S_n: $\{$permutations of $1, \ldots, n\}$

(i,j): the transposition $i \leftrightarrow j$ acting on **positions**, e.g. $(1,2)132 = 312$.

Let $w = a_1 a_2 \cdots a_n \in S_n$. Define

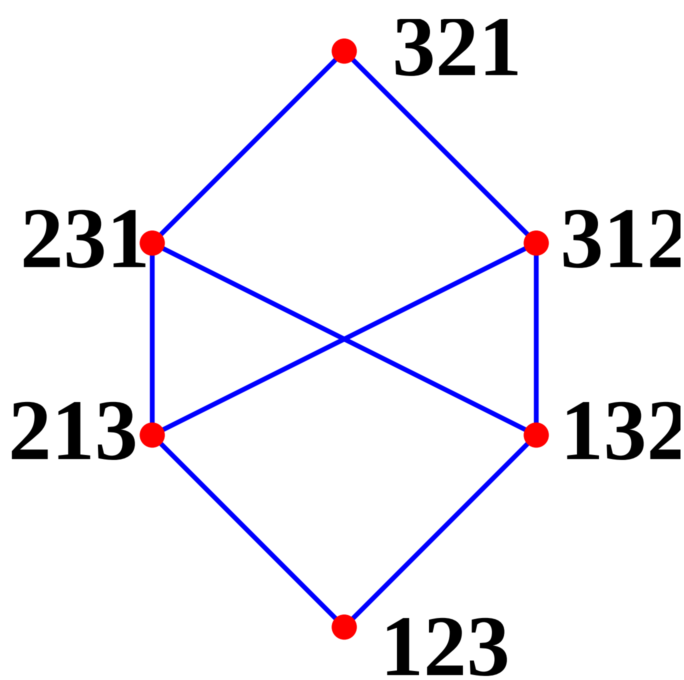
$$\boldsymbol{\ell(w)} = \#\{(i,j) \,:\, i < j,\ a_i > a_j\}.$$

Define a partial order $\leq$ on S_n, called (strong) **Bruhat order**, to be the transitive and reflexive ($w \leq w$) closure of

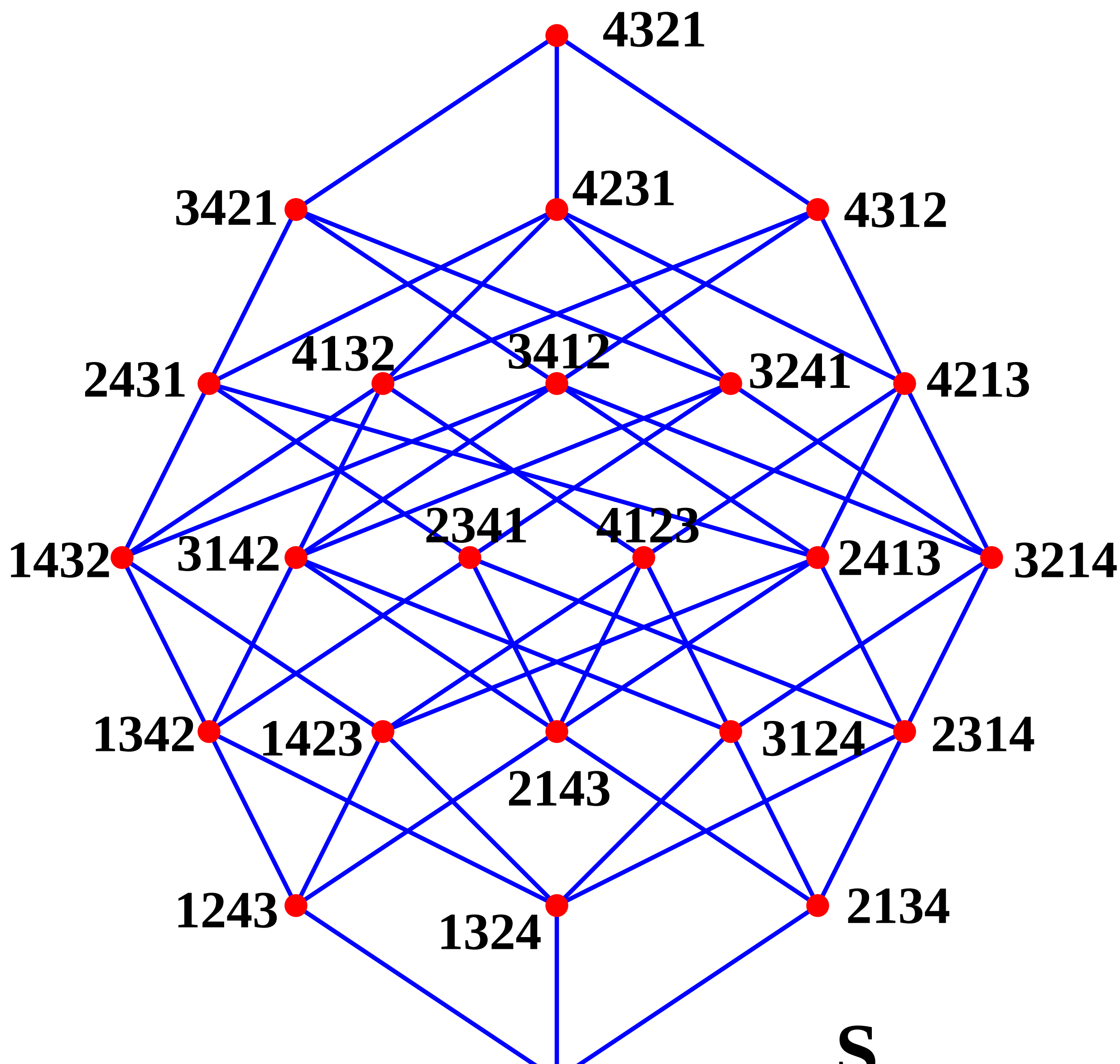
$$u < (i,j)u, \quad \text{if } \ell((i,j)u) = 1 + \ell(u).$$

$$6\mathbf{2}\underbrace{718}_{\text{all} < \text{ or } >2,4}\mathbf{4}53 < 6\mathbf{4}718\mathbf{2}53$$

$v \prec w$: w **covers** v, i.e., $w > v$ and $\ell(w) = 1 + \ell(v)$



S_3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S_4

S_n is a graded poset, where $\mathrm{rank}(w) = \ell(w)$. Thus the **rank-generating function** of S_n is given by

$$\boldsymbol{F(S_n, q)} := \sum_{w \in S_n} q^{\mathrm{rank}(w)}$$

$$= (1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1}).$$

Motivation. Let K be a field and $\boldsymbol{\mathcal{F}(K^n)}$ the set of all (complete) **flags**

$$0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = K^n$$

of subspaces of K^n (so $\dim V_i = i$).

For every such flag F, there are unique vectors $v_1, \ldots, v_n \in K^n$ such that:

- $\{v_1, \ldots, v_i\}$ is a basis for V_i

- The $n \times n$ matrix with rows $v_1, \ldots, v_n$ has the form

$$\begin{matrix} * & * & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & * & \mathbf{0} & * & * & \mathbf{1} \\ \mathbf{0} & * & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & * & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{matrix}.$$

The positions of the **1**'s define a permutation $\boldsymbol{w_F} = 316452$. The number of $*$'s is $\ell(w_F)$.

For $w \in S_n$ define the **Bruhat cell**

$$\boldsymbol{\Omega_w} = \{F \in \mathcal{F}(K^n) \, : \, w = w_F\}.$$

Thus

$$\mathcal{F}(K^n) = \bigsqcup_{w \in S_n} \Omega_w,$$

the **Bruhat decomposition** of $\mathcal{F}(K^n)$.

$\overline{\boldsymbol{\Omega}}_{\boldsymbol{w}}$: **closed** Bruhat cell

Theorem (Ehresmann, 1934)

$$\overline{\Omega}_v \subseteq \overline{\Omega}_w \Leftrightarrow v \leq w$$

(Bruhat order).

$\boldsymbol{P}$: finite poset (partially ordered set), say with a top element $\hat{\mathbf{1}}$

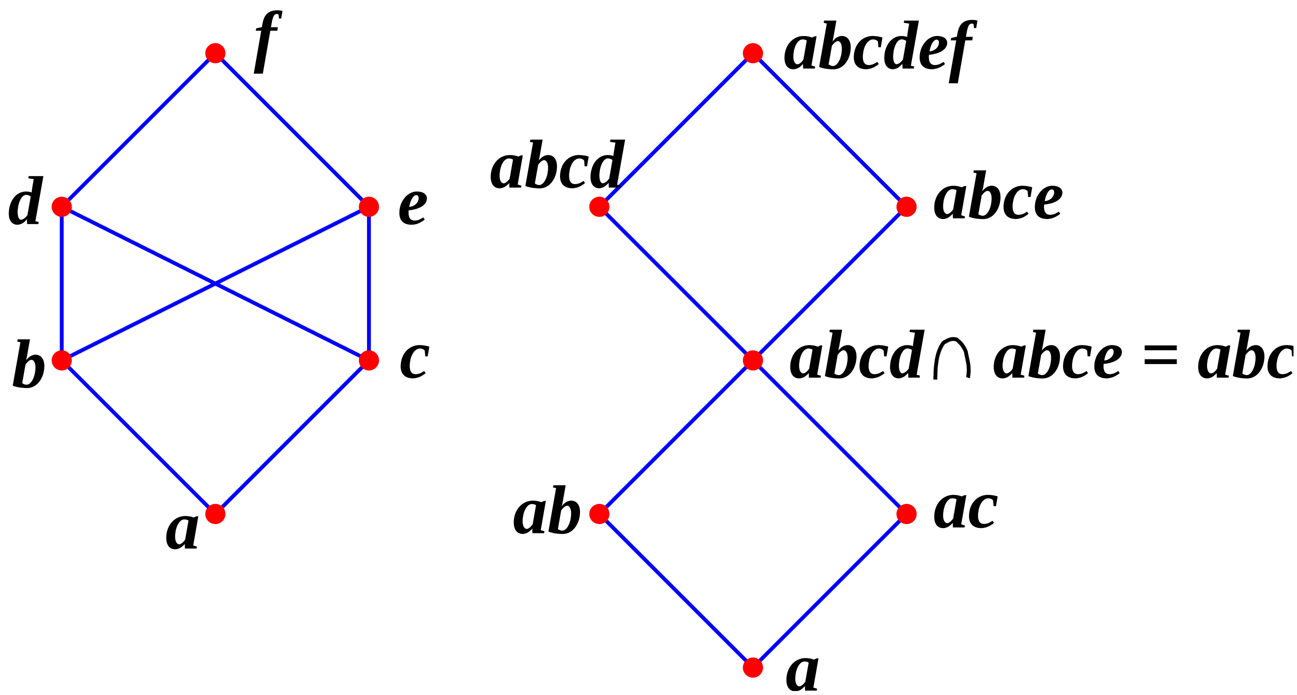
principal order ideal Λ_x for $x \in P$:

$$\boldsymbol{\Lambda_x} = \{y \in P \,:\, y \leq x\}$$

lattice: a poset for which every two elements x, y have a greatest lower bound (**meet**) $\boldsymbol{x \wedge y}$ and least upper bound (**join**) $\boldsymbol{x \vee y}$

$\boldsymbol{L_P}$: all subsets of P which are intersections of Λ_x's, the **MacNeille completion** of P. It is the "smallest" lattice containing P as a subposet and preserving any meets and joins existing in P.

E.g., $L_P \cong P \Leftrightarrow P$ is a lattice.


f
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e
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c
a
abcdef
abcd
abce
abcd ∩ abce = abc
ab
ac
a

Theorem (Ehresmann) *Let* $w = a_1 \cdots a_n \in S_n$*. Define*

$$\boldsymbol{T_w} = \begin{array}{ccccccccc} 1 & & 2 & & 3 & & \cdots & & n \\ & b_1 & & b_2 & & \cdots & & b_{n-1} & \\ & & c_1 & & \cdots & & c_{n-2} & & \\ & & & & \cdots & & & & \\ & & & & a_1 & & & & \end{array},$$

where

$$\begin{aligned} (b_1, \ldots, b_{n-1}) &= \{a_1, \ldots, a_{n-1}\}_{\text{sorted}} \\ (c_1, \ldots, c_{n-2}) &= \{a_1, \ldots, a_{n-2}\}_{\text{sorted}} \\ &\text{etc.} \end{aligned}$$

Then

$$v \leq w \Leftrightarrow T_v \leq T_w$$

(componentwise).

Example. $v = 35124$, $w = 45123$

$$T_v = \begin{array}{ccccccccc} 1 & & 2 & & 3 & & 4 & & 5 \\ & 1 & & 2 & & 3 & & 5 & \\ & & 1 & & 3 & & 5 & & \\ & & & 3 & & 5 & & & \\ & & & & 3 & & & & \end{array}$$

$$\leq \begin{array}{ccccccccc} 1 & & 2 & & 3 & & 4 & & 5 \\ & 1 & & 2 & & 4 & & 5 & \\ & & 1 & & 4 & & 5 & & \\ & & & 4 & & 5 & & & \\ & & & & 4 & & & & \end{array} = T_w$$

T_w is a (special) **monotone triangle**.

(general) monotone triangle:

$$\begin{array}{ccccccccc} 1 & & 2 & & 3 & & 4 & & 5 \\ & 1 & & 3 & & 4 & & 5 & \\ & & 2 & & 3 & & 5 & & \\ & & & 2 & & 4 & & & \\ & & & & 4 & & & & \end{array}$$

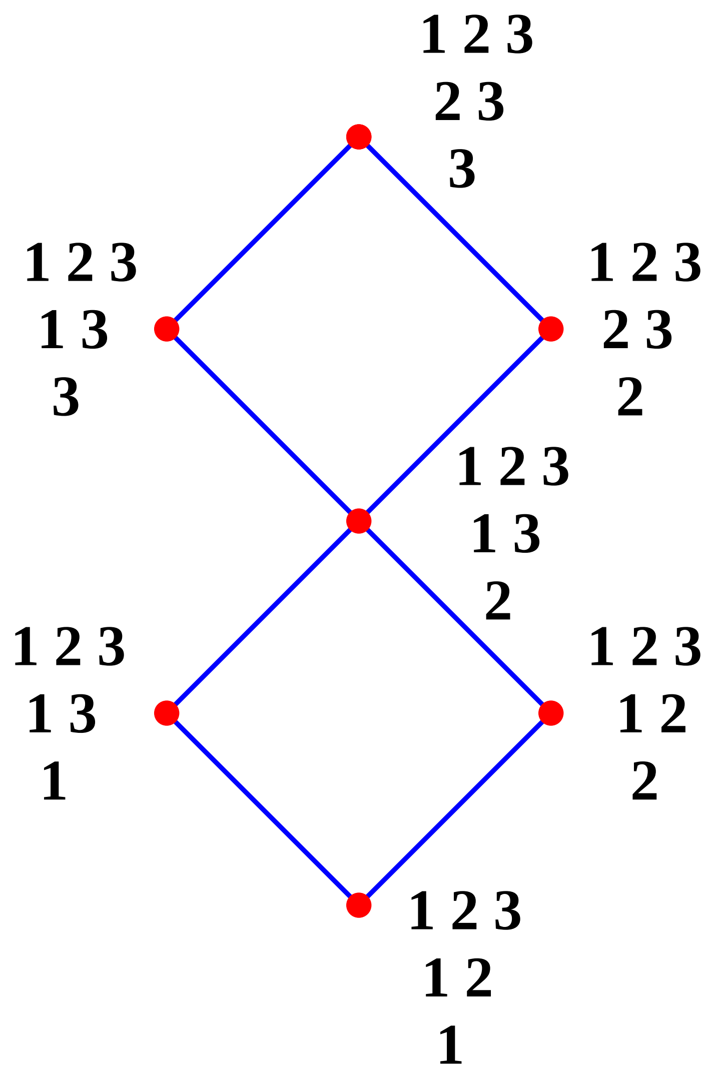
monotone triangle $\leftrightarrow$ ASM

$T_w \leftrightarrow$ permutation matrix

$\mathfrak{M}_n$: set of all monotone triangles of length n, ordered componentwise

Thus S_n is a subposet of $\mathfrak{M}_n$.

Theorem (Lascoux-Schützenberger, 1996) *$\mathfrak{M}_n$ is the MacNeille completion of S_n.*



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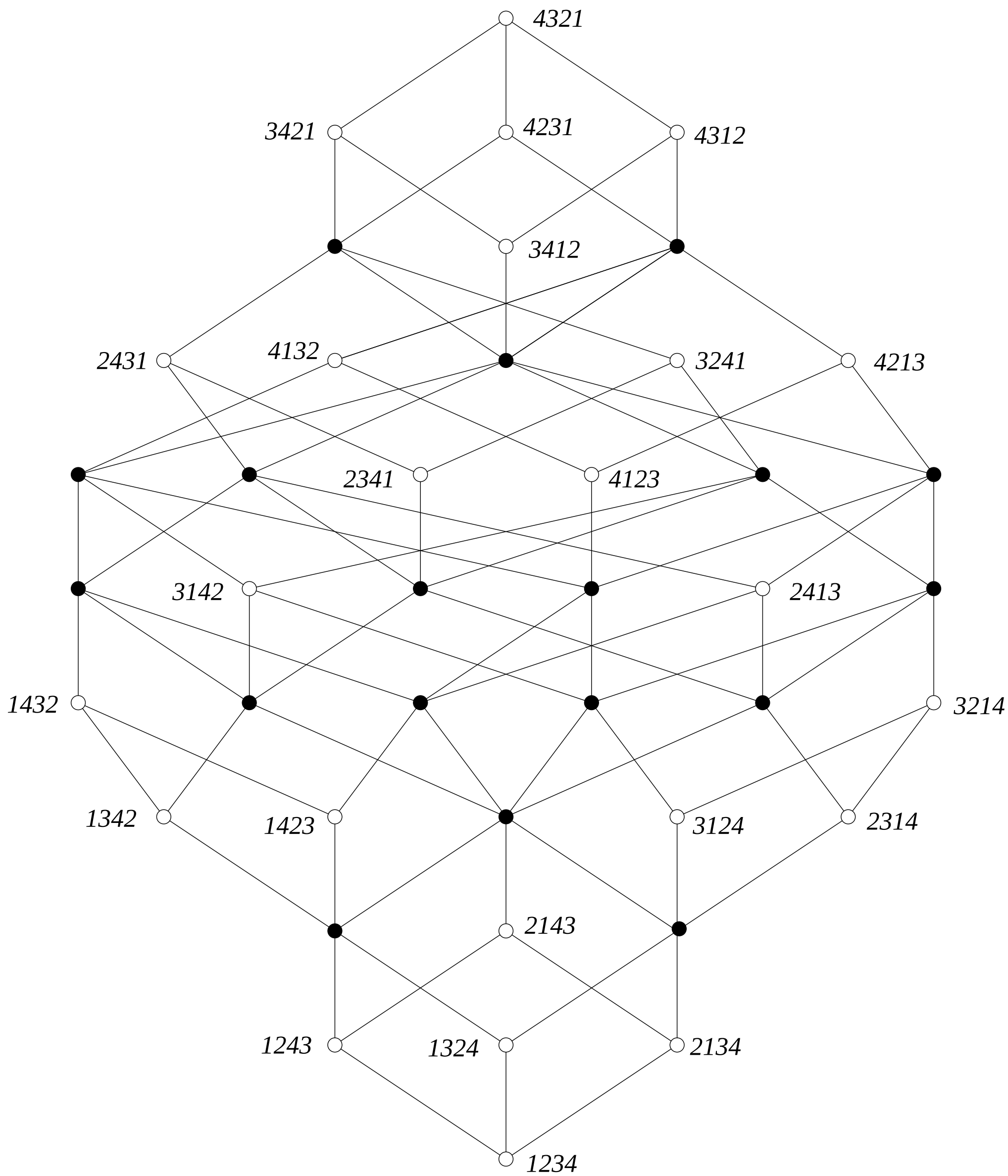
1 2
1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Topology of the Bruhat order

P: finite poset

Define the **Möbius function**

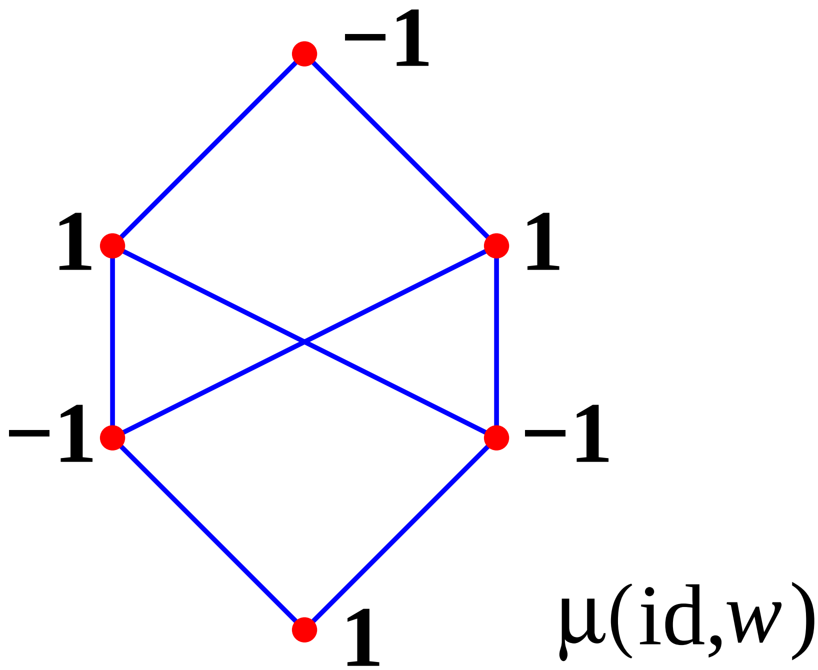
$$\boldsymbol{\mu} : P \times P \to \mathbb{Z}$$

recursively by:

$$\mu(x,y) = \begin{cases} 0, \text{ unless } x \leq y \\ 1, \ x = y \\ -\sum_{x \leq z < y} \mu(x,z), \ x < y \end{cases}$$

Thus

$$x < y \Rightarrow \sum_{x \leq z \leq y} \mu(x,z) = 0.$$



Theorem (Verma, 1971) *For* $v \leq w$ *in* S_n *we have*

$$\mu(v, w) = (-1)^{\ell(w)-\ell(v)}.$$

For $x \le y$ in any finite poset P, let $\boldsymbol{c_i}$ be the number of chains

$$x < x_0 < x_1 < \cdots < x_i < y,$$

with $c_{-1} = 1$.

Theorem (P. Hall, 1936)

$$\mu(x,y) = -c_{-1} + c_0 - c_1 + c_2 - \cdots$$

order complex $\boldsymbol{\Delta(x,y)}$: the abstract simplicial complex on the set

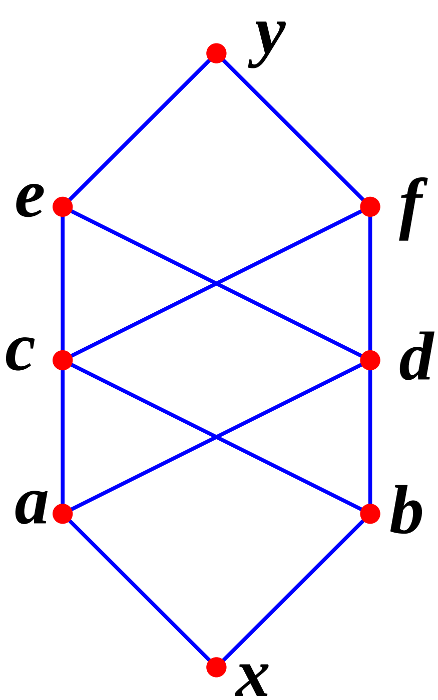
$$\boldsymbol{(x,y)} = \{z \in P \,:\, x < z < y\}$$

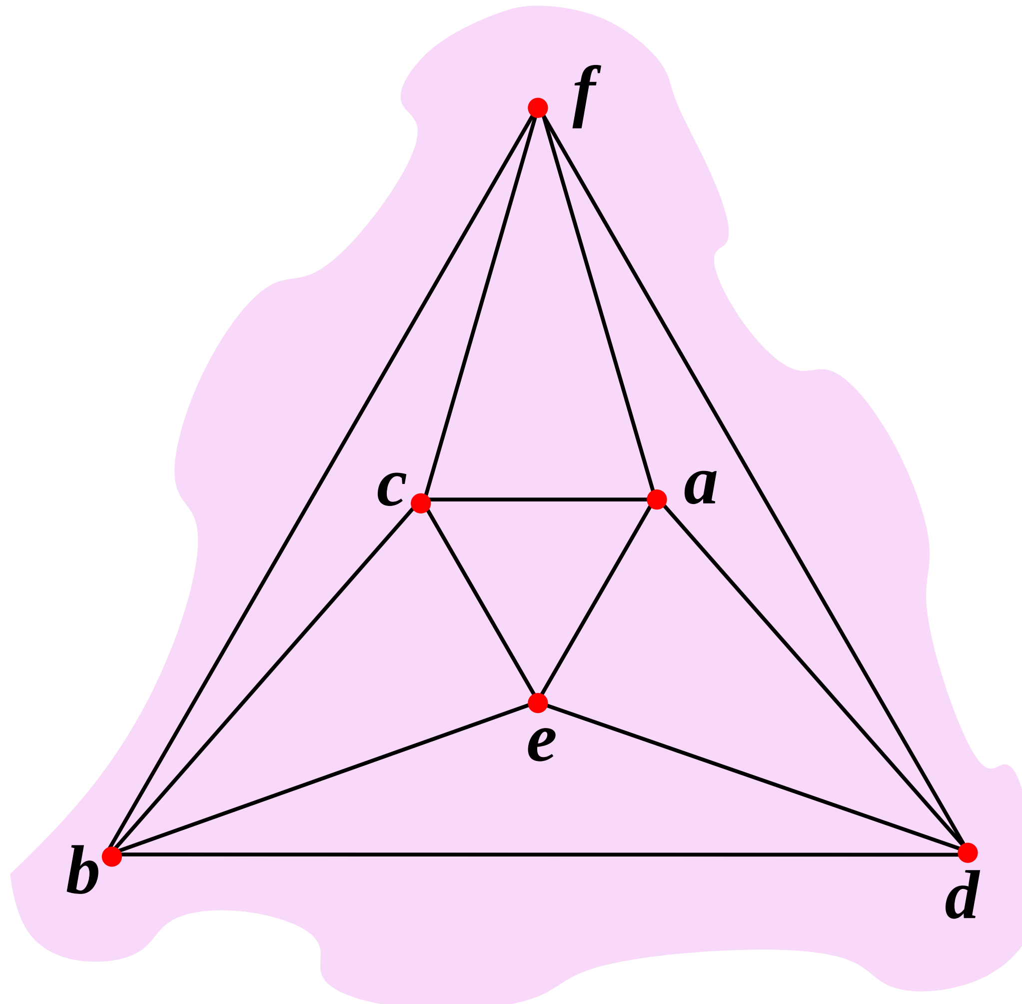
whose faces (simplices) are the chains in (x,y).

P. Hall's theorem restated:

$$\mu(x,y) = \tilde{\boldsymbol{\chi}}(\Delta(x,y)),$$

the **reduced Euler characteristic** of $\Delta(x,y)$.





$\Delta(x,y)$

Verma's theorem on μ for S_n suggests:

Conjecture. For all $v \leq w$ in S_n, $\Delta(x, y)$ is a triangulation of a sphere.

Note. Given an abstract simplicial complex Δ, it is **undecidable** whether Δ triangulates a sphere.

Basic tool: **lexicographic shellability** (Björner, Wachs). Let P be a finite graded poset with $\hat{0}$ and $\hat{1}$, with

$$\mu(x,y) = (-1)^{\mathrm{rank}(y)-\mathrm{rank}(x)} \quad \forall x \leq y$$

(i.e., P is **Eulerian**). Let

$$\boldsymbol{\lambda} : \mathcal{E}_P \to \{1, 2, , \ldots\}$$

be a labeling of the edges of the (Hasse) diagram of P satisfying

- For all $x < y$, $\exists$ a unique saturated **increasing** chain

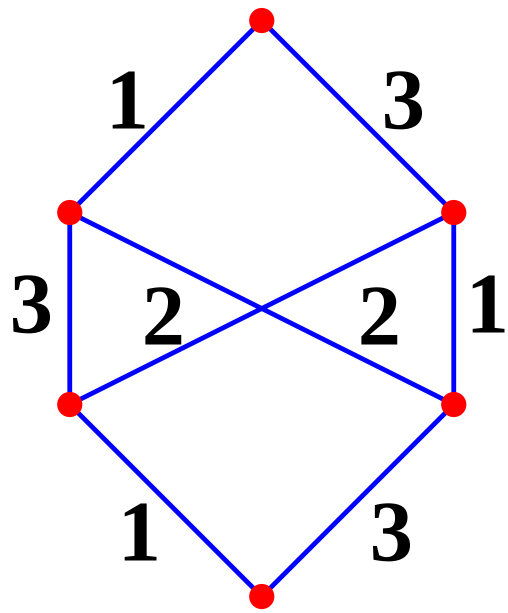
  $$\boldsymbol{C} \; : \; x_0 \prec x_1 \prec \cdots \prec x_r = y, \text{ i.e.,}$$

  $$\lambda(x_0, x_1) \leq \lambda(x_1, x_2) \leq \cdots \leq \lambda(x_{r-1}, x_r).$$

- The label sequence of C lexicographically precedes that of all other saturated chains from x to y.

Call λ an **EL-labeling**.

Theorem (Björner) *Let P be a finite Eulerian poset with an EL-labeling. Then for all $x < y$ in P, $\Delta(x, y)$ triangulates a sphere.*



First EL-labeling of S_n due to Edelman (1981): Let $\boldsymbol{\tau_1, \tau_2, \ldots, \tau_{\binom{n}{2}}}$ be the transpositions in S_n in lexicographic order. E.g., $n = 4$:

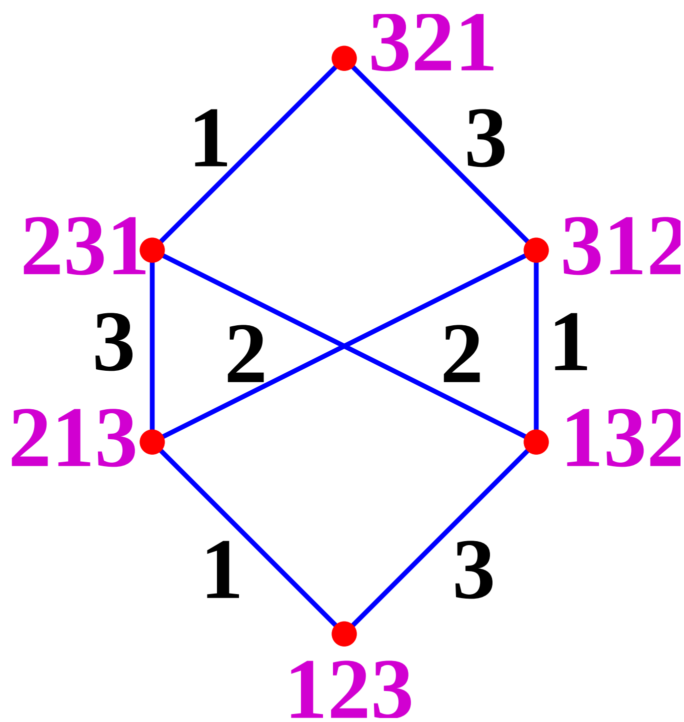
$$\tau_1 = (1,2),\ \tau_2 = (1,3), \tau_3 = (1,4)$$

$$\tau_4 = (2,3),\ \tau_5 = (2,4), \tau_6 = (3,4)$$

Let $w \succ v$ in S_n. Define

$$\boldsymbol{\lambda(v,w)} = j \ \text{ if } \ \tau_j v = w.$$

Theorem (Edelman). *λ is an EL-labeling of S_n, so $\forall\, v < w$ in S_n, $\Delta(v,w)$ triangulates a sphere.*


321
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2
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213
132
1
3
123

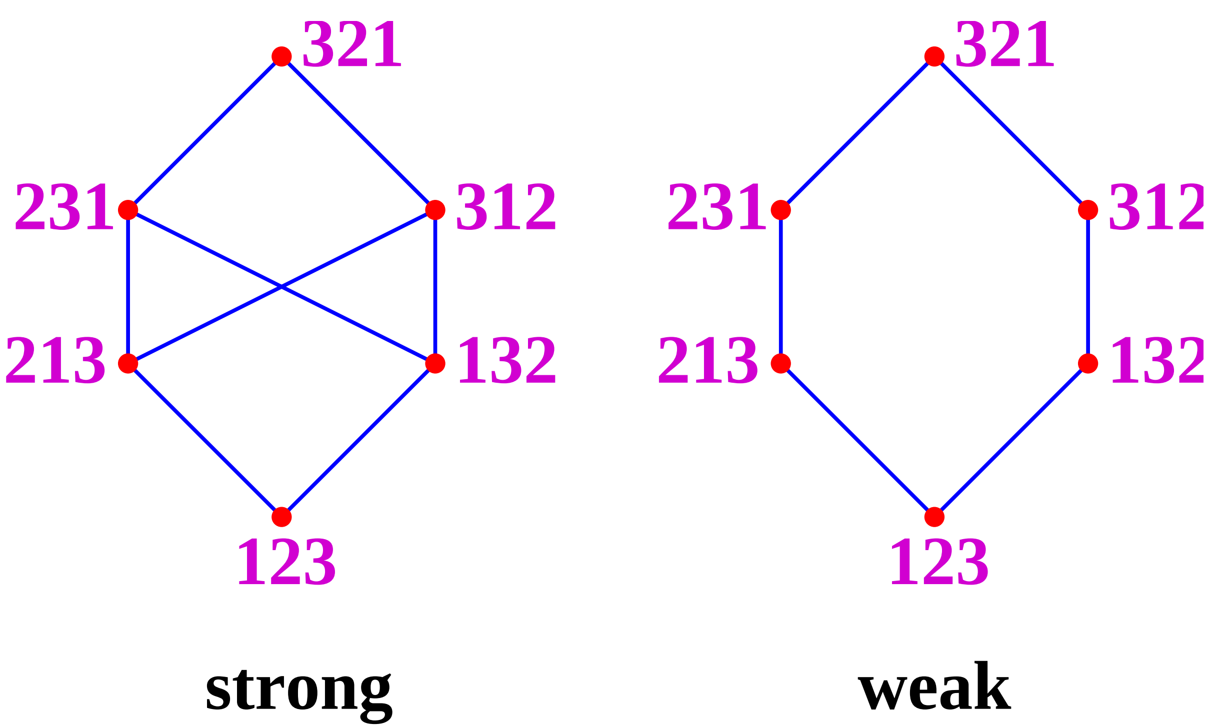
Counting maximal chains in S_n

weak (Bruhat) order WS_n on S_n: transitive, reflexive closure of

$$u < (i, i+1)u, \text{ if } \ell((i,i+1)u) = 1 + \ell(u).$$

Compare ordinary (strong) order:

$$u < (i,j)u, \text{ if } \ell((i,j)u) = 1 + \ell(u).$$



Theorem (RS, 1984). *The number* $\boldsymbol{M_n}$ *of maximal chains of* $W\mathfrak{S}_n$ *(i.e., the number of ways to move from* $1, 2, \ldots, n$ *to* $n, n-1, \ldots, 1$*) by* $\binom{n}{2}$ *adjacent transpositions) is given by*

$$M_n = \#\text{ SYT of shape } (n-1, n-2, \ldots, 1)$$

$$= \frac{\binom{n}{2}!}{1^{n-1}\,3^{n-2}\,5^{n-3}\cdots(2n-3)^1}$$

Is there something analogous for S_n (strong order)?

If $(i,j)v = w \succ v$ in S_n, define the **weight**

$$\mathbf{wt}(\boldsymbol{v},\boldsymbol{w}) = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1}.$$

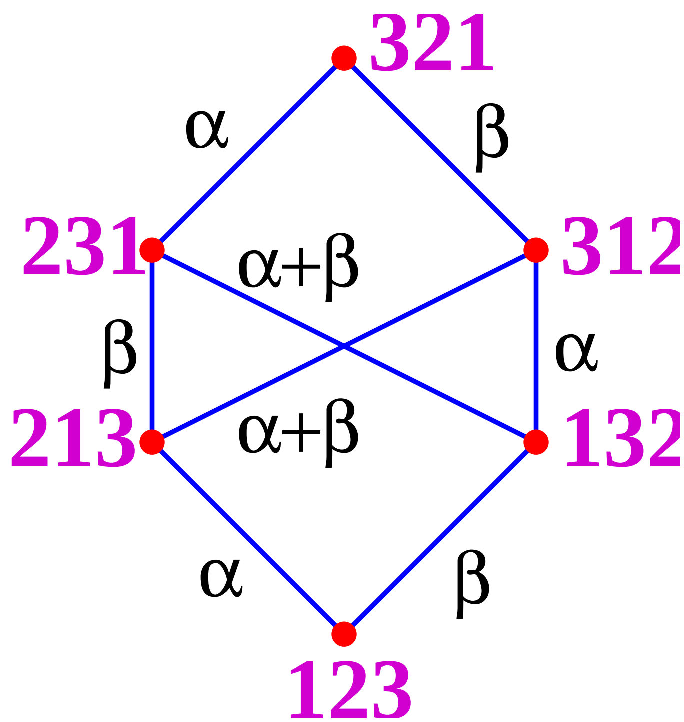
If $\boldsymbol{C}$: $\mathrm{id} = v_0 \prec v_1 \prec \cdots \prec v_{\binom{n}{2}} = w_0$ is a maximal chain in S_n, define

$$\mathbf{wt}(\boldsymbol{C}) = \mathrm{wt}(v_0, v_1)\,\mathrm{wt}(v_1, v_2) \cdots \mathrm{wt}(v_{\binom{n}{2}-1}, v_{\binom{n}{2}}).$$

Theorem (Stembridge, 2001) *We have*

$$\sum_C \mathrm{wt}(C) = \frac{\binom{n}{2}!}{1^{n-1}2^{n-2}\cdots(n-1)^1} \cdot \prod_{1\le i<j\le n} (\alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1}).$$

(extends to any Weyl group)



$$\sum_C \mathrm{wt}(C) = \alpha^2\beta + \alpha\beta^2 + 2\alpha\beta(\alpha+\beta)$$
$$= 3\alpha\beta(\alpha+\beta).$$

Alternative proof (sketch). Based on **Schubert polynomials**. Let $\boldsymbol{s_i}$ be the **adjacent transposition** (or **simple reflection**) $(i, i+1)$. Let $w \in S_n$ and $\boldsymbol{\ell} = \ell(w)$.

reduced decomposition of w: a sequence $(a_1, \dots, a_\ell)$, $1 \le a_j \le n-1$, such that

$$w = s_{a_1} s_{a_2} \cdots s_{a_\ell}.$$

divided difference operator $\boldsymbol{\partial_i}$:

$$\partial_i f = \frac{f(x_i, x_{i+1}) - f(x_{i+1}, x_i)}{x_i - x_{i+1}}$$

Define $\boldsymbol{w_0} = n, n-1, \dots, 1 \in S_n$ (the longest permutation in S_n, of length $\binom{n}{2}$).

Let $(a_1, \ldots, a_\ell)$ be a reduced decomposition of $v \in S_n$. Define

$$\boldsymbol{\partial_v} = \partial_{a_1} \cdots \partial_{a_\ell}$$

(independent of choice of reduced decomposition).

Define the **Schubert polynomial** $\mathfrak{S}_w$ by

$$\boldsymbol{\mathfrak{S}_w} = \partial_{w^{-1}w_0}(x_1^{n-1} x_2^{n-2} \cdots x_{n-1}).$$

$\mathfrak{S}_w$ is homogeneous of degree $\ell(w)$ in $x_1, \ldots, x_{n-1}$.

$$\mathfrak{S}_{w_0} = x_1^{n-1} x_2^{n-2} \cdots x_{n-1}$$

Example. $w = 4132 = s_2 s_3 s_2 s_1$

$$\begin{aligned} w_0 &= s_2 s_3 s_2 s_1 s_3 s_4 \\ w^{-1} w_0 &= s_3 s_4 \end{aligned}$$

$$\begin{aligned} \mathfrak{S}_{4132} &= \partial_3 \, \partial_4 \, x_1^3 x_2^2 x_3 \\ &= \partial_3 \, x_1^3 x_2^2 \\ &= x_1^2 x_2 + x_1^3 x_3. \end{aligned}$$

Note. $\mathfrak{S}_{s_i} = x_1 + x_2 + \cdots + x_i$.

Monk's rule. $\mathfrak{S}_{s_r}\mathfrak{S}_w = \sum \mathfrak{S}_{(j,k)w}$, summed over all transpositions (j,k) such that

$$1 \leq j \leq r < k$$

$\ell((j,k)w) = \ell(w)+1$ (i.e., $(j,k)w \succ w$).

For $w \in S_n$ let

$$N(w) = \sum_C \mathrm{wt}(C),$$

where C ranges over all saturated chains

$$\mathrm{id} = v_0 \prec v_1 \prec \cdots \prec v_\ell = w.$$

Iteration of Monk's rule gives:

$$(\alpha_1 \mathfrak{S}_{s_1} + \alpha_2 \mathfrak{S}_{s_2} + \cdots + \alpha_{n-1} \mathfrak{S}_{s_{n-1}})^\ell = \sum_{\substack{w \in S_n \\ \ell(w)=\ell}} N(w) \mathfrak{S}_w.$$

Let

$$\boldsymbol{\beta_i} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{n-1},$$

so

$$\alpha_1 \mathfrak{S}_{s_1} + \alpha_2 \mathfrak{S}_{s_2} + \cdots + \alpha_{n-1} \mathfrak{S}_{s_{n-1}} = \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_{n-1} x_{n-1}$$

Lemma. *Fix $v \in S_n$. Then the nonzero polynomials $\partial_v \mathfrak{S}_w$ are linearly independent.*

Let $\boldsymbol{\Psi_i} f = f(\beta_i \leftarrow \beta_{i+1})$. Then e.g.

$$\begin{aligned} 0 &= \partial_i \Psi_i (\beta_1 x_1 + \cdots + \beta_{n-1} x_{n-1})^{\binom{n}{2}} \\ &= \sum_{\ell(w)=\binom{n}{2}} \Psi_i N(w) \partial_i \mathfrak{S}_w. \end{aligned}$$

But $\partial_i \mathfrak{S}_{w_0} \neq 0$, so

$$\Psi_i N(w_0) = 0.$$

Thus $(\beta_i - \beta_{i+1}) | N(w_0)$. Similarly,

$$(\beta_i - \beta_j) | N(w_0) \ \ \forall\, 1 \leq i < j \leq n$$

Since $\deg N(w_0) = \binom{n}{2}$, we get

$$N(w_0) = C_n \prod_{1 \leq i < j \leq n} (\alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1}).$$

To show:

$$C_n = 1^{n-1}2^{n-2}\cdots(n-1)^1.$$

Follows from: for every permutation $b_1 \cdots b_{\binom{n}{2}}$ of $\{1^{n-1}, 2^{n-2}, \ldots, (n-1)^1\}$, there is a unique maximal chain

$$\mathrm{id} = v_0 \prec v_1 \prec \cdots \prec v_{\binom{n}{2}} = w_0$$

in S_n such that $\forall i$,

$$v_i = (b_i, c_i)v_{i-1}, \quad \text{for some } c_i > b_i.$$

Example. $n = 4$, $(b_1, \ldots, b_6) = 211312$

$$1\,\mathbf{2}\,\mathbf{3}\,4$$
$$\mathbf{1}\,3\,\mathbf{2}\,4$$
$$\mathbf{2}\,\mathbf{3}\,1\,4$$
$$3\,2\,\mathbf{1}\,\mathbf{4}$$
$$\mathbf{3}\,2\,\mathbf{4}\,1$$
$$4\,\mathbf{2}\,\mathbf{3}\,1$$
$$4\,3\,2\,1$$