-1$.

$$\boldsymbol{\mu(n)}=\lim_{k\to\infty}\mu_k(n).$$

  Then $\mu(n)$ exists, and $0<\mu(n)<1$.

Second result

- Let $\alpha_n \in \mathbb{P}$.

  $\boldsymbol{\nu_k(n)}$: probability that the SNF of a random $A \in \mathrm{Mat}_k(n)$ satisfies $e_n = \alpha_n$.

  Then
$$\lim_{k\to\infty} \nu_k(n) = 0.$$

Sample result

$\boldsymbol{\mu_k(n)}$: probability that the SNF of a random $A \in \mathrm{Mat}_k(n)$ satisfies $e_1 = 2$, $e_2 = 6$.

$$\boldsymbol{\mu(n)} = \lim_{k\to\infty} \mu_k(n).$$

Conclusion

$$
\begin{aligned}
\mu(n) \;&=\; 2^{-n^2}\left(1-\sum_{i=(n-1)^2}^{n(n-1)} 2^{-i} + \sum_{i=n(n-1)+1}^{n^2-1} 2^{-i}\right)\\
&\cdot\frac{3}{2}\cdot 3^{-(n-1)^2}(1-3^{(n-1)^2})(1-3^{-n})^2\\
&\cdot\prod_{p>3}\left(1-\sum_{i=(n-1)^2}^{n(n-1)} p^{-i} + \sum_{i=n(n-1)+1}^{n^2-1} p^{-i}\right).
\end{aligned}
$$

Cyclic cokernel

$\boldsymbol{\kappa(n)}$: probability that an $n \times n$ $\mathbb{Z}$-matrix has SNF diag$(e_1, e_2, \ldots, e_n)$ with $e_1 = e_2 = \cdots = e_{n-1} = 1$

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Theorem. $$\kappa(n) = \frac{\prod_p \left(1 + \frac{1}{p^2} + \frac{1}{p^3} + \cdots + \frac{1}{p^n}\right)}{\zeta(2)\zeta(3)\cdots}$$

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Corollary.
$$\lim_{n\to\infty} \kappa(n) = \frac{1}{\zeta(6)\prod_{j\geq 4}\zeta(j)}$$
$$\approx 0.846936\cdots .$$

Small number of generators

g: number of generators of cokernel (number of entries of SNF $\neq 1$) as $n \to \infty$

previous slide: $\mathrm{Prob}(g = 1) = 0.846936 \cdots$

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Theorem. $\text{Prob}(g \leq \ell) =$

$$1 - (3.46275\cdots)2^{-(\ell+1)^2}(1 + O(2^{-\ell}))$$

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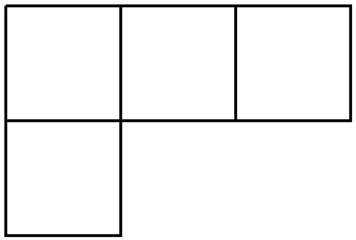
$3.46275\cdots$

$$3.46275\cdots = \frac{1}{\prod_{j\geq 1}\left(1-\frac{1}{2^j}\right)}$$

Example of SNF computation

$\boldsymbol{\lambda}$: a partition $(\lambda_1, \lambda_2, \dots)$, identified with its Young diagram



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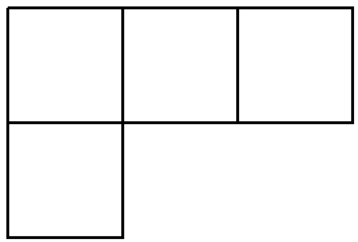

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λ*: λ extended by a border strip along its entire boundary

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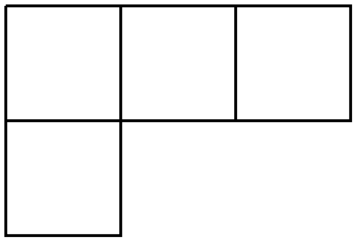

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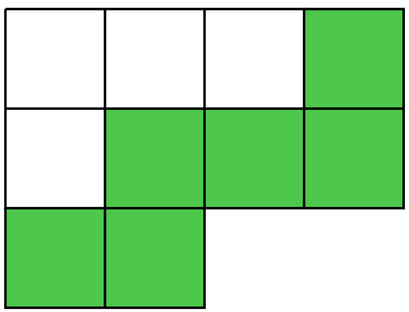
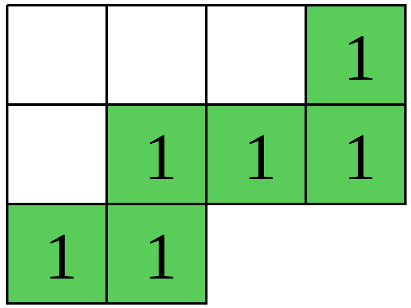


(3,1)* = (4,4,2)

Initialization

Insert 1 into each square of λ^*/λ.



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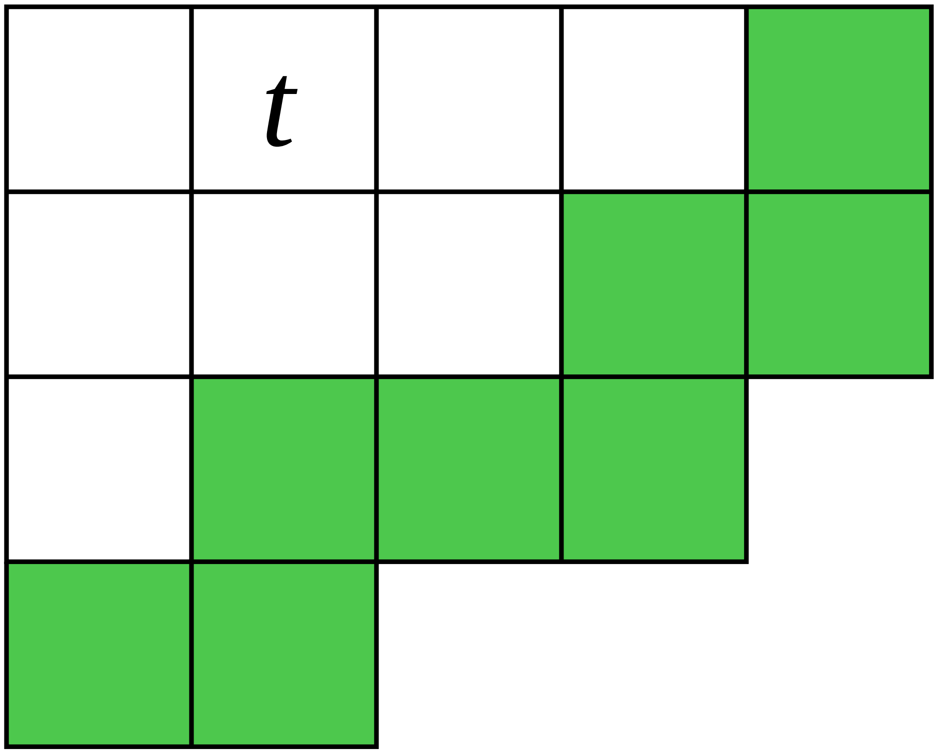

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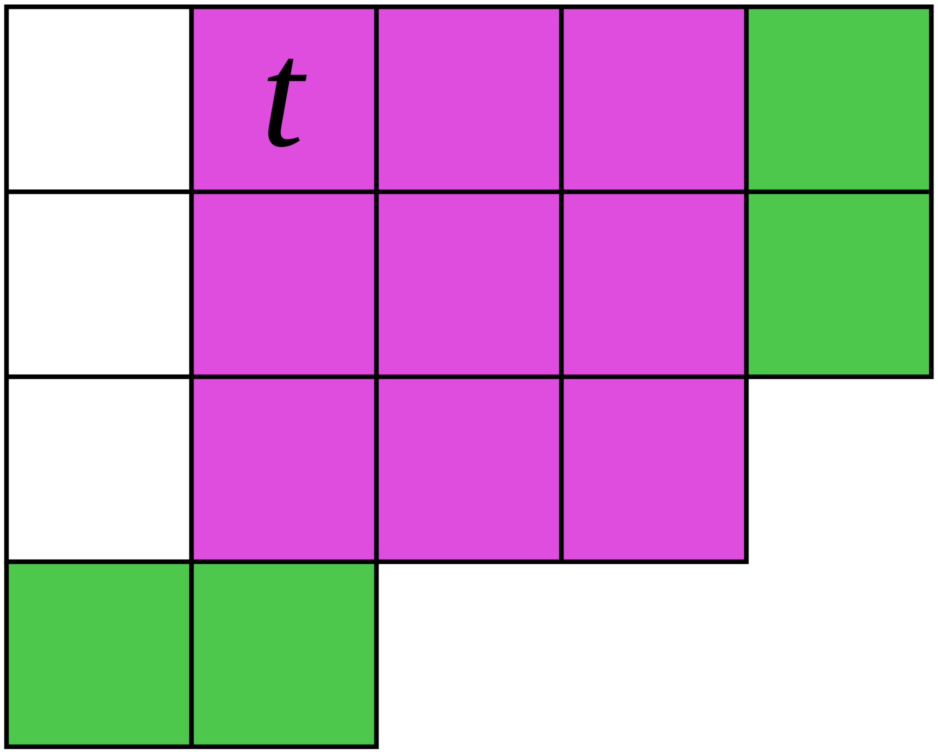

Determinantal algorithm

Suppose all squares to the southeast of t have been filled. Insert into t the number $\boldsymbol{n_t}$ so that $\det M_t = 1$.

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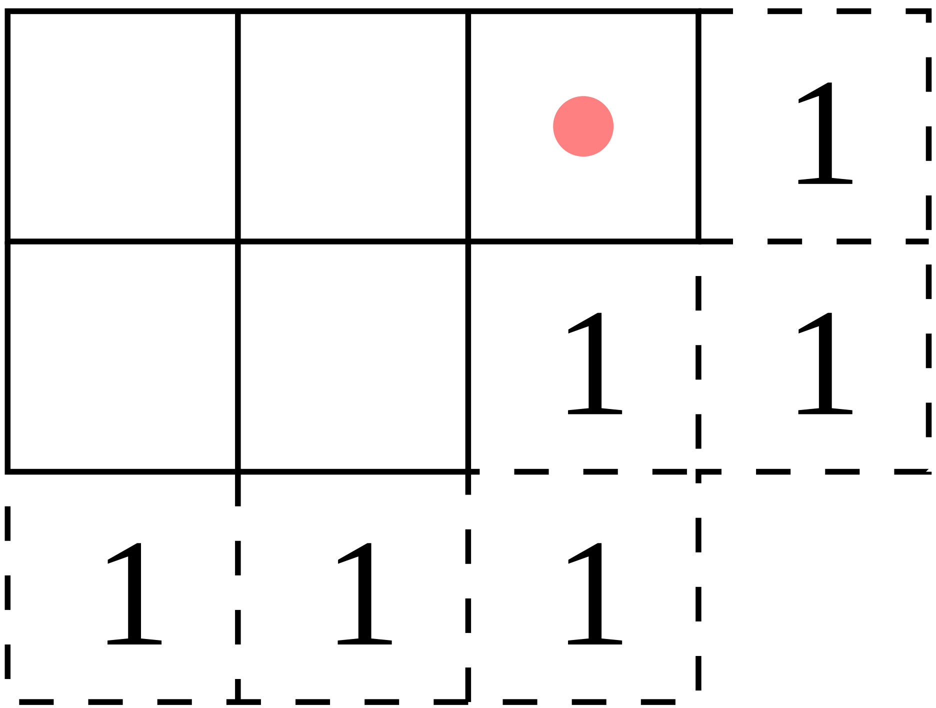

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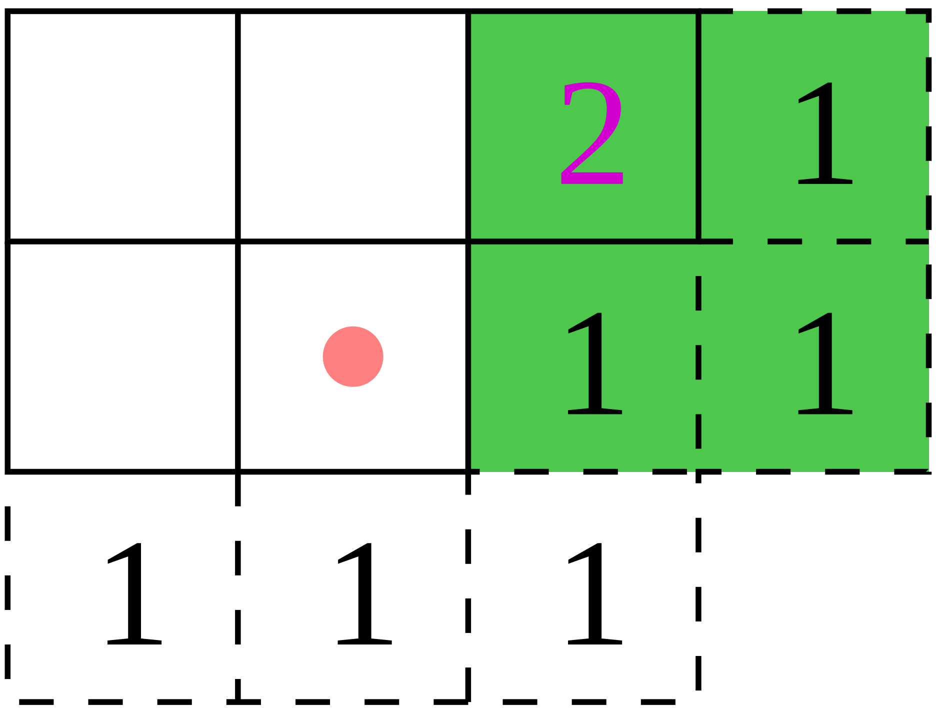

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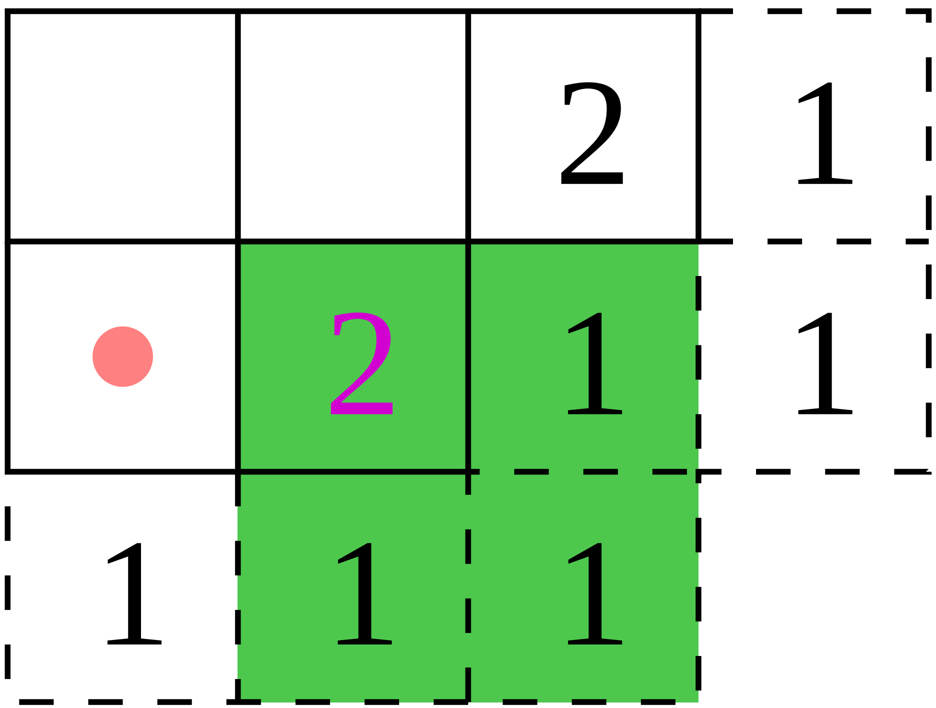

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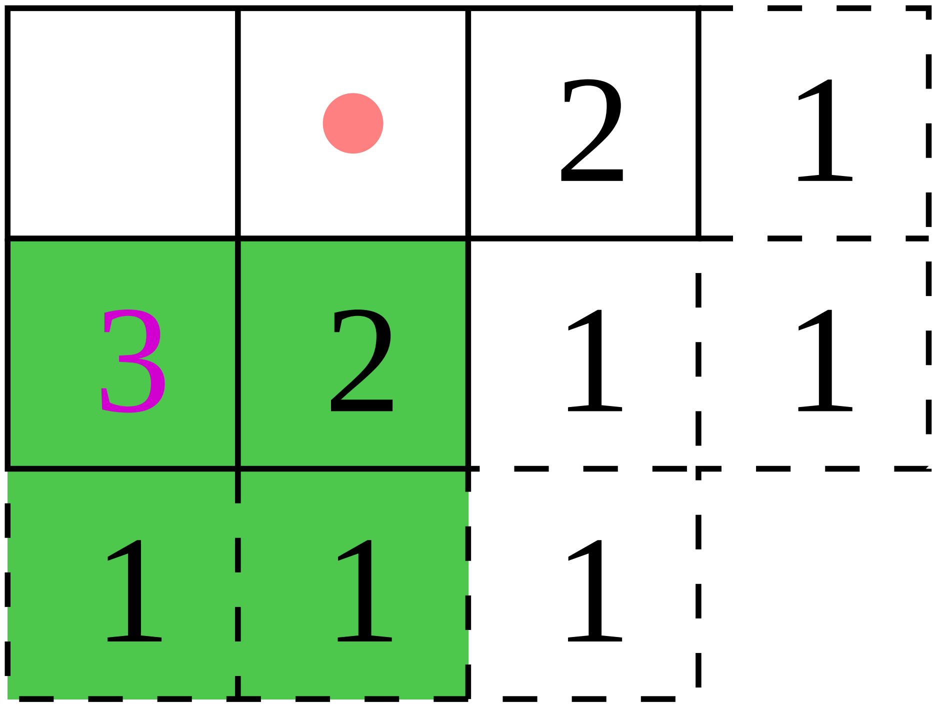

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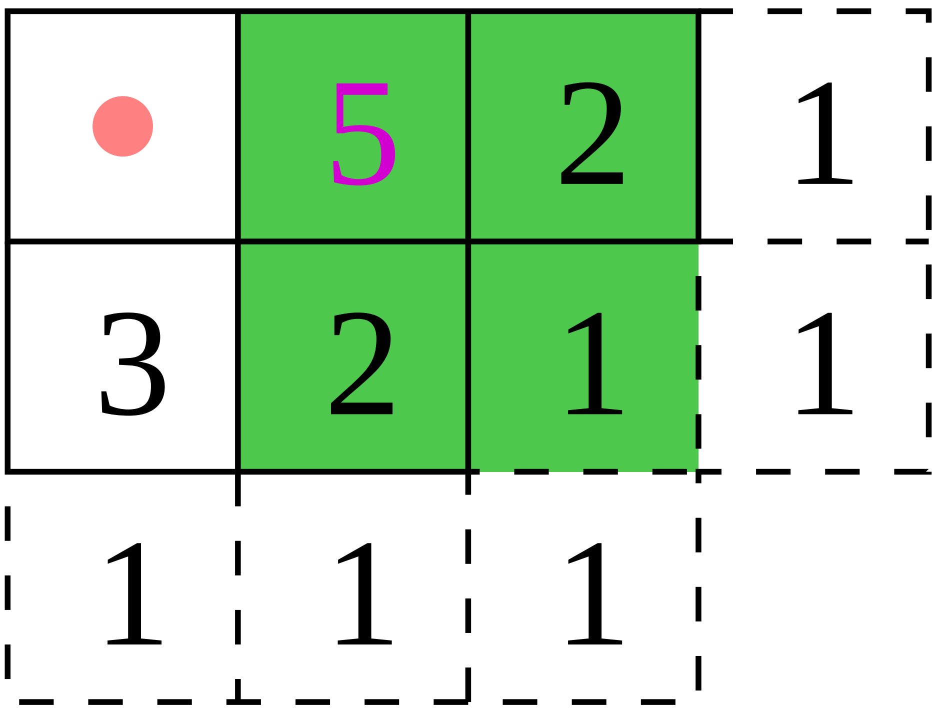

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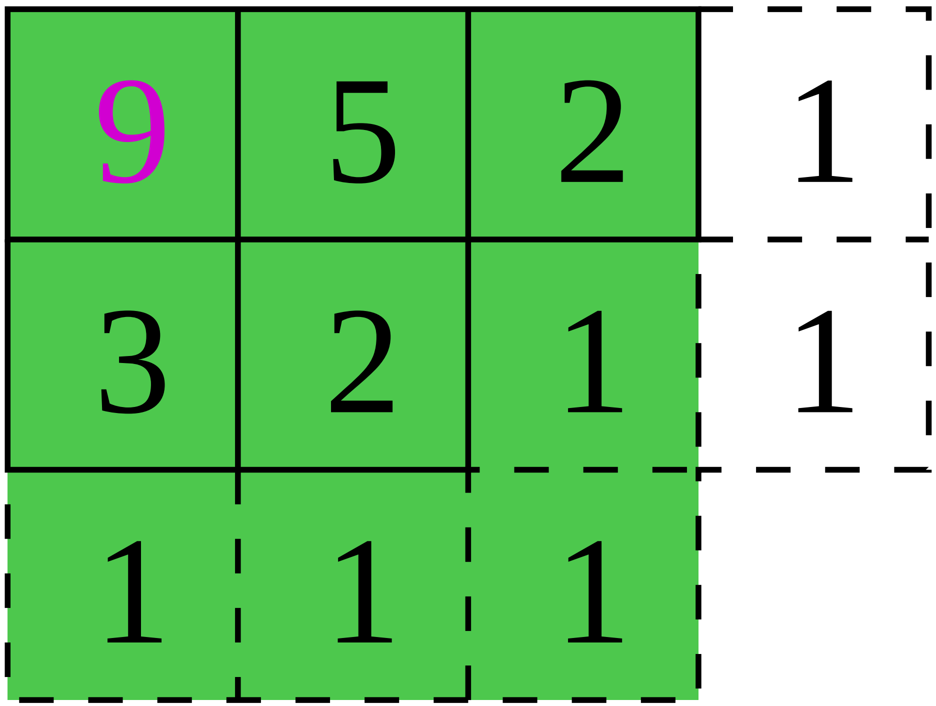

Uniqueness

Easy to see: the numbers n_t are well-defined and unique.

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Why? Expand $\det M_t$ by the first row. The coefficient of n_t is 1 by induction.

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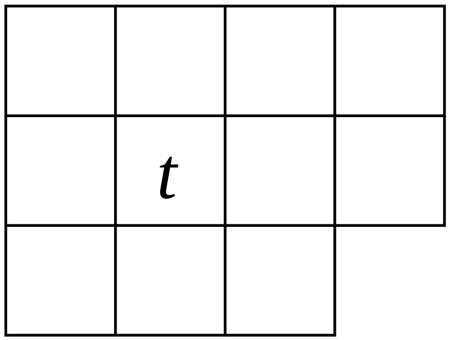

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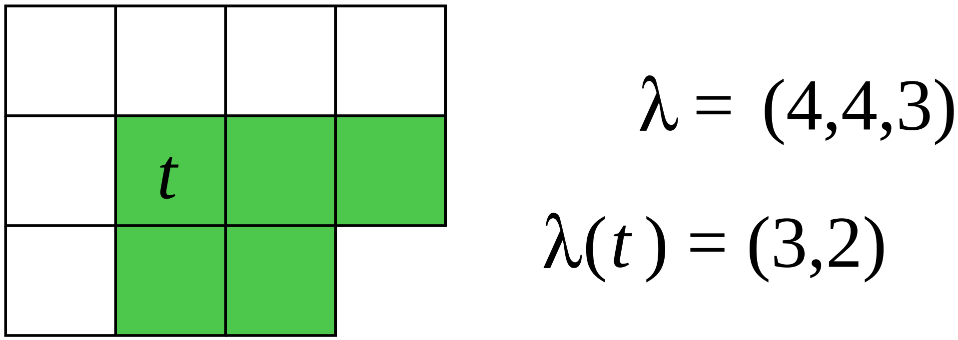

u_λ

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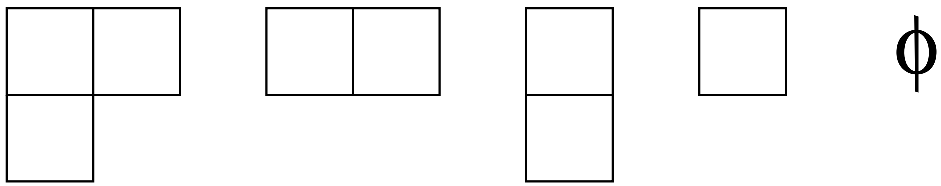
Example. $u_{(2,1)} = 5$:



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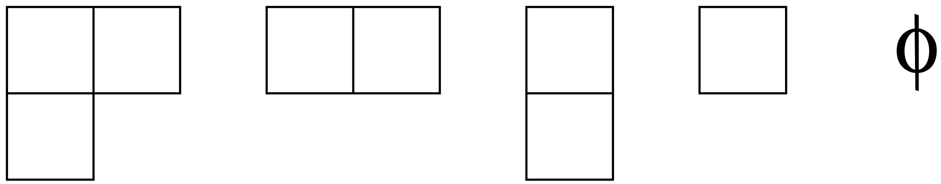

There is a determinantal formula for u_λ, due essentially to **MacMahon** and later **Kreweras** (not needed here).

Carlitz-Scoville-Roselle theorem

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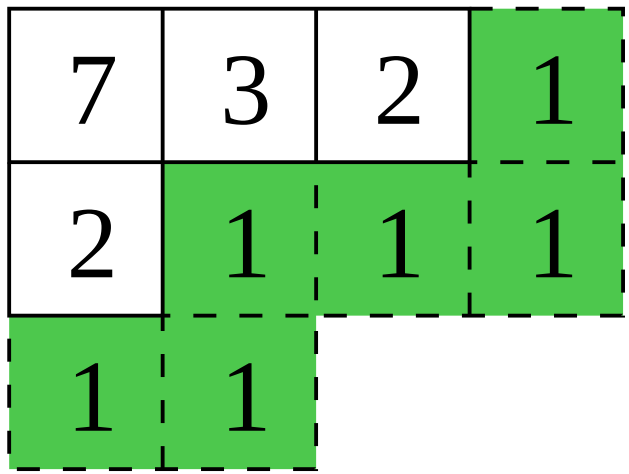
Proofs. 1. Induction (row and column operations).

2. Nonintersecting lattice paths.

$\square$

An example



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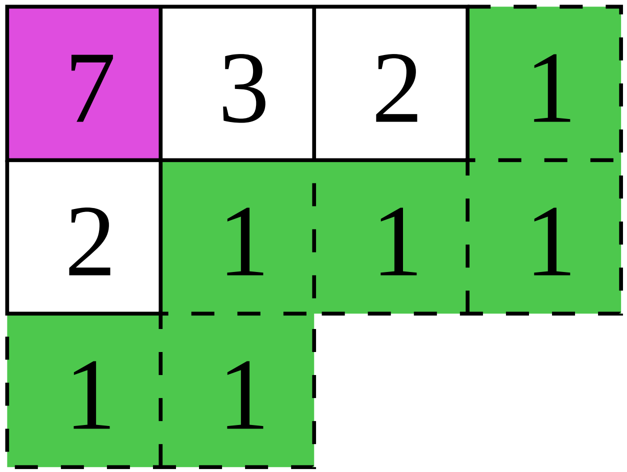

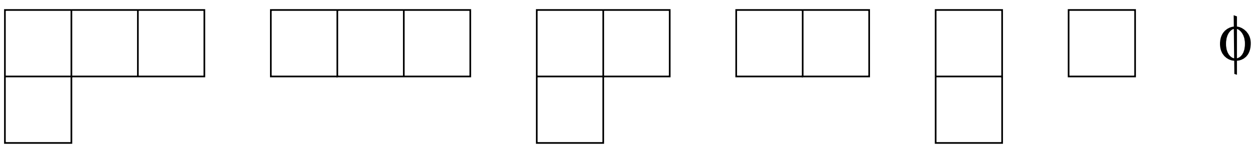

A q-analogue

Weight each $\mu \subseteq \lambda$ by $q^{|\lambda/\mu|}$.

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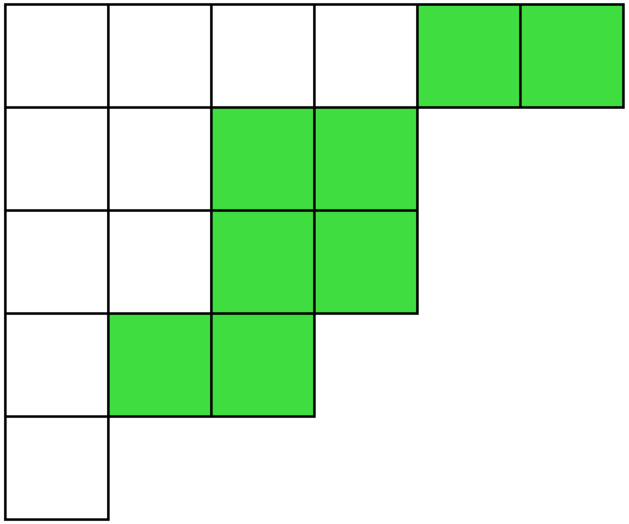
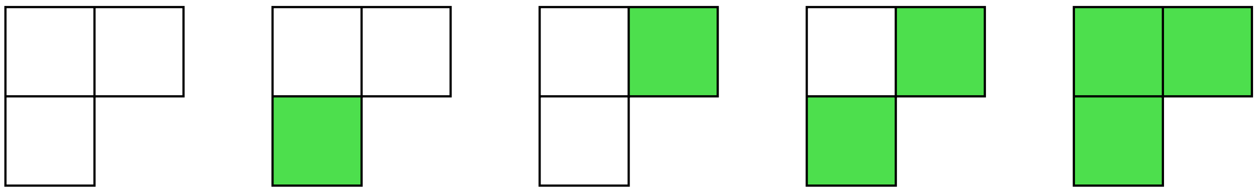


$$\lambda = 64431, \quad \mu = 42211, \quad q^{\lambda/\mu} = q^8$$

$u_\lambda(q)$

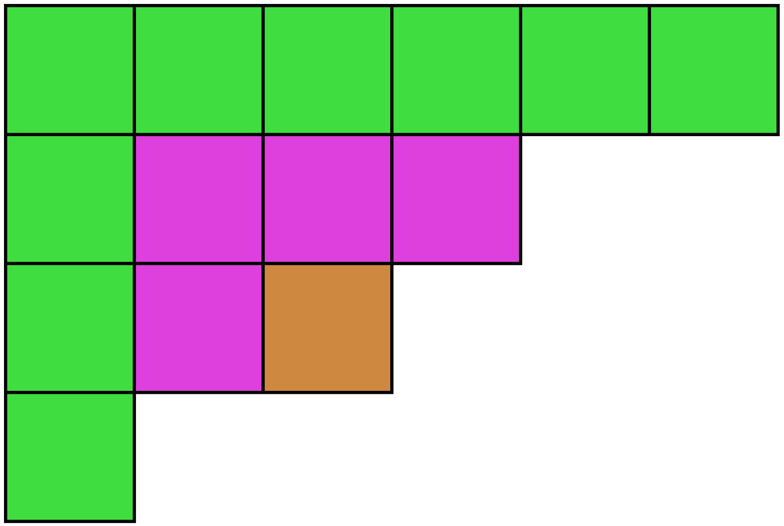
$$u_\lambda(q) = \sum_{\mu \subseteq \lambda} q^{|\lambda/\mu|}$$

$$u_{(2,1)}(q) = 1 + 2q + q^2 + q^3 :$$



Diagonal hooks

$$d_i(\lambda) = \lambda_i + \lambda'_i - 2i + 1$$



$$d_1 = 9, \quad d_2 = 4, \; d_3 = 1$$

Main result (with C. Bessenrodt)

Theorem. *M_t has an SNF over $\mathbb{Z}[q]$. Write $d_i = d_i(\lambda_t)$. If M_t is a $(k+1)\times(k+1)$ matrix then M_t has SNF*

$$\mathrm{diag}(1, q^{d_k}, q^{d_{k-1}+d_k}, \dots, q^{d_1+d_2+\cdots+d_k}).$$

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Theorem. *M_t has an SNF over $\mathbb{Z}[q]$. Write $d_i = d_i(\lambda_t)$. If M_t is a $(k+1)\times(k+1)$ matrix then M_t has SNF*

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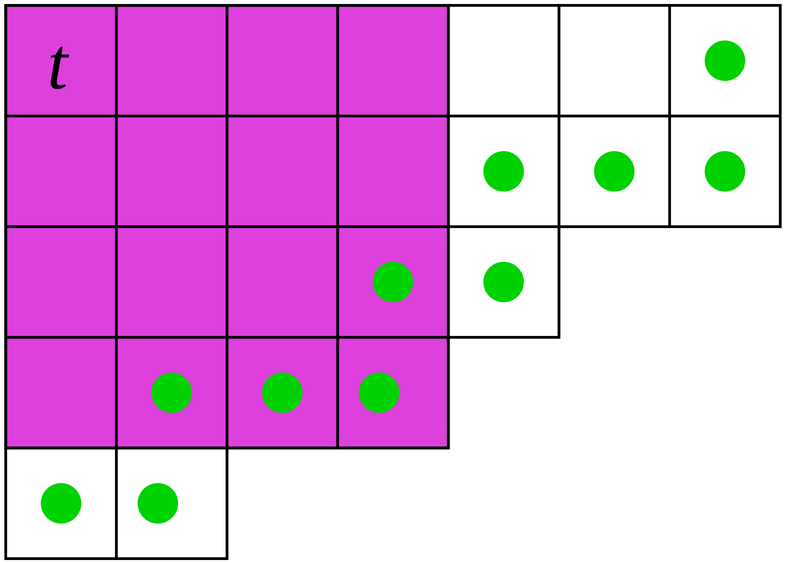
Note. There is a multivariate generalization.

An example



$$\lambda = 6431, \quad d_1 = 9, \quad d_2 = 4, \quad d_3 = 1$$

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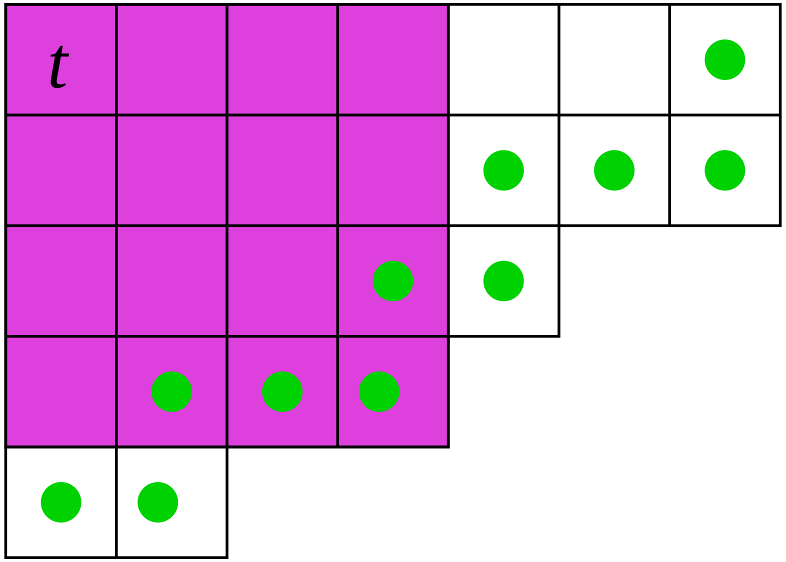

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$$\text{SNF of } M_t: \ (1, q, q^5, q^{14})$$

A special case

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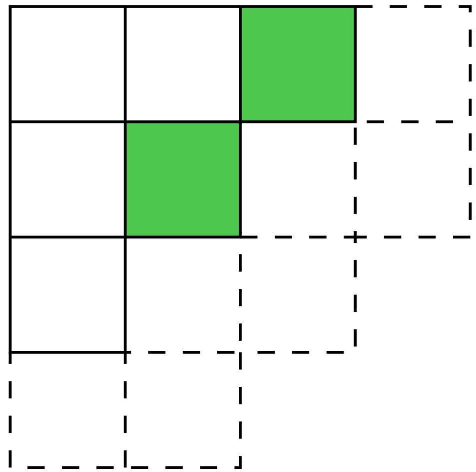

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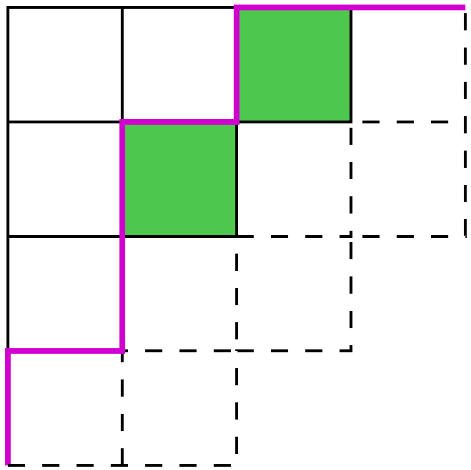
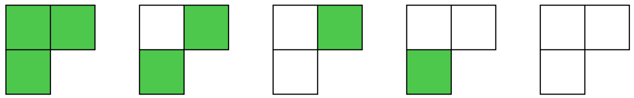


$u_{\delta_{n-1}}(q)$ counts Dyck paths of length $2n$ by (scaled) area, and is thus the well-known q-analogue $C_n(q)$ of the Catalan number C_n.

A q-Catalan example



$$C_3(q) = q^3 + q^2 + 2q + 1$$

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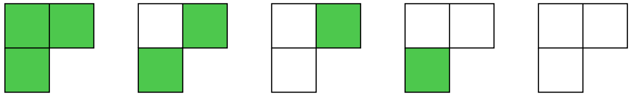

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$$\begin{vmatrix} C_4(q) & C_3(q) & 1+q \\ C_3(q) & 1+q & 1 \\ 1+q & 1 & 1 \end{vmatrix} \overset{\text{SNF}}{\sim} \operatorname{diag}(1, q, q^6)$$

since $d_1(3,2,1) = 1$, $d_2(3,2,1) = 5$.

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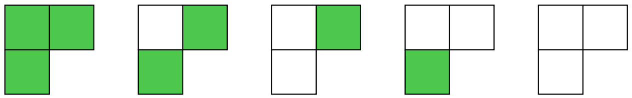

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- q-Catalan determinant previously known
- SNF is new

Ramanujan

$\sum_{n\geq 0} C_n(q)x^n =$

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Ramanujan

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THE END