Products of Cycles

Richard P. Stanley

M.I.T.

A simple question

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Let $n \geq 2$. Choose $w \in \mathfrak{S}_n$ (uniform distribution). What is the probability $\sigma_2(n)$ that $1, 2$ are in the same cycle of w?

The “fundamental bijection”

Write w as a product of disjoint cycles, least element of each cycle first, decreasing order of least elements:

$$(\mathbf{6},8)(\mathbf{4})(\mathbf{2},7,3)(\mathbf{1},5).$$

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The map $\boldsymbol{f}: \mathfrak{S}_n \to \mathfrak{S}_n$, $f(w) = \widehat{w}$, is a bijection (**Foata**).

Answer to question

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\begin{aligned}
w &= (6,8)(4)(2,7,3)(1,5) \\
\widehat{w} &= 68427315
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$\Rightarrow$ **Theorem.** $\sigma_2(n) = 1/2$

α-separation

Let $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_j)$ be a **composition** of k, i.e., $\alpha_i \geq 1$, $\sum \alpha_i = k$.

Let $n \geq k$. Define $w \in \mathfrak{S}_n$ to be **$\boldsymbol{\alpha}$-separated** if $1, 2, \dots, \alpha_1$ are in the same cycle C_1 of w, $\alpha_1 + 1, \alpha_1 + 2, \dots, \alpha_1 + \alpha_2$ are in the same cycle $C_2 \neq C_1$ of w, etc.

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Example. $w = (\mathbf{1}, \mathbf{2}, 10)(\mathbf{3}, 12, 7)(\mathbf{4}, 6, \mathbf{5}, 9)(8, 11)$ is $(2, 1, 2)$-separated.

Generalization of $\sigma_2(n) = 1/2$

Let $\boldsymbol{\sigma_\alpha(n)}$ be the probability that a random permutation $w \in \mathfrak{S}_n$ is α-separated, $\alpha = (\alpha_1, \ldots, \alpha_j)$, $\sum \alpha_i = k$.

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Similar argument gives:

Theorem.

$$\sigma_\alpha(n) = \frac{(\alpha_1 - 1)! \cdots (\alpha_j - 1)!}{k!}.$$

Conjecture of M. Bóna

Conjecture (**Bóna**). Let u, v be random n-cycles in $\mathfrak{S}_n$, n **odd**. The probability $\boldsymbol{\pi_2(n)}$ that uv is (2)-separated (i.e., 1 and 2 appear in the same cycle of uv) is 1/2.

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Corollary. Probability that uv is $(1,1)$-separated:

$$\pi_{(1,1)}(n) = 1 - \frac{1}{2} = \frac{1}{2}.$$

$n = 3$ and even n

Example ($n = 3$).

$$
\begin{aligned}
(1,2,3)(1,3,2) &= (1)(2)(3): \quad (1,1) \text{ – separated} \\
(1,3,2)(1,2,3) &= (1)(2)(3): \quad (1,1) \text{ – separated} \\
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What about n even?

Probability $\pi_2(n)$ that uv is (2)-separated:

n	2	4	6	8	10
$\pi_2(n)$	0	7/18	9/20	33/70	13/27

Theorem on (2)-separation

Theorem. *We have*

$$\pi_2(n) = \begin{cases} \frac{1}{2}, & n \text{ odd} \\ \frac{1}{2} - \frac{2}{(n-1)(n+2)}, & n \text{ even.} \end{cases}$$

Sketch of proof

Let $w \in \mathfrak{S}_n$ have cycle type $\boldsymbol{\lambda} \vdash n$, i.e.,

$$\lambda = (\lambda_1, \lambda_2, \dots),\ \lambda_1 \geq \lambda_2 \geq \cdots \geq 0,\ \sum \lambda_i = n,$$

cycle lengths $\lambda_i > 0$.

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cycle lengths $\lambda_i > 0$.

$$\text{type}((1,3)(2,9,5,4)(7)(6,8)) = (4,2,2,1)$$

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Easy:

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E.g., $q_{(1,1,\ldots,1)} = 0$.

a_λ

Let a_λ be the number of pairs (u, v) of n-cycles in $\mathfrak{S}_n$ for which uv has type λ (a **connection coefficient**).

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Easy: $\pi_2(n) = \frac{1}{(n-1)!^2} \sum_{\lambda \vdash n} a_\lambda q_\lambda.$

The key lemma

Let $\boldsymbol{n!/z_\lambda} = \#\{w \in \mathfrak{S}_n \ : \ \text{type}(w) = \lambda\}$. E.g.,

$$\frac{n!}{z_{(1,1,\dots,1)}} = 1, \quad \frac{n!}{z_{(n)}} = (n-1)!.$$

Lemma (**Boccara**, 1980).

$$a_\lambda = \frac{n!(n-1)!}{z_\lambda} \int_0^1 \prod_i \left(x^{\lambda_i} - (x-1)^{\lambda_i}\right) dx.$$

A “formula” for $\pi_2(n)$

$$\pi_2(n) = \frac{1}{(n-1)!^2} \sum_{\lambda \vdash n} \frac{n!}{z_\lambda} \left(\sum_i \frac{\lambda_i(\lambda_i - 1)}{n(n-1)} \right)$$

$$\cdot (n-1)! \int_0^1 \prod_i \left(x^{\lambda_i} - (x-1)^{\lambda_i} \right) dx$$

$$= \frac{1}{n-1} \sum_{\lambda \vdash n} z_\lambda^{-1} \left(\sum_i \lambda_i(\lambda_i - 1) \right)$$

$$\cdot \int_0^1 \prod_i \left(x^{\lambda_i} - (x-1)^{\lambda_i} \right) dx.$$

The exponential formula

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Let $\boldsymbol{p_m(x)} = x_1^m + x_2^m + \cdots$,

$$\boldsymbol{p_\lambda(x)} = p_{\lambda_1}(x) p_{\lambda_2}(x) \cdots .$$

“Exponential formula, permutation version”

$$\exp \sum_{m \geq 1} \frac{1}{m} p_m(x) = \sum_\lambda z_\lambda^{-1} p_\lambda(x).$$

The “bad” factor

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Compare

$$\pi_2(n) = \frac{1}{n-1} \sum_{\lambda \vdash n} z_\lambda^{-1} \left(\sum_i \lambda_i(\lambda_i - 1) \right)$$

$$\cdot \int_0^1 \prod_i \left(x^{\lambda_i} - (x-1)^{\lambda_i} \right) dx.$$

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$$\cdot \int_0^1 \prod_i \left(x^{\lambda_i} - (x-1)^{\lambda_i} \right) dx.$$

Bad: $\sum \lambda_i(\lambda_i - 1)$

A trick

Straightforward: Let $\boldsymbol{\ell(\lambda)}$ = number of parts.

$$2^{-\ell(\lambda)+1}\left(\frac{\partial^2}{\partial a^2}-\frac{\partial^2}{\partial a\partial b}\right)p_\lambda(a,b)\Big|_{a=b=1}=\sum\lambda_i(\lambda_i-1).$$

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Exponential formula gives:

$$\sum(n-1)\pi_2(n)t^n = 2\int_0^1\left(\frac{\partial^2}{\partial a^2}-\frac{\partial^2}{\partial a\partial b}\right)$$

$$\exp\left[\sum_{k\geq 1}\frac{1}{k}\left(\frac{a^k+b^k}{2}\right)(x^k-(x-1)^k)t^k\right]\Bigg|_{a=b=1} dx.$$

Miraculous integral

Get:

$$\begin{aligned}\sum (n-1)\pi_2(n)t^n &= \int_0^1 \frac{t^2(1-2x-2tx+2tx^2)}{(1-t(x-1))(1-tx)^3}dx \\ &= \frac{1}{t^2}\log(1-t^2)+\frac{3}{2}+\frac{-\frac{1}{2}+t}{(1-t)^2}\end{aligned}$$

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$$= \frac{1}{t^2}\log(1-t^2) + \frac{3}{2} + \frac{-\frac{1}{2}+t}{(1-t)^2}$$

(coefficient of t^n)$/(n-1)$:

$$\pi_2(n) = \begin{cases} \frac{1}{2}, & n \text{ odd} \\ \frac{1}{2} - \frac{2}{(n-1)(n+2)}, & n \text{ even.} \end{cases}$$

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Combinatorial proof **open**, even for n odd.

Generalizations, with R. Du (杜若霞)

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Some simple relations hold, e.g.,

$$\pi_3(n) = \pi_4(n) + \pi_{3,1}(n).$$

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First step: generalize

$$2^{-\ell(\lambda)+1}\left(\frac{\partial^2}{\partial a^2} - \frac{\partial^2}{\partial a \partial b}\right) p_\lambda(a,b)\Big|_{a=b=1} = \sum \lambda_i(\lambda_i - 1).$$

The case $\alpha = (3)$

$$3^{-\ell(\lambda)+1}\left(\frac{\partial^3}{\partial a^3} - 3\frac{\partial^3}{\partial a^2 \partial b} + 2\frac{\partial^3}{\partial a \partial b \partial c}\right) p_\lambda(a,b,c)\Big|_{a=b=c=1}$$

$$= \sum \lambda_i(\lambda_i - 1)(\lambda_i - 2)$$

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Theorem.

$$\pi_3(n) = \begin{cases} \frac{1}{3} + \frac{1}{(n-2)(n+3)}, & n \text{ odd} \\ \frac{1}{3} - \frac{3}{(n-1)(n+2)}, & n \text{ even} \end{cases}$$

$\pi_{(1^k)}(n)$

Theorem. *Let $n \geq k \geq 2$. Then*

$$\pi_{(1^k)}(n) = \begin{cases} \dfrac{1}{k!}, & n-k \text{ od} \\ \dfrac{1}{k!} + \dfrac{2}{(k-2)!(n-k+1)(n+k)}, & n-k \text{ ev} \end{cases}$$

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Combinatorial proof, especially for $n-k$ odd?

A general result

Recall: $\boldsymbol{\sigma_\alpha(n)}$ $=$ probability that a random permutation $w \in \mathfrak{S}_n$ is α-separated $= (\alpha_1 - 1)! \cdots (\alpha_j - 1)!/k!$.

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Theorem. *Fix $m \geq 0$, and let α be a composition of k. Then there exist rational functions $R_\alpha(n)$ and $S_\alpha(n)$ of n such that for n sufficiently large,*

$$\pi_\alpha(n) = \begin{cases} R_\alpha(n), & n \text{ even} \\ S_\alpha(n), & n \text{ odd.} \end{cases}$$

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$$\pi_\alpha(n) = \begin{cases} R_\alpha(n), & n \text{ even} \\ S_\alpha(n), & n \text{ odd.} \end{cases}$$

Moreover, $\pi_\alpha(n) = \sigma_\alpha(n) + O(1/n)$.

Not the whole story

$$\pi_{(2,2,2)} = \begin{cases} \frac{1}{720} - \frac{n^2+n-32}{20(n-3)(n+4)(n-5)(n+6)}, & n \text{ even} \\ \frac{1}{720} - \frac{n^2+n-26}{20(n-2)(n+3)(n-4)(n+5)}, & n \text{ odd} \end{cases}$$

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$$\pi_{(4,2)} = \begin{cases} \frac{1}{120} - \frac{n^4+2n^3-38n^2-39n+234}{5(n-1)(n+2)(n-3)(n+4)(n-5)(n+6)}, & n \text{ even} \\ \frac{1}{120} - \frac{3n^2+3n-58}{10(n-2)(n+3)(n-4)(n+5)}, & n \text{ odd} \end{cases}$$

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Obvious conjecture. Is there an explicit formula or generating function?

n-cycle times $(n-j)$-cycle

Let $\boldsymbol{\lambda} \vdash n$, $0 \leq j < n$. Let $\boldsymbol{a_{\lambda,j}}$ be the number of pairs $(u,v) \in \mathfrak{S}_n \times \mathfrak{S}_n$ for which u is an n-cycle, v is an $(n-j)$-cycle, and uv has type λ.

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Theorem (Boccara).

$$a_{\lambda,j} = \frac{n!(n-j-1)!}{z_\lambda\, j!} \int_0^1 \frac{d^j}{dx^j} \prod_i \left(x^{\lambda_i} - (x-1)^{\lambda_i}\right) dx.$$

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Easy case:

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Easy case: $j = 1$

The case $j = 1$

$$
\begin{aligned}
\alpha_{\lambda,1} &= \frac{n!(n-2)!}{z_\lambda} \int_0^1 \frac{d}{dx} \prod_i \left(x^{\lambda_i} - (x-1)^{\lambda_i}\right) dx \\
&= \begin{cases} \frac{2n!(n-2)!}{z_\lambda}, & \lambda \text{ odd type} \\ 0, & \lambda \text{ even type.} \end{cases}
\end{aligned}
$$

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In other words, if u is an n-cycle and v is an $(n-1)$-cycle, then uv is equidistributed on odd permutations.

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In other words, if u is an n-cycle and v is an $(n-1)$-cycle, then uv is equidistributed on odd permutations.

Bijective proof known (**A. Machì**, 1992).

Explicit formula

Let $\boldsymbol{u} \in \mathfrak{S}_n$ be a random n-cycle and $\boldsymbol{v} \in \mathfrak{S}_n$ a random $(n-j)$-cycle. Let $\boldsymbol{\pi_\alpha(n,j)}$ be the probability that uv is α-separated.

Theorem.

$$\pi_\alpha(n,1) = \frac{(\alpha_1-1)!\cdots(\alpha_\ell-1)!}{(j-2)!}$$

$$\times\left(\frac{1}{j(j-1)} + (-1)^{n-j}\frac{1}{n(n-1)}\right)$$

General j

Recall (Boccara):

$$a_{\lambda,j} = \frac{n!(n-j-1)!}{z_\lambda\, j!} \int_0^1 \frac{d^j}{dx^j} \prod_i \left(x^{\lambda_i} - (x-1)^{\lambda_i}\right) dx.$$

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$$= c(n,j)\, \frac{d^{j-1}}{dx^{j-1}} \prod_i \left(x^{\lambda_i} - (x-1)^{\lambda_i}\right) dx \Big|_0^1$$

Taylor series

$$a_{\lambda,j} = c(n,j)\,\frac{d^{j-1}}{dx^{j-1}} \prod_i \left(x^{\lambda_i} - (x-1)^{\lambda_i}\right) dx \Big|_0^1$$

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Can treat all $j \geq 1$ at one time using

$$\sum_{j\geq 0}\frac{d^j}{dx^j}f(a)\frac{x^j}{j!} = f(x+a).$$

Some results

Theorem. *Fix $m \geq 0$, and let α be a composition of k. Then there exist rational functions $R_\alpha(n)$ and $S_\alpha(n)$ of n such that for n sufficiently large,*

$$\pi_\alpha(n,j) = \begin{cases} R_\alpha(n), \ n \text{ even} \\ S_\alpha(n), \ n \text{ odd.} \end{cases}$$

Some results

Theorem. *Fix $m \geq 0$, and let α be a composition of k. Then there exist rational functions $R_\alpha(n)$ and $S_\alpha(n)$ of n such that for n sufficiently large,*

$$\pi_\alpha(n,j) = \begin{cases} R_\alpha(n), & n \text{ even} \\ S_\alpha(n), & n \text{ odd.} \end{cases}$$

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Moreover, $\pi_\alpha(n,j) = \sigma_\alpha(n,j) + O(1/n)$.

Probably $O(1/n^2)$.

The case $\alpha = (1, 1)$

Theorem.

$$\pi_{(1,1)}(n,j) = \frac{1}{2} + \begin{cases} \frac{j}{(n-j+1)(n-1)}, & n-j \text{ odd} \\ \frac{2(n-j+1)-j(n-j)}{(n-j+1)(n-j+2)(n-1)}, & n-j \text{ odd} \end{cases}$$

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Note the case $j = 0$ and n odd: $\pi_{(1,1)}(n) = 1/2$ (Bóna's conjecture).

Open problems

Many additional open problems remain, e.g.:

- Combinatorial proofs
- Nice denominators
- Product of $(n-1)$-cycle and $(n-1)$-cycle, for instance

Some data

$\boldsymbol{P_n} + \frac{1}{2} =$ probability that uv is (2)-separated, where u, v are $(n-1)$-cycles in $\mathfrak{S}_n$

n	3	4	5	6	7
P_n	1/6	-1/8	1/25	-19/432	19/980

II. Number of cycles

$\boldsymbol{\kappa(w)}$: number of cycles of w

$$\boldsymbol{P_\lambda(q)} = \sum_{\text{type}(w)=\lambda} q^{\kappa((1,2,\ldots,n)\cdot w)}.$$

$$\boldsymbol{(a)_n} = a(a-1)\cdots(a-n+1)$$

$$\boldsymbol{E} f(q) = f(q-1)$$

Formula for $P_\lambda(q)$

Let

$$g_\lambda(t) = \frac{1}{1-t}\prod_{j=1}^{\ell}(1-t^{\lambda_j}).$$

Theorem. $P_\lambda(q) = z_\lambda^{-1} g_\lambda(E)(q+n-1)_n$

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Proof based on symmetric functions. Equivalent to a result of **D. Zagier** (1995).

An example

$$\lambda = (5,2,2,1), \quad z_\lambda = 40$$

$$P_{5221}(q) = 360q^7 + 13860q^5 + 59220q^3 + 17280q$$

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approximate zeros of $P_{5221}(q)$:

$$\pm 5.80i, \quad \pm 2.13i, \quad \pm 0.561i, \quad 0$$

A lemma on zeros

Lemma. *Let $g(t)$ be a complex polynomial of degree exactly d, such that every zero of $g(t)$ lies on the circle $|z| = 1$. Suppose that the multiplicity of 1 as a root of $g(t)$ is $m \geq 0$. Let $P(q) = g(E)(q + n - 1)_n$.*

(a) *If $d \leq n - 1$, then*

$$P(q) = (q + n - d - 1)_{n-d}\, Q(q),$$

where $Q(q)$ is a polynomial of degree $d - m$ for which every zero has real part $(d - n + 1)/2$.

Lemma (continued)

(b) *If* $d \geq n - 1$*, then* $P(q)$ *is a polynomial of degree* $n - m$ *for which every zero has real part* $(d - n + 1)/2$*.*

An application to $P_\lambda(q)$

$\boldsymbol{\ell(\lambda)}$: number of parts of λ

Corollary. *The polynomial* $P_\lambda(q)$ *has degree* $n - \ell(\lambda) + 1$*, and every zero of* $P_\lambda(q)$ *has real part 0.*

Parity

Simple parity argument gives $P_\lambda(q) = (-1)^n P_\lambda(-q)$. Thus

$$P_\lambda(q) = \begin{cases} R_\lambda(q^2), & n \text{ even} \\ qR_\lambda(q^2), & n \text{ odd}, \end{cases}$$

for some polynomial $R_\lambda(q)$.

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Reformulation of previous corollary: $R_\lambda(q)$ has (nonpositive) real zeros.

$\Rightarrow$ The coefficients of $R_\lambda(q)$ are log-concave with no external zeros, and hence unimodal.

The case $\lambda = (n)$

$$P_{(n)}(q) = \frac{1}{n(n+1)}((q+n)_{n+1} - (q)_{n+1}).$$

$\boldsymbol{c(n,k)}$: number of $w \in \mathfrak{S}_n$ with k cycles (signless Stirling number of the first kind)

Corollary. *The number of n-cycles $w \in \mathfrak{S}_n$ for which $w \cdot (1,2,\dots,n)$ has exactly k cycles is 0 if $n-k$ is odd, and is otherwise equal to $c(n+1,k)/\binom{n+1}{2}$.*

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Combinatorial proof by **V. Féray** and **E. A. Vassilieva**.

A generalization?

Let $\lambda, \mu \vdash n$.

$$\boldsymbol{P_{\lambda,\mu}(q)} = \sum_{\rho(w)=\lambda} q^{\kappa(w_\mu \cdot w)},$$

where $\boldsymbol{w_\mu}$ is a fixed permutation of cycle type μ.

Does $P_{\lambda,\mu}(q)$ have purely imaginary zeros?

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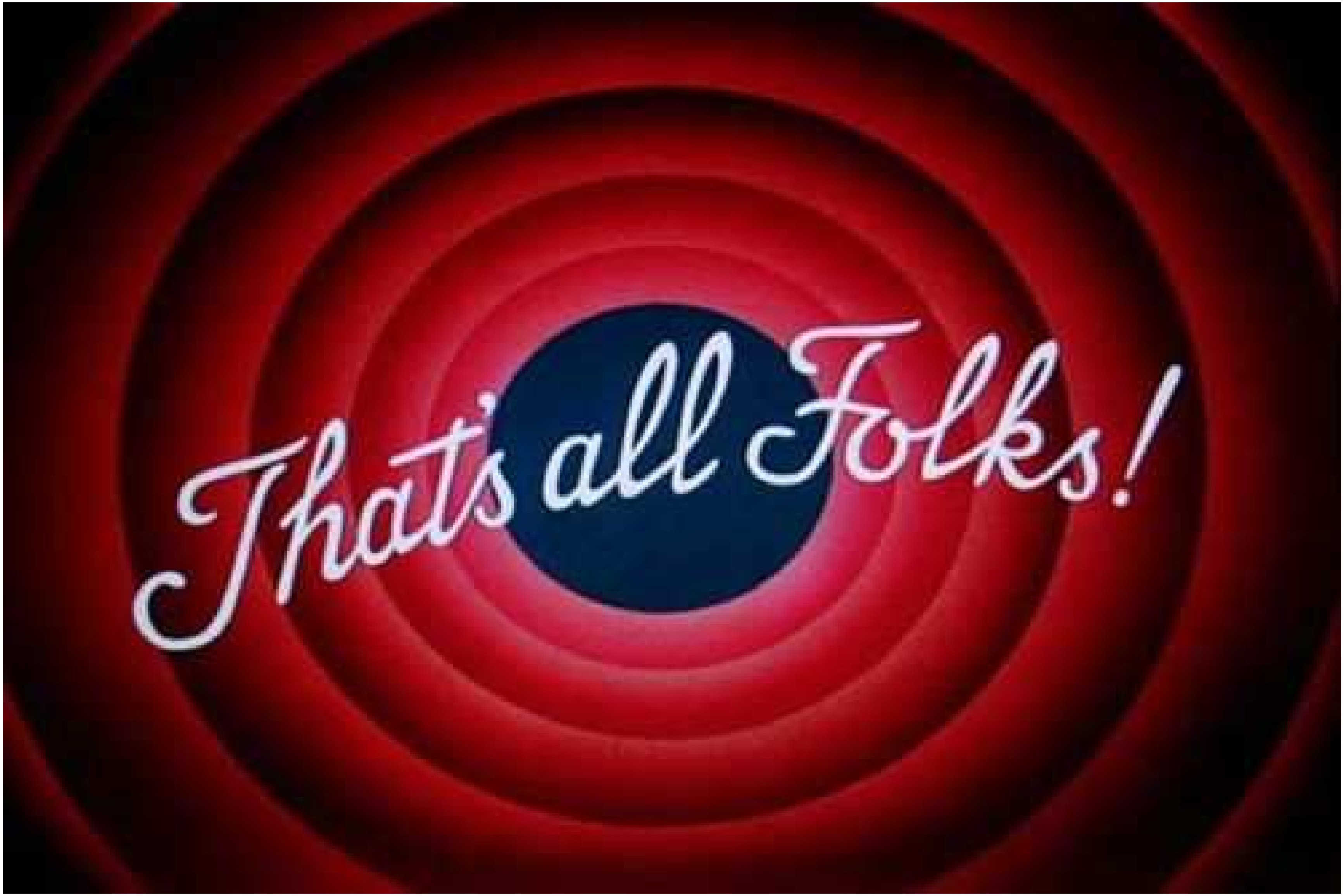
Does $P_{\lambda,\mu}(q)$ have purely imaginary zeros?

Alas, $P_{332,332}(q) = q^8 + 35q^6 + 424q^4 + 660q^2$, four of whose zeros are approximately

$$\pm 1.11366 \pm 4.22292i.$$


That's all Folks!