BORDER STRIPS, SNAKES AND CODES OF SKEW PARTITIONS

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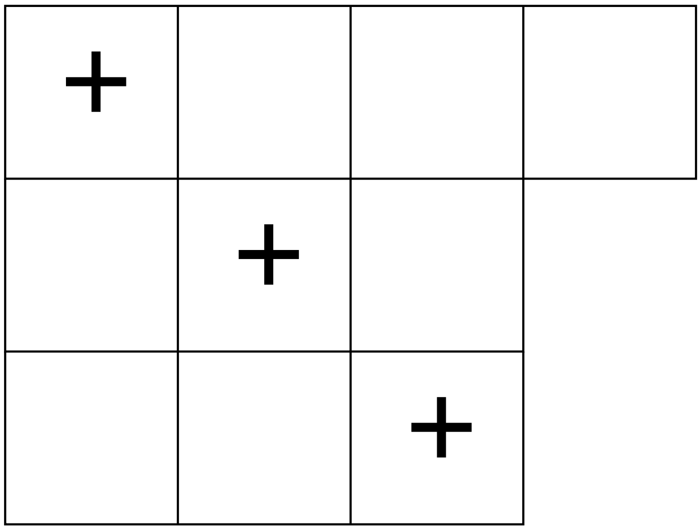
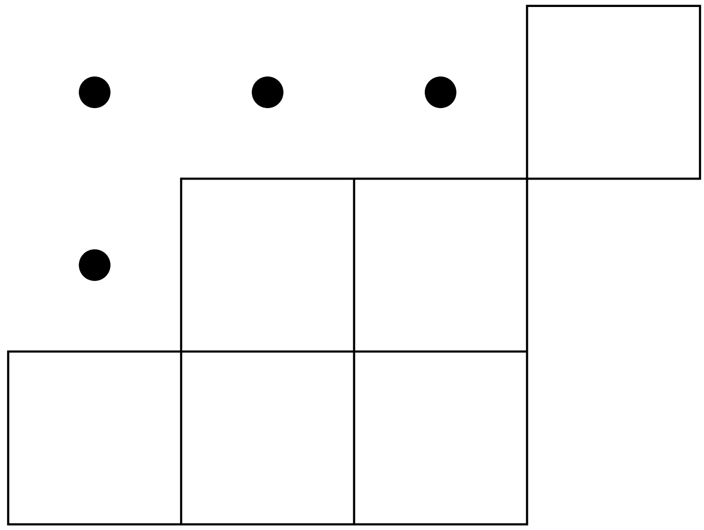
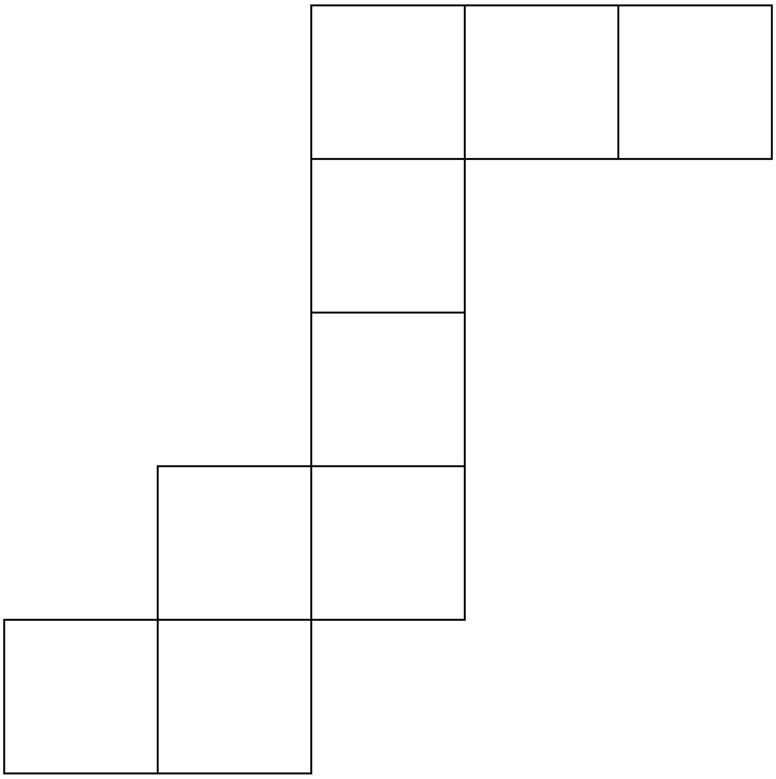


partition or **shape** 433

rank$(433) = 3$



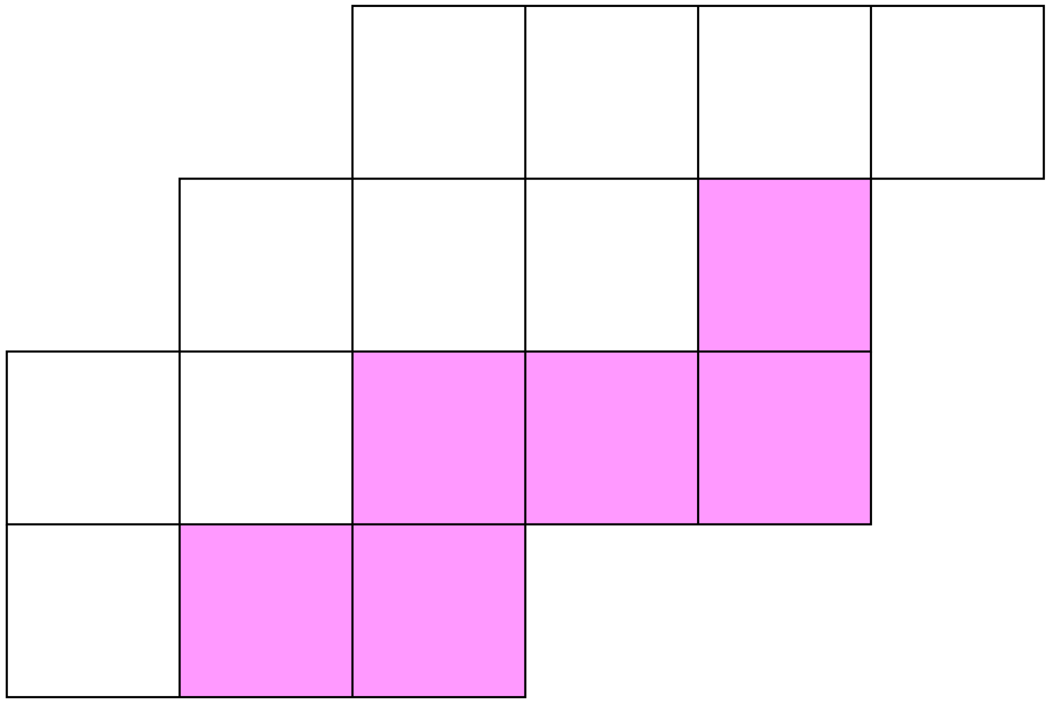
skew partition or **skew shape** $433/31$



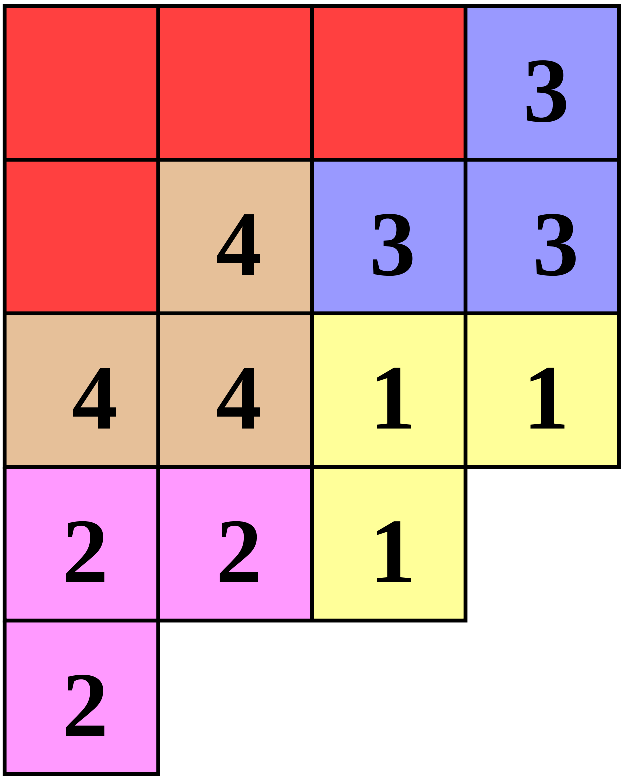
border strip (**ribbon**, **rim hook**)

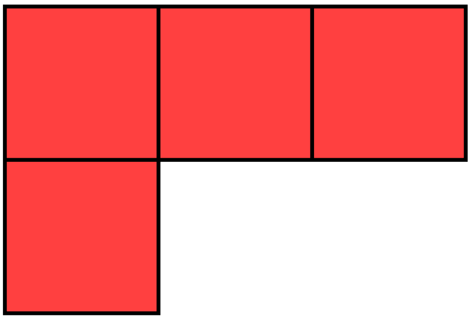
connected, no 2×2 square

determined by row lengths

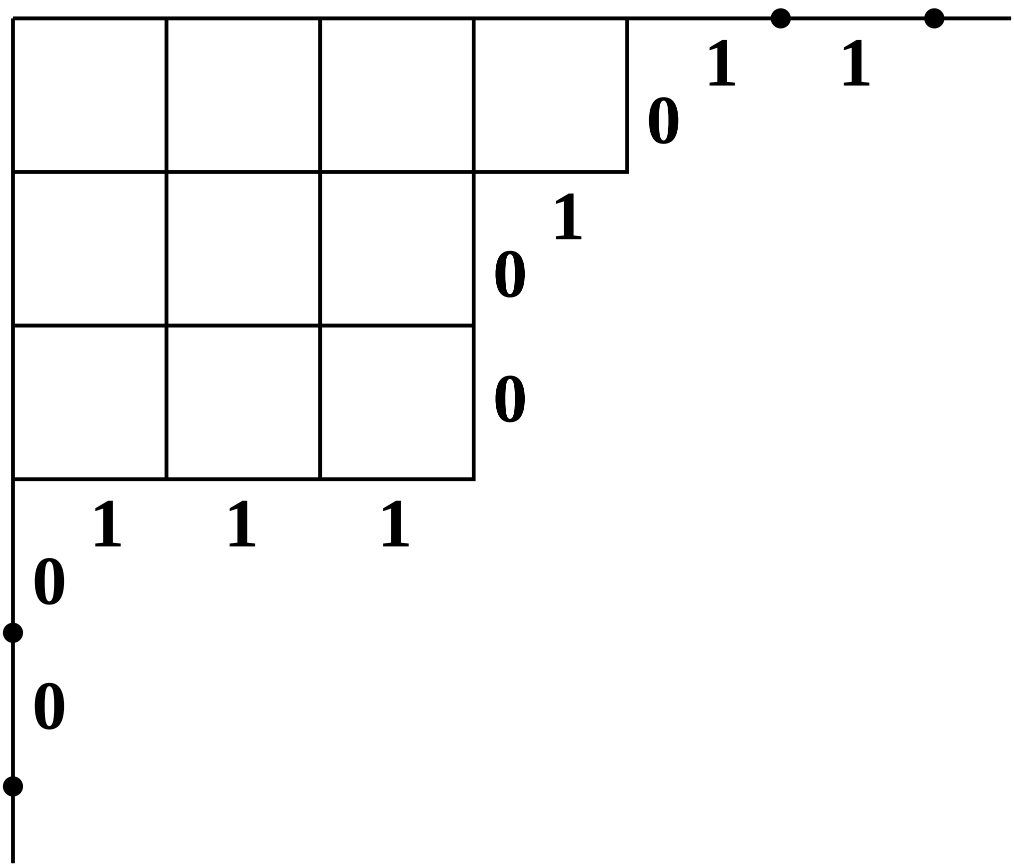


border strip B of $\lambda/\mu = 6553/21$



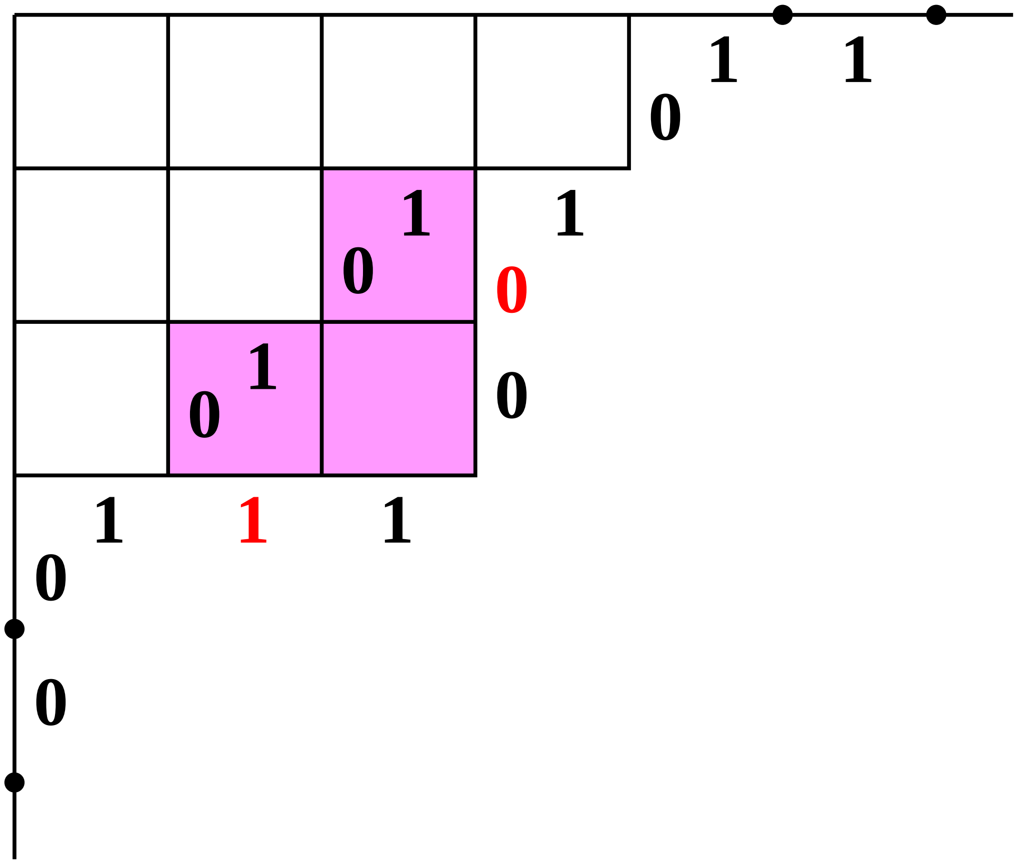


3−core(44431)



(Comét) code:

$$\text{code}(433) = \cdots 00111001011 \cdots$$

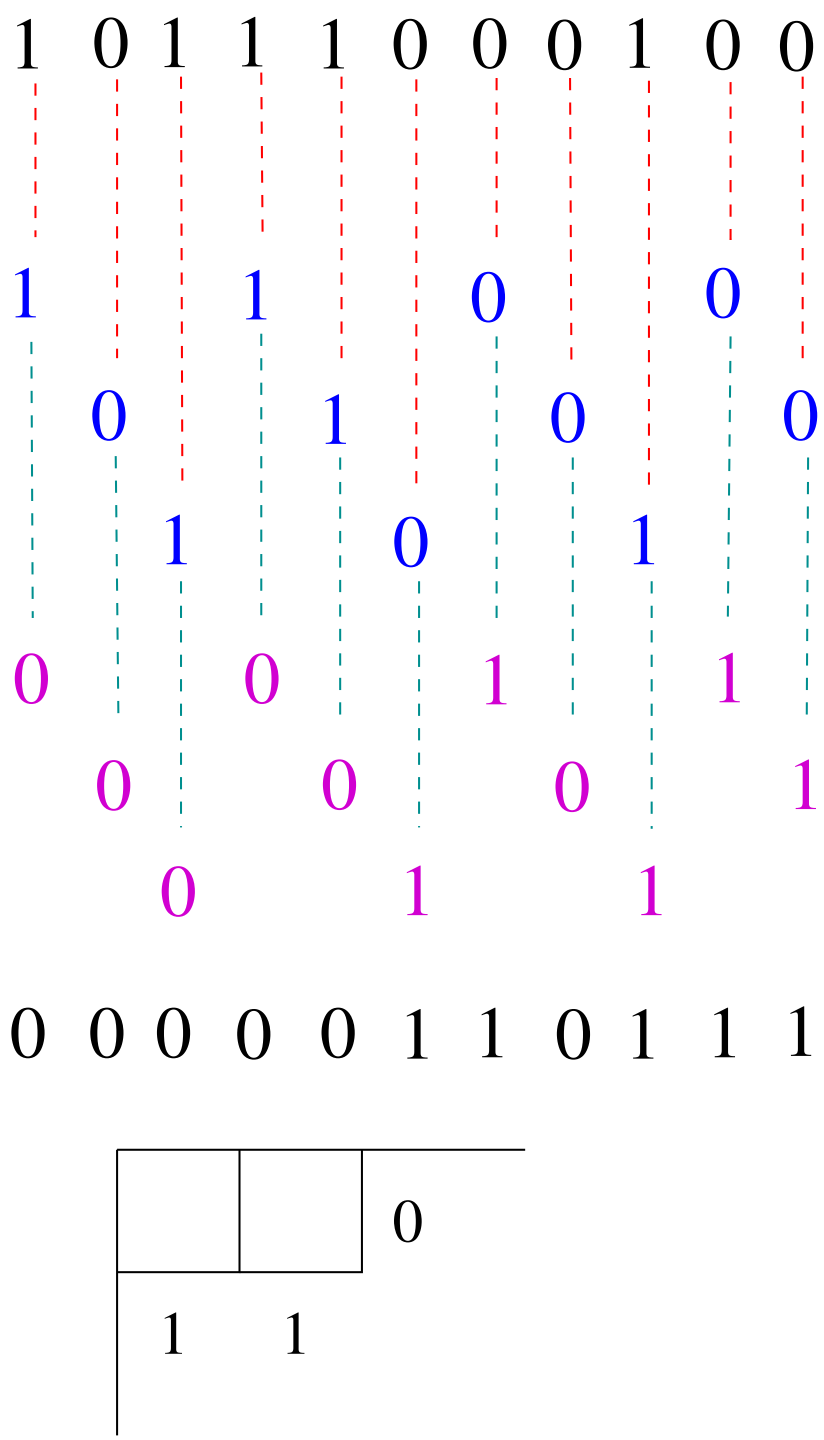


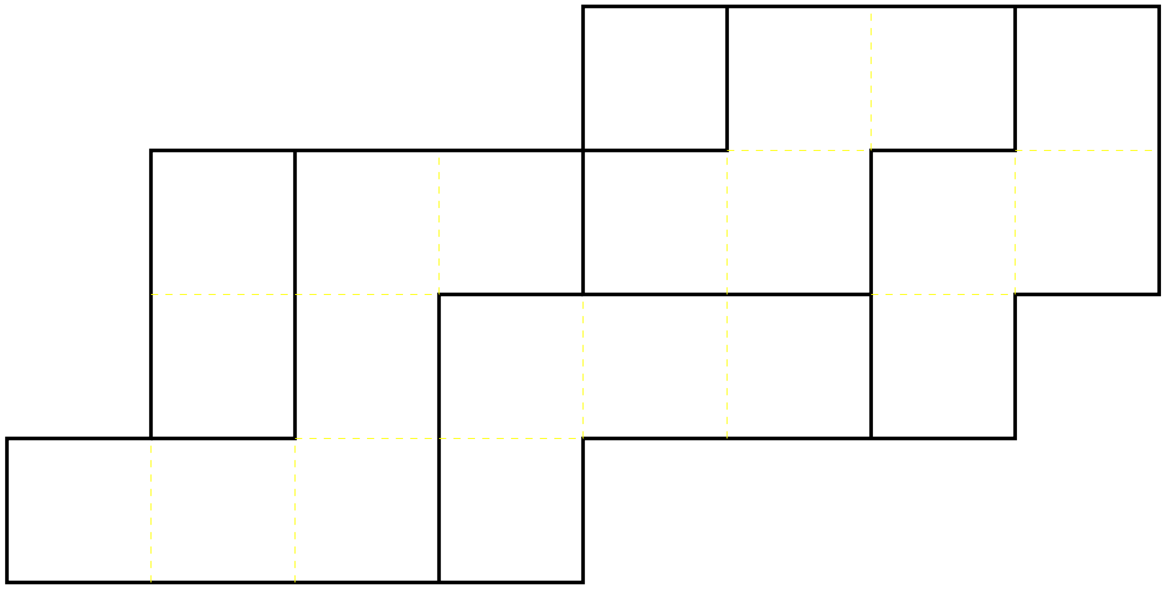
$$\cdots 001\overbrace{1100}1011\cdots$$

$$\cdots 00101011011\cdots$$

$\Rightarrow$ core uniqueness

$p = 3$

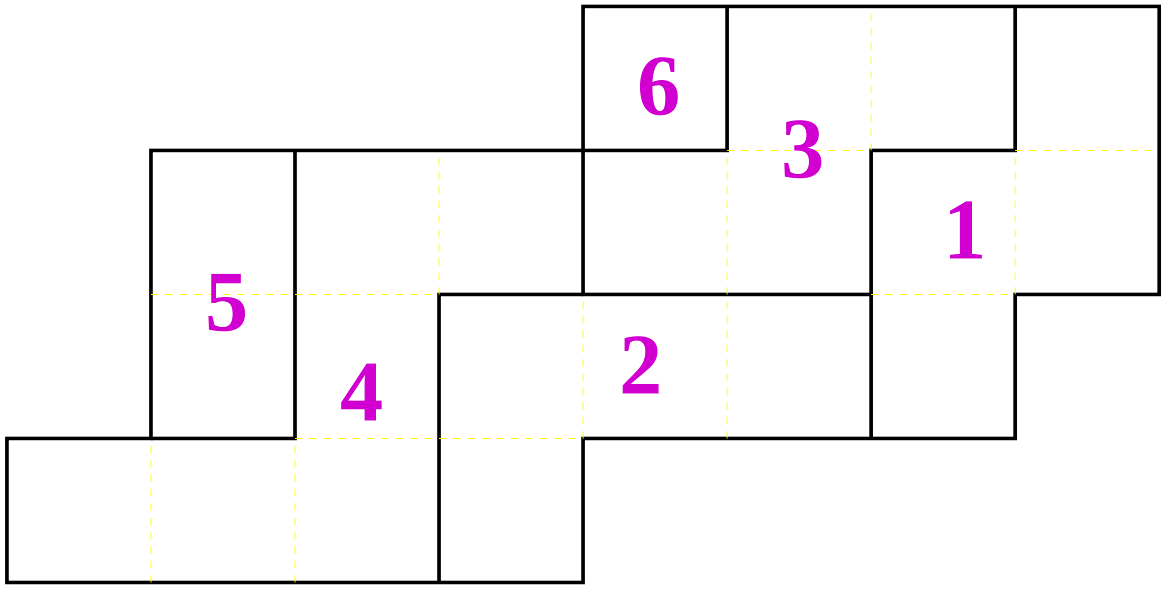




border strip decomposition (BSD) $\boldsymbol{D}$

$$\textbf{type}(\boldsymbol{D}) = 644421$$

$$\textbf{sh}(\boldsymbol{D}) = 8874/411$$



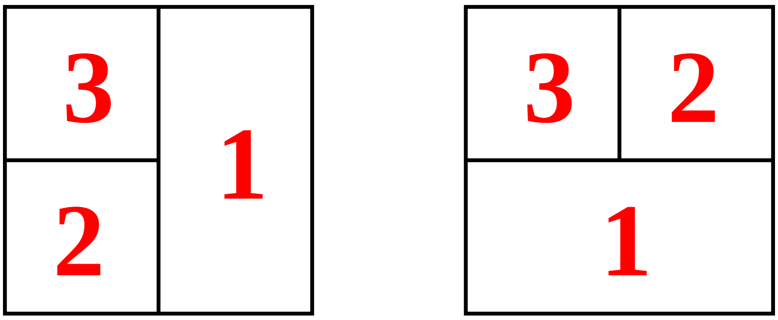
border strip tableau $(\mathbf{BST})$ $\boldsymbol{T}$

$$\begin{aligned} \mathbf{type}(\boldsymbol{T}) &= 444621 \\ \mathbf{sh}(\boldsymbol{T}) &= 8874/411 \end{aligned}$$

$$\begin{aligned} \mathrm{ht}(B) &= \#\mathrm{rows} - 1 \\ \mathrm{ht}(\boldsymbol{T}) &= \sum_{B \in \boldsymbol{T}} \mathrm{ht}(B) \\ &= 2 + 1 + 1 + 2 + 1 + 0 = 7. \end{aligned}$$

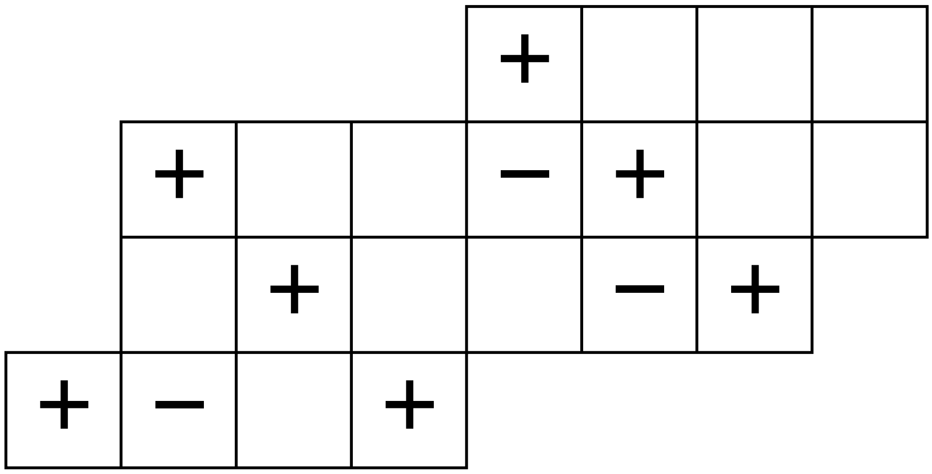
Murnaghan-Nakayama rule:

$$\chi^{\lambda/\mu}(\nu) = \sum_{\substack{\text{type}(\mathbf{T})=\nu \\ \text{sh}(\mathbf{T})=\lambda/\mu}} (-1)^{\text{ht}(\mathbf{T})}$$



ht = 1 **ht = 0**

$$\chi^{22}(211) = -1 + 1 = 0$$



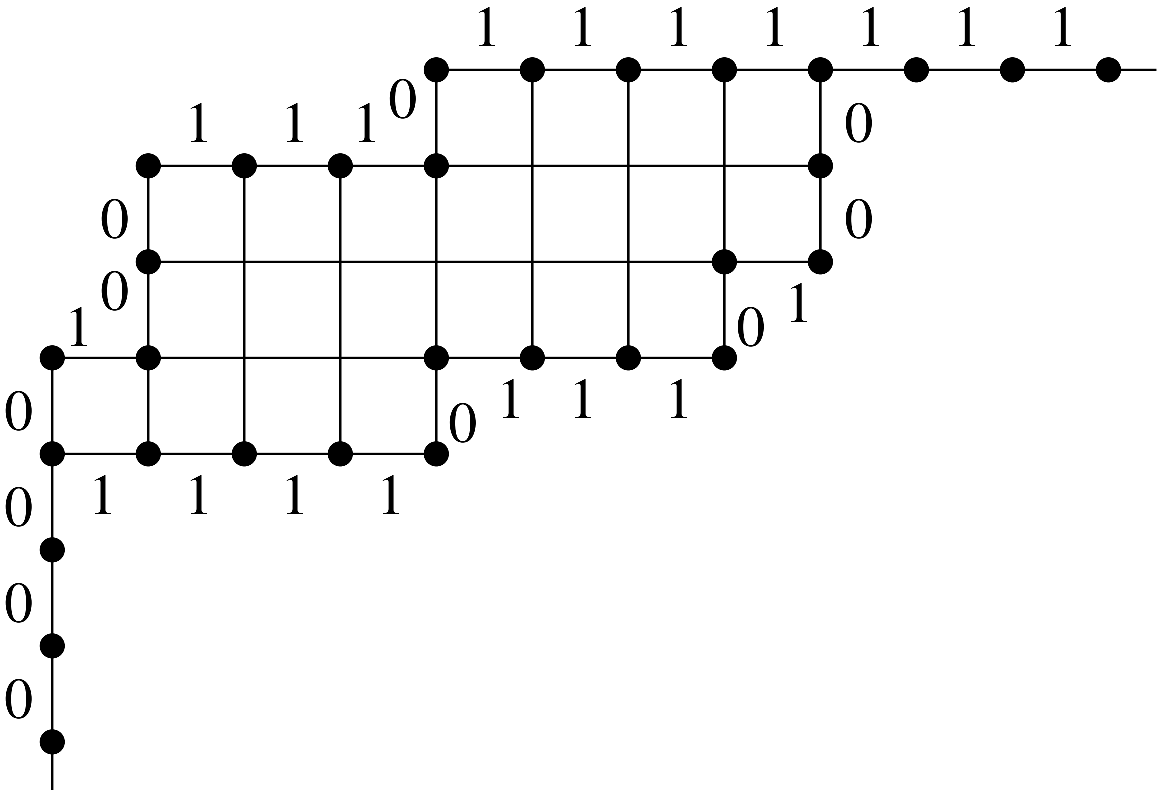
inside $(-)$ and **outside** $(+)$ diagonals

$$\text{rank}(8874/41) = 3+3+1-(2+1) = 4$$

$$\boldsymbol{(\lambda/\mu)^{\natural}} = \lambda/\mu \text{ rotated } 180^\circ$$

Nazarov-Tarasov (2000):

$$\text{rank}(\lambda/\mu) = \text{rank}\left((\lambda/\mu)^{\natural}\right)$$



$$\mathrm{code}(\lambda/\mu) = \cdots \begin{matrix} 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \end{matrix} \cdots$$

Theorem. *Following are equal:*

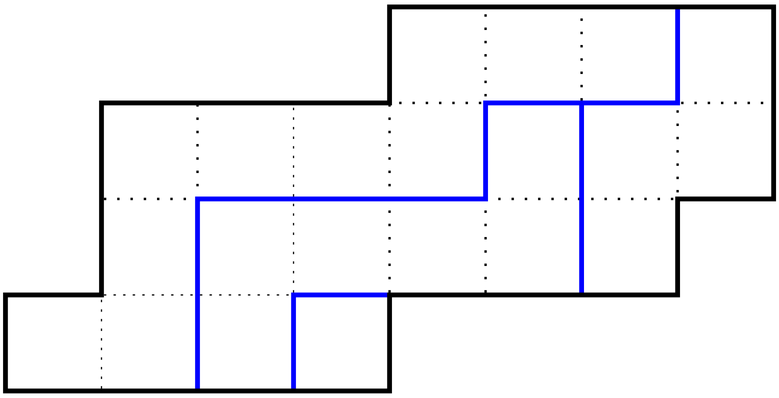
- $\mathrm{rank}(\lambda/\mu)$
- *# columns of* $\mathrm{code}(\lambda/\mu)$ *equal to* $\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}$ *(or to* $\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}$*)*
- *minimum size of a BSD of* λ/μ

Suggests looking at minimal BSD's (**MBSD**).

Note. $s_{\lambda/\mu}(1^t) \in \mathbb{Z}[t]$

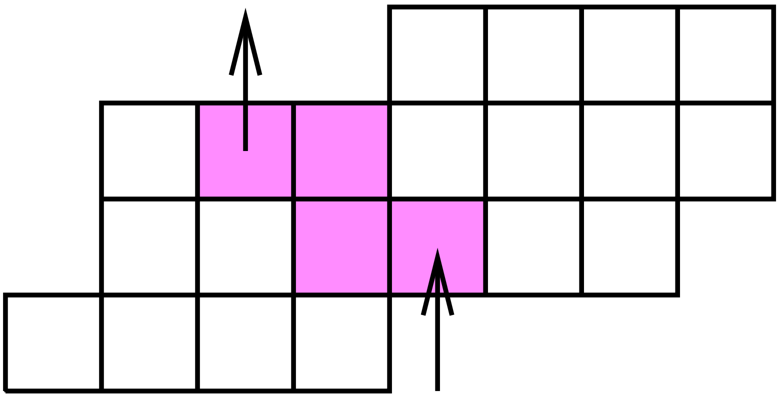
$$t^{\mathrm{rank}(\lambda/\mu)} | s_{\lambda/\mu}(1^t)$$

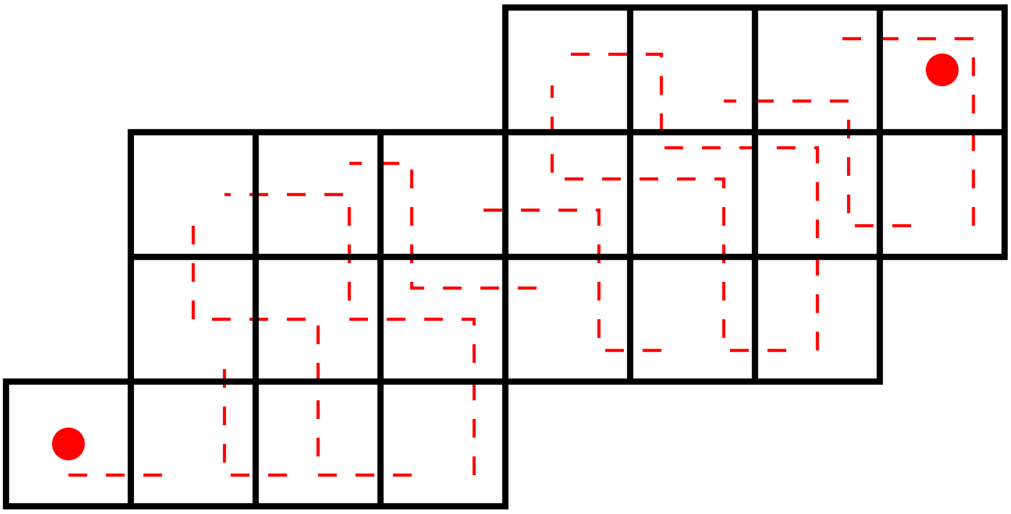
Open: $t^{\mathrm{rank}(\lambda/\mu)+1} \not| s_{\lambda/\mu}(1^t)$
(true for $\mu = \emptyset$)



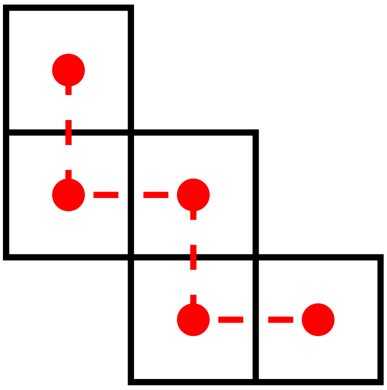
MBSD

snake:





snakes for 8874/411



link: **two consecutive squares of a snake**

link of a BSD $\boldsymbol{D}$: two consecutive squares of a border strip $B \in \boldsymbol{D}$

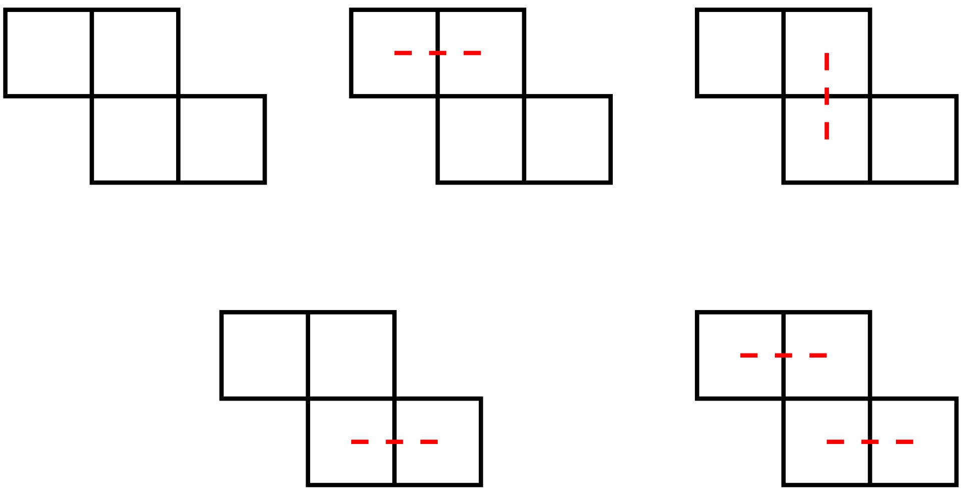
Theorem. (a) $\boldsymbol{D}$ *is uniquely determined by its links (obvious).*

(b) *Links of* $\boldsymbol{D}$ *can be any set of non-consecutive links of snakes.*

Let $\ell(S) = \#\text{links} = |S| - 1$.

k-snake: $|S| = k$, so $\ell(S) = k - 1$

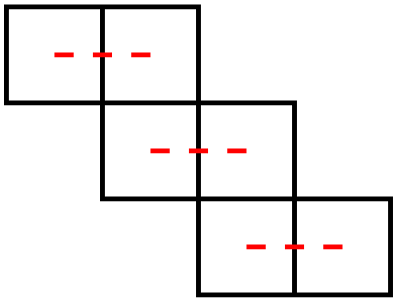
A k-snake has F_{k+1} ways to choose non-consecutive links ($F_1 = F_2 = 1, F_{n+1} = F_n + F_{n-1}$).



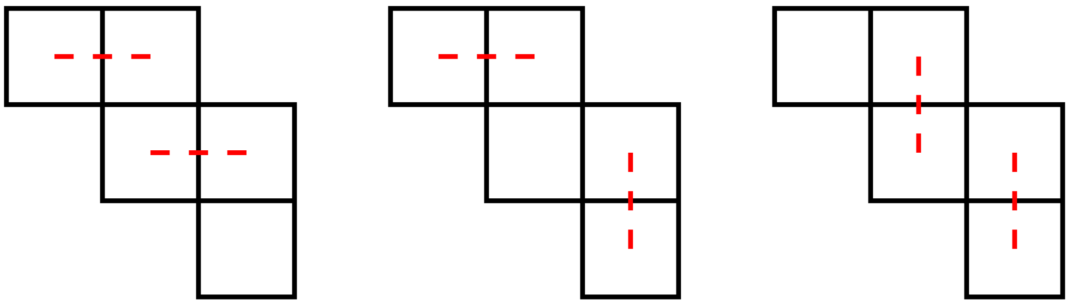
Corollary (EC2, Exercise 7.66(a)).

$$\mathrm{bsd}(\lambda/\mu) = \prod_S F_{\ell(S)+2}$$

minimal $\boldsymbol{D} \leftrightarrow$ maximum # of links

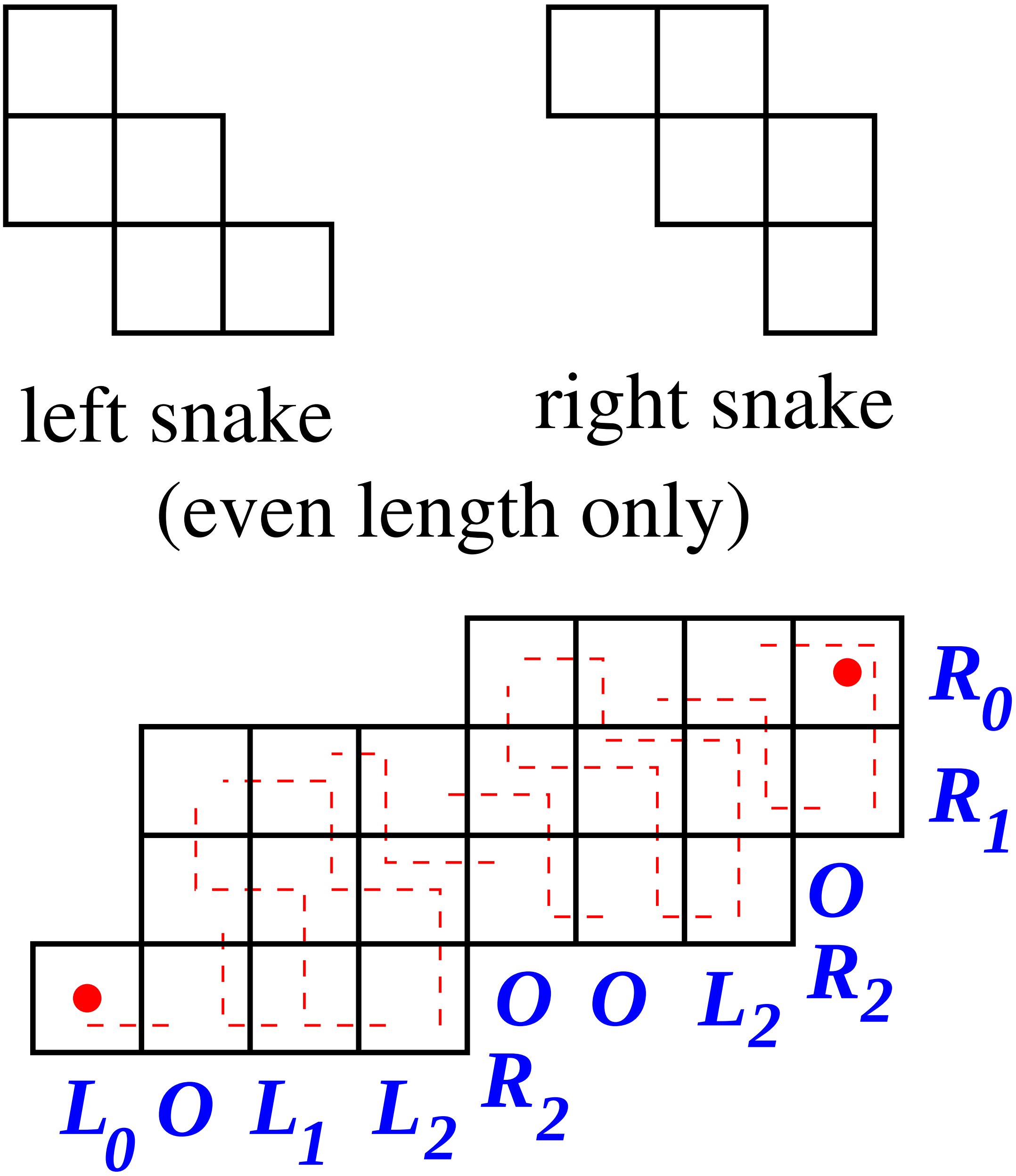


odd length
(one way)



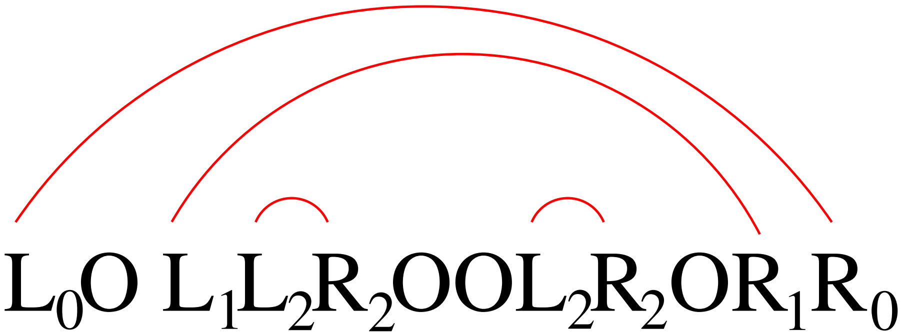
even length 2*k* (*k* + 1 ways)

Corollary. $\mathrm{mbsd}(\lambda/\mu) = \prod_{\ell(S)\ \text{even}} \left(\frac{1}{2}\ell(S) + 1\right)$



snake sequence:

$$SS(8874/411) = L_0 O L_1 L_2 R_2 O O L_2 R_2 O R_1 R_0$$



SS(λ/μ) is "well-parenthesized."

Corollary.

$$\mathrm{mbsd}(\lambda/\mu) = \prod_{\substack{\text{left snakes} \\ S}} \left(\frac{1}{2}\ell(S) + 1\right)^2$$

Interval set:

$$\mathcal{I} = \{(1,5),(3,12),(4,9),(8,11)\}$$

$$\textbf{type}(\mathcal{I}) = 9543$$

Number of interval sets of λ/μ:

$$\mathrm{is}(\lambda/\mu) = \prod_{\substack{\text{left snakes} \\ S}} \left(\frac{1}{2}\ell(S) + 1\right),$$

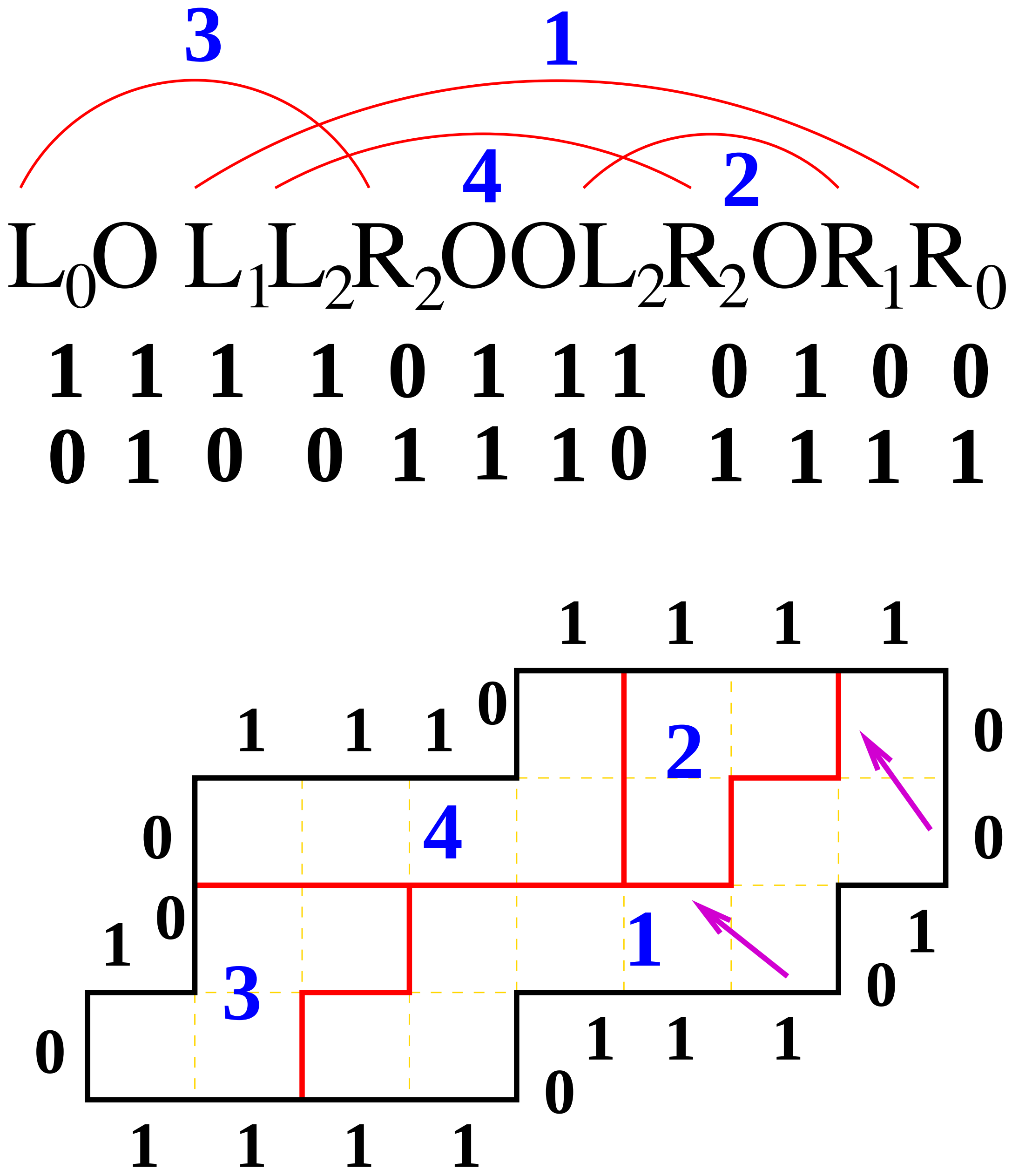
so

$$\mathrm{mbsd}(\lambda/\mu) = \mathrm{is}(\lambda/\mu)^2.$$

MBST's obtained by linearly ordering the pairs of an interval set, so

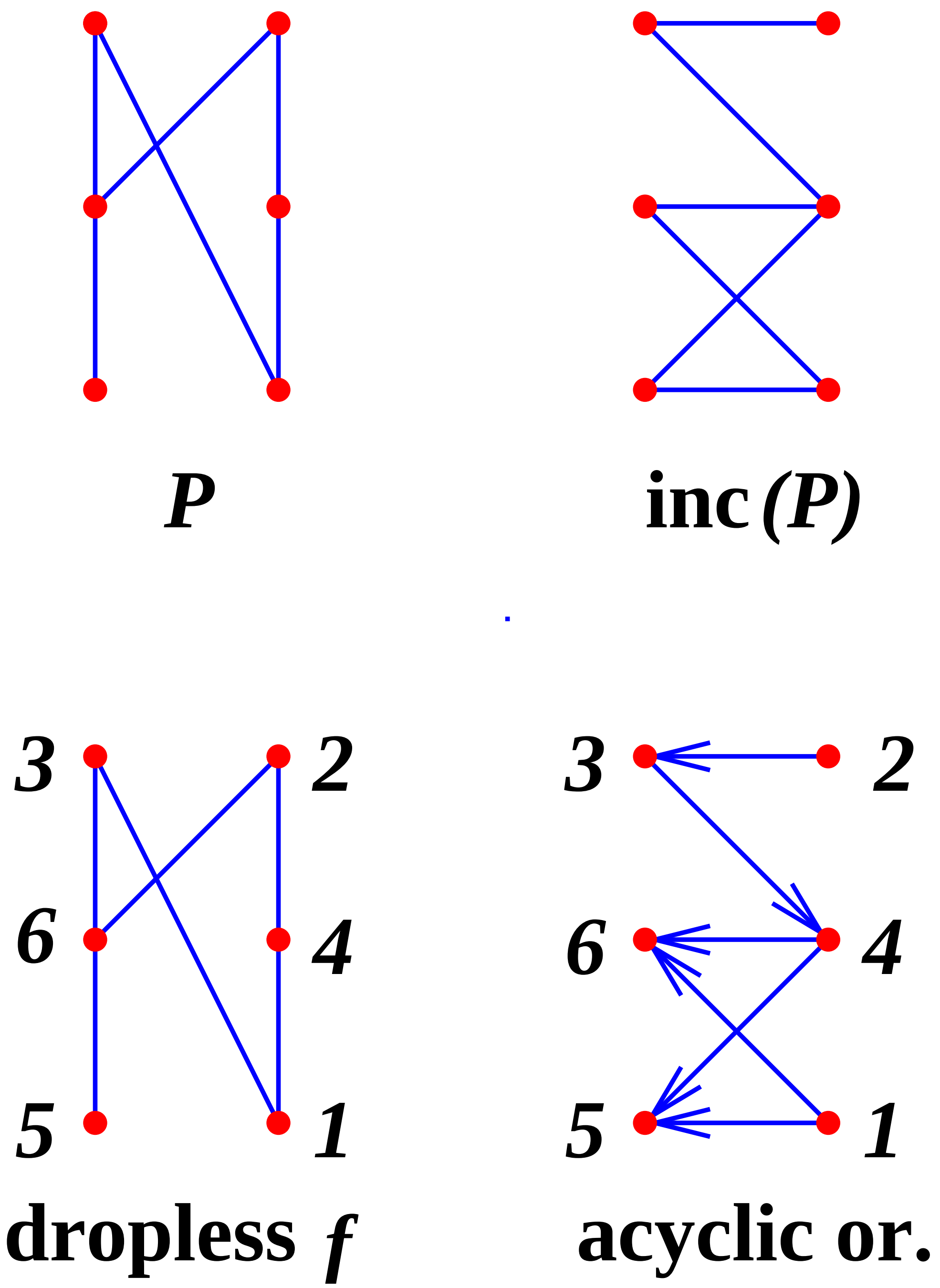
$$\mathrm{mbst}(\lambda/\mu) = r!\,\mathrm{is}(\lambda/\mu),$$

where $r = \mathrm{rank}(\lambda/\mu)$.


3
1
4
2
L0O L1L2R2OOL2R2OR1R0
1 1 1 1 0 1 1 1 0 1 0 0
0 1 0 0 1 1 1 0 1 1 1 1
1 1 1 1
1 1 1 0
0 0
2
4
0
1 0
1
1
3
0
1 1 1
0
1 1 1 1
0

Theorem. *Among the* $r!$ *MBST's corresponding to a given interval set* $\mathcal{I}$, *the number of* ***distinct*** *MBSD's they define is* $\mathrm{is}(\lambda/\mu)$.

Key lemma: Let P be a p-element poset. A bijection $f : P \to \{1, \dots, p\}$ is **dropless** if we never have $f^{-1}(i+1) < f^{-1}(i)$ in P. Then the number of dropless f is the number of acyclic orientations of the incomparability graph $\mathrm{inc}(P)$.



(Goldman-Joichi-White, Buhler-Graham, Steingrimsson)

Corollary. *For* $\sigma \vdash n = |\lambda/\mu|$, *let*

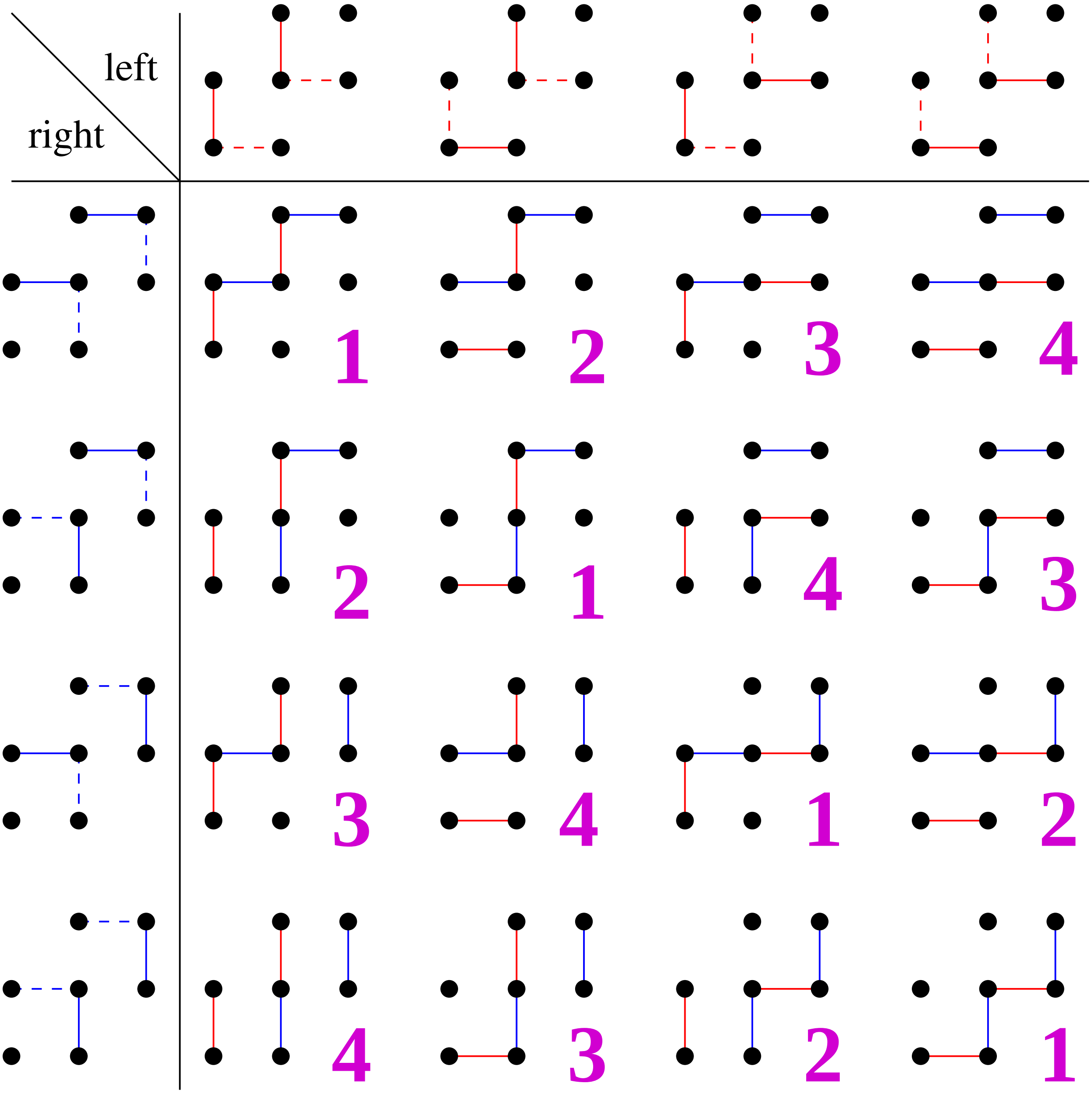
$\mathrm{is}_\sigma(\lambda/\mu) = \#$interval sets of λ/μ of type σ

$\mathrm{mbsd}_\sigma(\lambda/\mu) = \#$MBSD's of λ/μ of type σ.

Then

$$\mathrm{mbsd}_\sigma(\lambda/\mu) = \mathrm{is}_\sigma(\lambda/\mu)\mathrm{is}(\lambda/\mu).$$

Can achieve by explicitly partitioning the MBSD's of λ/μ according to the links of their left snakes (or right snakes).



$$\mathcal{I}_1 = \{(1,6),(2,3),(4,5)\},\ \mathcal{I}_2 = \{(1,3),(2,6),(4,5)\}$$
$$\mathcal{I}_3 = \{(1,5),(2,3),(4,6)\},\ \mathcal{I}_4 = \{(1,3),(2,5),(4,6)\}.$$

Application to $\chi^{\lambda/\mu}(\nu)$. Recall

$$\chi^{\lambda/\mu}(\nu) = \sum_{\substack{\text{type}(\mathbf{T})=\nu \\ \text{sh}(\mathbf{T})=\lambda/\mu}} (-1)^{\text{ht}(\mathbf{T})}$$

Corollary.

$$\chi^{\lambda/\mu}(\nu) \neq 0 \Rightarrow \ell(\nu) \geq r = \text{rank}(\lambda/\mu)$$

Theorem. *Let* $\nu = (1^{m_1} 2^{m_2} \cdots)$, *with* $\ell(\nu) = r = \text{rank}(\lambda/\mu)$. *Then*

$$\chi^{\lambda/\mu}(\nu) = \pm m_1!\, m_2! \cdots \sum_{\text{type}(\mathcal{I})=\nu} (-1)^{c(\mathcal{I})},$$

where $c(\mathcal{I})$ *is the number of crossings of the interval set* $\mathcal{I}$.

Corollary. $\ell(\nu) = \text{rank}(\lambda/\mu) \Rightarrow$

$$m_1!\, m_2! \cdots | \chi^{\lambda/\mu}(\nu)$$

Restatement. Recall that

$$s_{\lambda/\mu} = \sum_{\nu} z_{\nu}^{-1} \chi^{\lambda/\mu}(\nu) p_{\nu},$$

where p_{ν} is a power sum symmetric function. Define

$$\hat{s}_{\lambda/\mu} = \sum_{\ell(\nu)=r} \chi^{\lambda/\mu}(\nu) p_{\nu}.$$

Let $\tilde{p}_i = \frac{p_i}{i}$. Then $\pm \hat{s}_{\lambda/\mu}$ can be expressed as a $2r \times 2r$ Pfaffian with entries 0 or $\tilde{p}_i$. E.g.,

$$\hat{s}_{443/2} = \mathrm{Pf} \begin{pmatrix} 0 & \tilde{p}_3 & 0 & \tilde{p}_5 & \tilde{p}_6 \\ & \tilde{p}_2 & 0 & \tilde{p}_4 & \tilde{p}_5 \\ & & 0 & 0 & 0 \\ & & & \tilde{p}_1 & \tilde{p}_2 \\ & & & & 0 \\ & & & & \end{pmatrix}.$$