More interesting polytopes

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Origin (Postnikov & RS): Let

$$\boldsymbol{M_n} = x_{12}x_{23}\cdots x_{n-1,n}.$$

Continually apply

$$x_{ij}x_{jk} \to x_{ik}(x_{ij}+x_{jk}),$$

ending with $P_n(x_{ij})$.

An example

Example.

$$\begin{aligned}
\boldsymbol{x_{12}}\boldsymbol{x_{23}}x_{34} &\longrightarrow \boldsymbol{x_{13}}x_{12}\boldsymbol{x_{34}} + \boldsymbol{x_{13}}x_{23}\boldsymbol{x_{34}} \\
&\longrightarrow x_{14}x_{13}x_{12} + x_{14}x_{34}x_{12} \\
&\qquad + x_{14}x_{13}x_{23} + x_{14}\boldsymbol{x_{34}}\boldsymbol{x_{23}} \\
&\longrightarrow x_{14}x_{13}x_{12} + x_{14}x_{34}x_{12} \\
&\qquad + x_{14}x_{13}x_{23} + x_{14}x_{24}x_{23} + x_{14}x_{24}x_{34} \\
&= P_3(x_{ij}).
\end{aligned}$$

Invariance of $P_n(x_{ij})$

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Theorem. We have

$$P_n(1, 1, \ldots, 1) = C_n = \frac{1}{n+1}\binom{2n}{n},$$

a **Catalan number**.

Full root polytopes

e_i: ith unit vector in $\mathbb{R}^{n+1}$

A_n^+: the positive roots

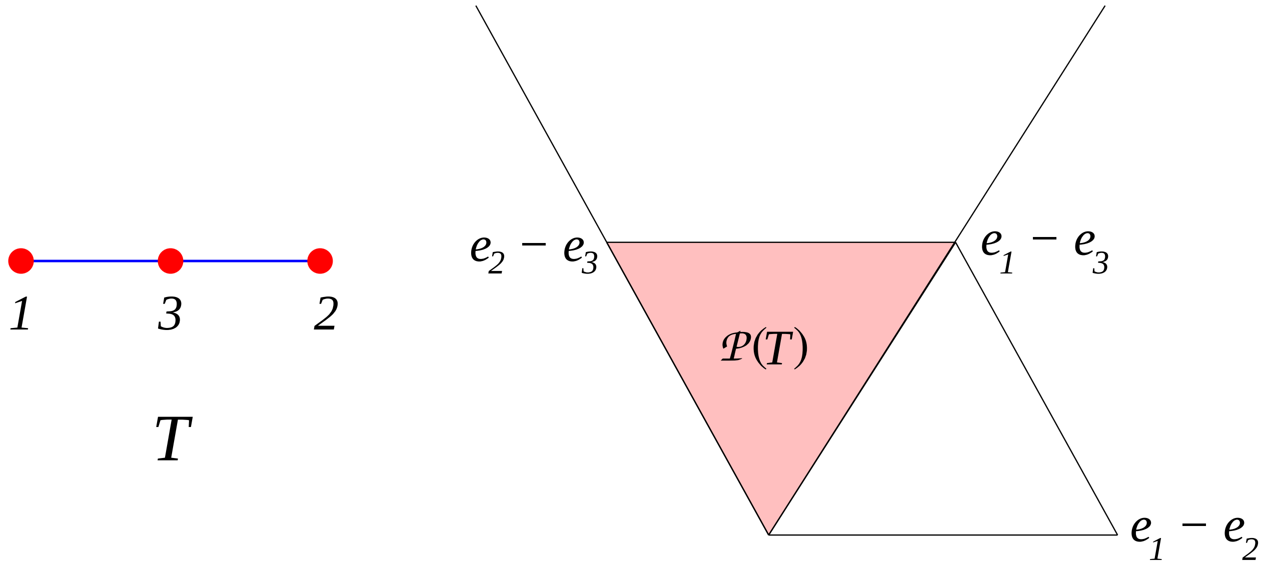
$$\{e_i - e_j \;:\; 1 \le i < j \le n+1\}$$

full root polytope $\mathcal{P}(A_n^+)$: convex hull of A_n^+ and the origin in $\mathbb{R}^{n+1}$ (**Gelfand-Graev-Postnikov**)

Root polytopes

T: a tree on the vertex set $[n+1]$

root polytope $\mathcal{P}(T)$ (of type A_n): intersection of $\mathcal{P}(A_n^+)$ with the cone generated by $e_i - e_j$, where $ij \in E(T)$, $i < j$



Noncrossing alternating trees

A graph G on $[n+1]$ is **noncrossing** if $\nexists$ vertices $i < j < k < l$ such that $ik \in E(G)$ and $jl \in E(G)$.

G is **alternating** if $\nexists$ $i < j < k$ such that $ij \in E(G)$ and $jk \in E(G)$.

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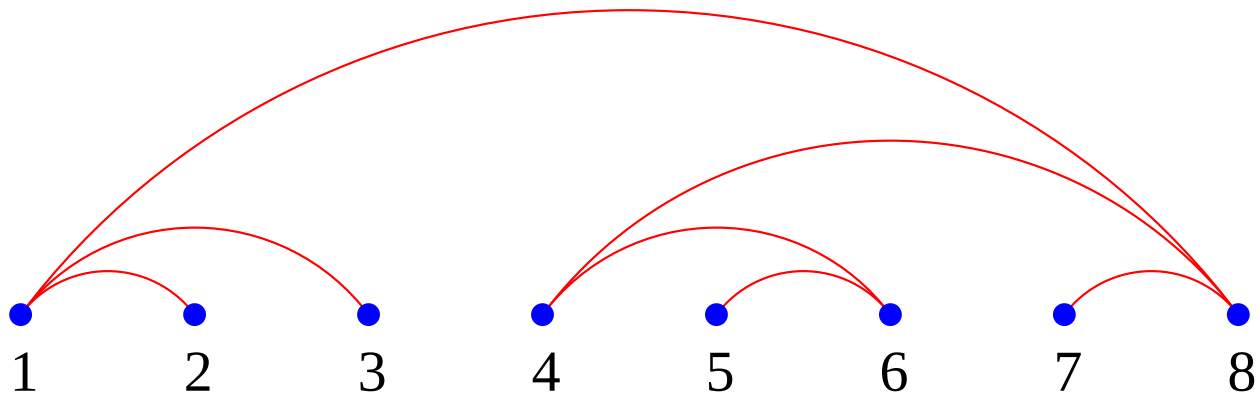

Some notation

$\overline{G}$: graph with vertex set $[n+1]$ and edge set

$\{ij : \exists\, ii_1, i_1i_2, \ldots, i_kj \in E(G),\ i < i_1 < \cdots < i_k < j\},$

the **transitive closure** of G

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the **transitive closure** of G

T: a noncrossing tree on $[n+1]$

$T_1, \ldots, T_k$: noncrossing, alternating spanning trees of $\overline{T}$

Volume of $\mathcal{P}(T)$

Theorem. *The root polytopes* $\mathcal{P}(T_1), \ldots, \mathcal{P}(T_k)$ *are* n*-simplices with disjoint interior and union* $\mathcal{P}(T)$*. Moreover,*

$$\operatorname{vol} \mathcal{P}(T) = \frac{f_T}{n!},$$

where $\boldsymbol{f_T}$ *is the number of noncrossing alternating spanning trees of* $\overline{T}$*.*

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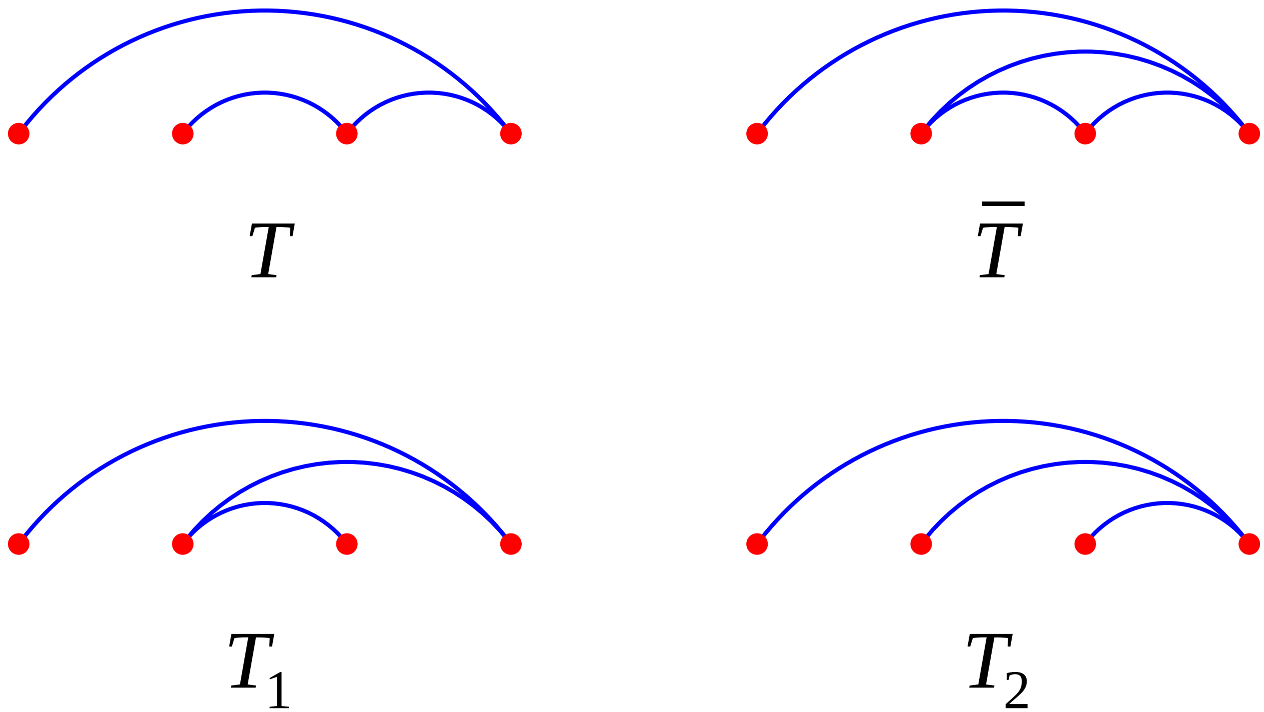
(several generalizations)

Example



$$\operatorname{vol} \mathcal{P}(T) = \frac{2}{3!}$$

Yang-Baxter algebras

Proof of theorem:

$\mathcal{B}(A_n)$: **quasi-classical Yang-Baxter algebra** or **bracket algebra of type A** (Anatol Kirillov). It is an associative algebra over $\mathbb{Q}[\beta]$ (β a central indeterminate) generated by

$$\{x_{ij} : 1 \leq i < j \leq n+1\},$$

with relations

$$\begin{aligned} x_{ij}x_{jk} &= x_{ik}x_{ij} + x_{jk}x_{ik} + \beta x_{ik} \\ x_{ij}x_{kl} &= x_{kl}x_{ij}, \text{ if } i,j,k,l \text{ are distinct.} \end{aligned}$$

Subdivision algebra

$\boldsymbol{\mathcal{S}(A_n)}$: **subdivision algebra** (Meszaros). It is $\mathcal{B}(A_n)$ made commutative, i.e.,

$$x_{ij}x_{kl} = x_{kl}x_{ij} \text{ for } \textbf{\textit{all}} \ i, j, k, l.$$

Reduction rule

Treat the first relation as a **reduction rule**:

$$x_{ij}x_{jk} \longrightarrow x_{ik}x_{ij} + x_{jk}x_{ik} + \beta x_{ik}.$$

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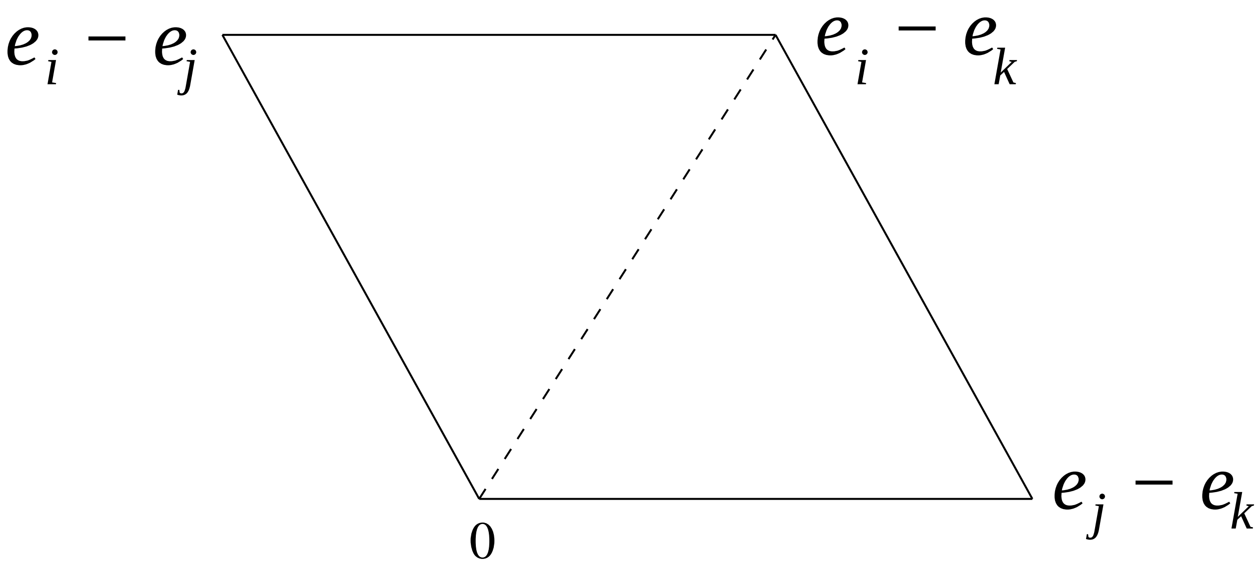

Reduced forms

A **reduced form** of the monomial m in $\mathcal{B}(A_n)$ or $\mathcal{S}(A_n)$ is a polynomial obtained from m by applying successive reductions until no longer possible.

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For $\mathcal{S}(A_n)$ and $\beta = 0$, same as reduction of Postinikov and RS.

A reduction redux

$$\begin{aligned}
\mathbf{x_{12}x_{23}}x_{34} &\longrightarrow \mathbf{x_{13}}x_{12}\mathbf{x_{34}} + \mathbf{x_{13}}x_{23}\mathbf{x_{34}} \\
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\end{aligned}$$

Reduced form of a graph monomial

G: graph on vertex set $[n+1]$

$$m_G = \prod_{ij \in E(G)} x_{ij} \in \mathcal{S}(A_n)$$

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$$\boldsymbol{m_G} = \prod_{ij \in E(G)} x_{ij} \in \mathcal{S}(A_n)$$

Theorem. *Let $\boldsymbol{T}$ be a noncrossing tree on $[n+1]$ and $\boldsymbol{P_T}$ a reduced form of m_G. Then*

$$P_T(x_{ij}=1, \beta=0) = f_T,$$

the number of noncrossing alternating spanning trees of $\overline{T}$.

Relation to root polytopes

The monomials appearing in the reduced form $P_T(x_{ij}, \beta = 0)$ correspond to the facets in a triangulation of $\mathcal{P}(A_n)$.

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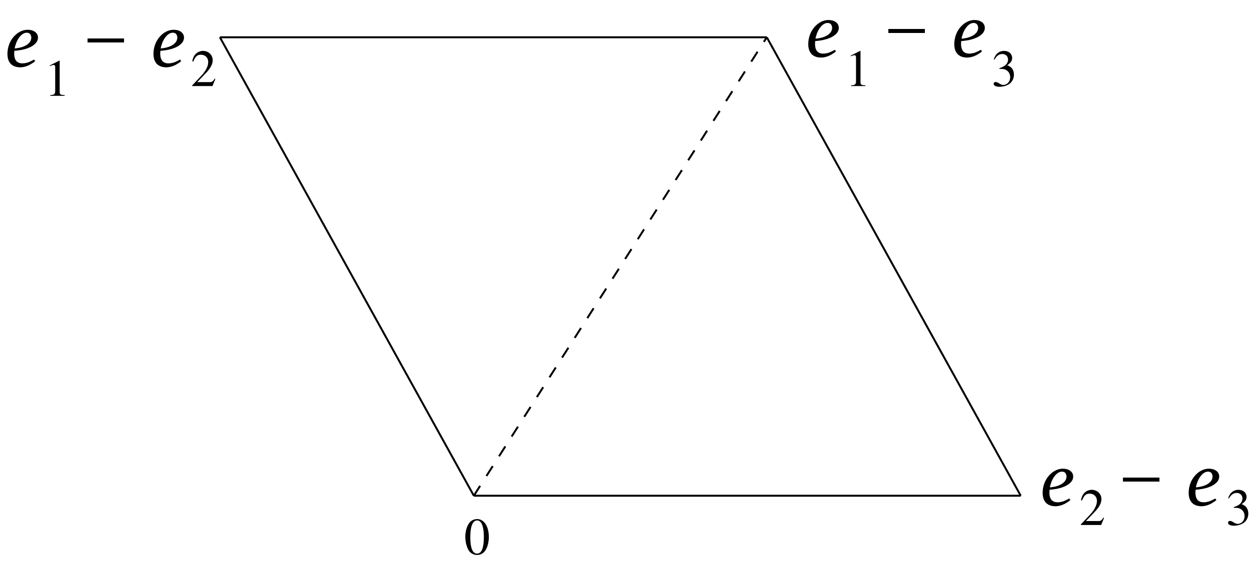
$$x_{12}x_{23} \longrightarrow x_{12}x_{13} + x_{23}x_{13}$$



Interior faces of $\mathcal{P}(A_n)$

The interior faces (not necessarily facets) of $\mathcal{P}(A_n)$ correspond to the terms in the reduced form of $P_T(x_{ij}, \beta)$.

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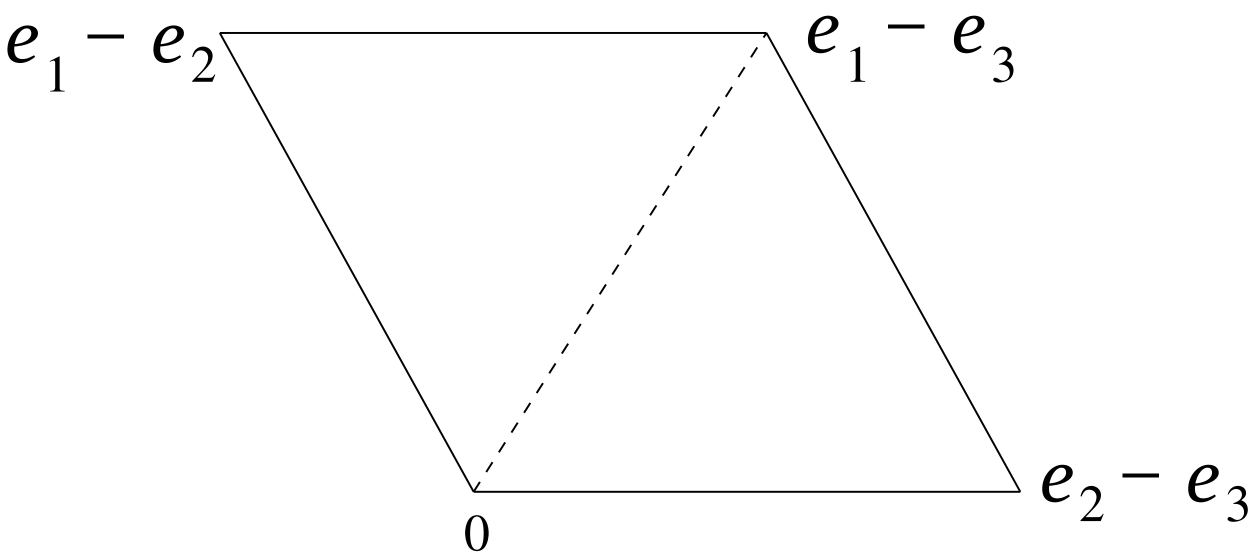
$$x_{12}x_{23} \to x_{12}x_{13} + x_{23}x_{13} + \beta x_{13}$$



Uniqueness

In the ring $\mathcal{B}(A_n)$, the reduced form of any monomial m is **unique** (up to commutations).

Proof uses noncommutative Gröbner bases.

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Many generalizations $\ldots$

Flow polytopes

G = **acyclic** graph on vertex set

$$V(G) = \{1, 2, \ldots, m+1\},$$

with edge $i \xrightarrow{e} j$ only if $i < j$

$$E(G) = \text{edge set of } G$$

Flows

flow on G:

$$f : E(G) \longrightarrow R_{\geq 0},$$

such that for $1 < i < m + 1$,

$$\text{flow into } i = \text{ flow out of } i$$

size of f: flow out of 1 (or into $m + 1$)

$\mathbb{N}$-flows

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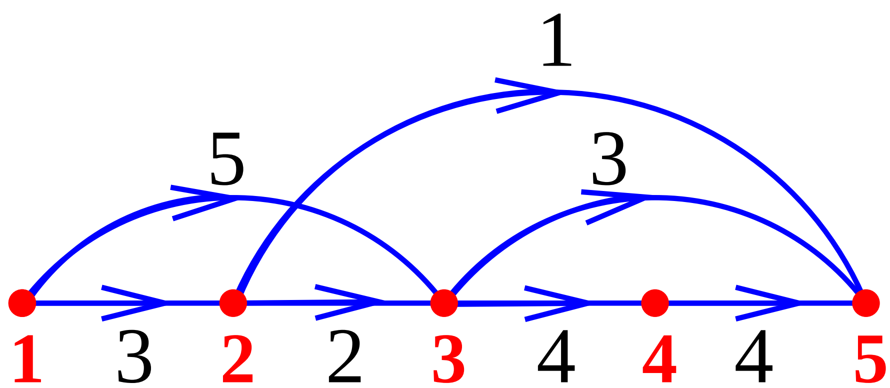

size = 8

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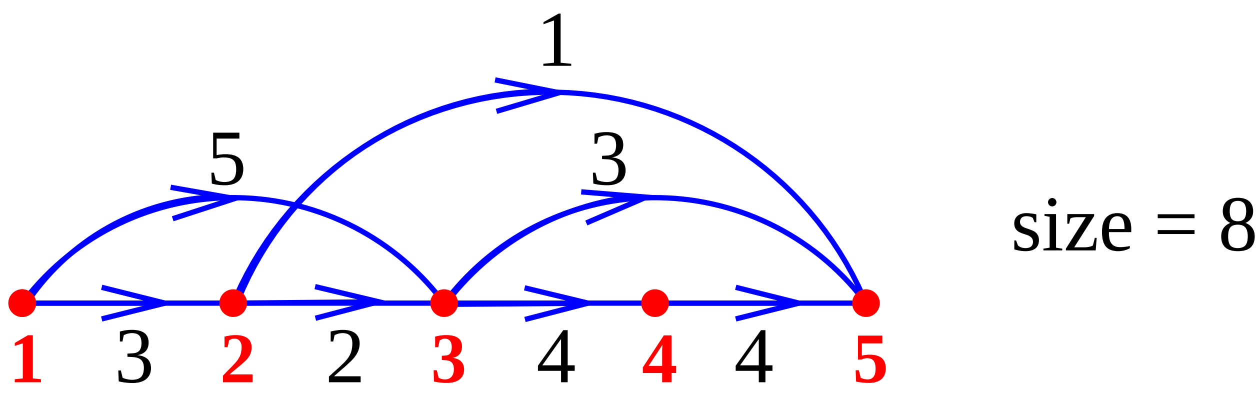

flow polytope $\mathcal{F}(G)$ $\subset \mathbb{R}^{E(G)}$:

$$\{\text{flows } f : E(G) \to \mathbb{R}_{\geq 0} \text{ of size } 1\}$$

Root polytopes vs. flow polytopes

Note. The root polytopes $\mathcal{P}(T)$ of Meszaros are special cases of flow polytopes $\mathcal{F}(G)$. In particular,

$$\mathcal{P}(A_n^+) = \mathcal{F}(K_n).$$

Vertices of $\mathcal{F}(G)$

vertices $\leftrightarrow$ **paths** in G from 1 to $m+1$

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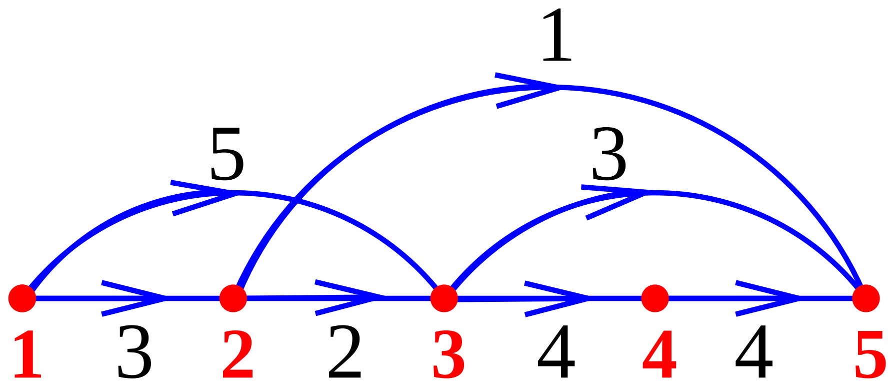

size = 8

12345 1235 125 1345 135

Excess flows

excess flow vector $\gamma = (a_1, \ldots, a_m) \in \mathbb{N}^m$

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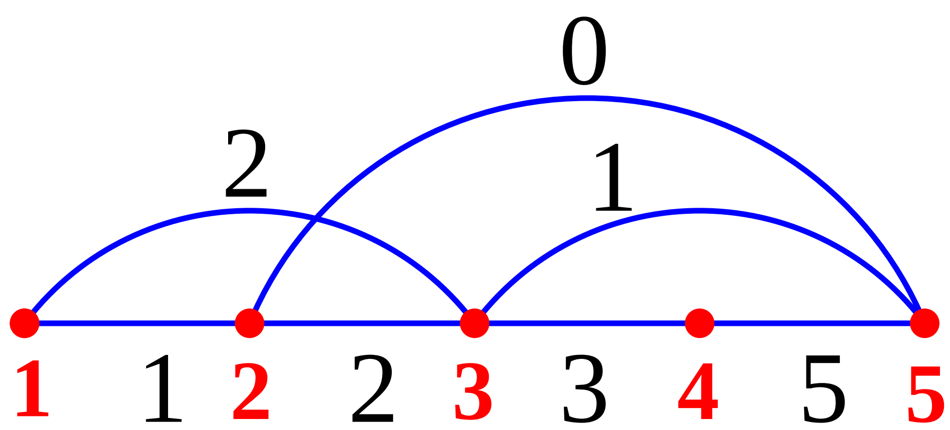

$\gamma = (3, 1, 0, 2)$

The positive roots A_m^+

Recall:

$$e_i = (0 \cdots 0 \overset{i}{1} 0 \cdots 0) \in \mathbb{R}^{m+1}$$

$$e_{ij} = e_i - e_j$$

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$$e_{ij} = e_i - e_j$$

$$A_m^+ = \{e_{ij} : 1 \leq i < j \leq m+1\} \subset \mathbb{Z}^{m+1}$$

(restricted) Kostant partition function

$$\boldsymbol{\nu} \in \mathbb{Z}^{m+1}, \quad \sum \nu_i = 0$$

$$\boldsymbol{A_m^+} = \{e_{ij} \,:\, 1 \leq i < j \leq m+1\} \subset \mathbb{Z}^{m+1}$$

$$\boldsymbol{S} \subseteq A_m^+$$

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$$\boldsymbol{S} \subseteq A_m^+$$

$$\boldsymbol{K_S(\nu)} = \# \left\{ (b_{ij})_{e_{ij} \in S} \,:\, \nu = \sum b_{ij} e_{ij} \right\}$$

$$\boldsymbol{K(\nu)} = K_{A_m^+}(\nu)$$

An example

Example.

$$S = \{e_{12}, e_{23}, e_{13}\} = A_3^+$$

$$(2, 0, -2) = 2e_{12} + 2e_{23} = e_{12} + e_{13} + e_{23} = 2e_{13}$$

$$\Rightarrow K_S(2, 0, -2) = K(2, 0, -2) = 3.$$

Flows and partitions

Proposition. *Let*

$$S = S(G) = \{e_{ij} \ : \ (i,j) \in E(G)\}.$$

The number of $\mathbb{N}$*-flows with excess* $(a_1, \ldots, a_m)$ *is equal to*

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Now let $d_i = \mathrm{outdeg}(i) - 1$.

Main thm. (D. Peterson for $S = A_m^+$

$$K_S\left(a_1, \ldots, a_m, -\sum a_i\right) =$$

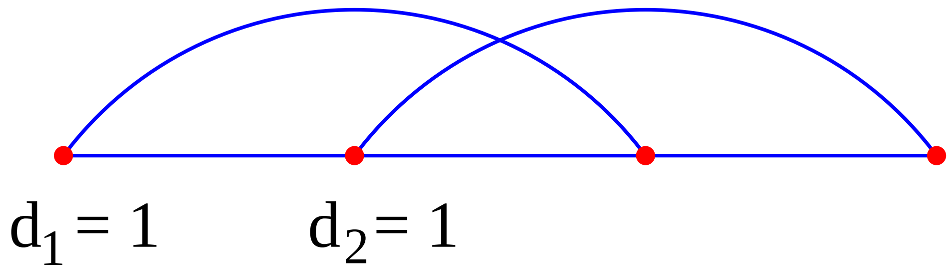
$$\sum K_S(\nu_1 - d_1, \ldots, \nu_{m-1} - d_{m-1}, 0, 0)$$

$$\cdot \binom{a_1 + d_1}{\nu_1} \cdots \binom{a_{m-1} + d_{m-1}}{\nu_{m-1}},$$

summed over all $\nu_1, \ldots, \nu_{m-1} \in \mathbb{N}$ satisfying

$$\begin{aligned} \nu_1 + \cdots + \nu_i &\geq d_1 + \cdots + d_i \\ \sum \nu_i &= d_1 + \cdots + d_{m-1}. \end{aligned}$$

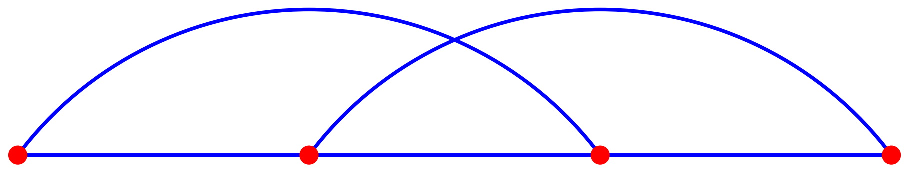
An example



$$(\nu_1, \nu_2) = (2,0), (1,1)$$

$$S = \{e_{12}, e_{13}, e_{23}, e_{24}, e_{34}\}, \ \ K_S(\alpha,\beta) = K_S(\alpha,\beta,0,0)$$

An example



$d_1 = 1$ $d_2 = 1$

$$(\nu_1, \nu_2) = (2,0), (1,1)$$

$$S = \{e_{12}, e_{13}, e_{23}, e_{24}, e_{34}\}, \;\; K_S(\alpha, \beta) = K_S(\alpha, \beta, 0, 0)$$

$$K_S(a, b, c, -a-b-c) =$$

$$K_S(1,-1)\binom{a+1}{2} + K_S(0,0)\binom{a+1}{1}\binom{b+1}{1}$$

$$= \binom{a+1}{2} + (a+1)(b+1).$$

(Piecewise) polynomiality

Corollary. *$K_S(a_1, \ldots, a_m, a_{m+1})$ is a* ***polynomial*** *function of $a_1, \ldots, a_{m+1}$ in the cone*

$$\mathcal{C}_S : \quad x_1, \ldots, x_m \geq 0, \quad x_{m+1} \leq 0.$$

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Note. K_S is **piecewise polynomial** on $\mathbb{Z}^{m+1}$. Mimimum number of cones of nonzero polynomiality not known. For $S = A_m^+$, we have:

m	2	3	4	5	6
# cones	2	7	48	820	51133

An example

Example. $m = 2$:

$$K(a, b, -a-b) = \begin{cases} a+1, & a, b \geq 0 \\ a+b+1, & 0 \leq -b \leq a \\ 0, & \text{otherwise.} \end{cases}$$

An example

Example. $m = 2$:

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Proof of polynomiality based on Elliott-MacMahon algorithm. (There are other proofs.)

Volume of flow polytope

Corollary. *Let* $d = \dim \mathcal{F}(G)$*. Then*

$$d! \cdot \mathrm{vol}(\mathcal{F}(G)) := \widetilde{V}(\mathcal{F}(G))$$

$$= K_S \left(d_{m-1}, d_{m-2}, \ldots, d_1, -\sum d_i \right).$$

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For $G = K_{m+1}$, we have

$$\widetilde{V}(\mathcal{F}_{K_{m+1}}) = K\left(1, 2, \ldots, m-2, -\binom{m-1}{2}\right).$$

Chan-Robbins conjecture

Theorem (**Zeilberger, Baldoni-Vergne**). *We have*

$$\widetilde{V}\left(\mathcal{F}_{K_{m+1}}\right) = C_1 \cdots C_{m-2},$$

where

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$ **(Catalan number)**.

Alternate formulation

Let $f(n)$ be the number of $n \times n$ $\mathbb{N}$-matrices A such that

- $A_{ij} = 0$ if $j > i + 1$
- row and column sum vector

$$\left(1, 3, 6, \ldots, \binom{n+1}{2}\right)$$

Alternate formulation (cont.)

$$\begin{bmatrix} 0 & 1 & & & \\ 0 & 1 & 2 & & \\ 0 & 0 & 2 & 4 & \\ 1 & 0 & 1 & 3 & 5 \\ 0 & 1 & 1 & 3 & 10 \end{bmatrix}$$

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Then $f(n) = C_1 C_2 \cdots C_n$.

Alternate formulation (cont.)

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Then $f(n) = C_1 C_2 \cdots C_n$.

No combinatorial proof known.

Divisibility properties I

Theorem (easy consequence of Ehrhart's law of reciprocity). $K\left(a_1, \ldots, a_m, -\sum a_i\right)$ *is divisible by*

$$(a_1 + 1)(a_1 + 2) \cdots (a_1 + m - 1).$$

Divisibility properties II

Theorem (**J. R. Schmidt and A. M. Bincer**, 1984) *Also divisible by*

$$a_1 + a_2 + \cdots + a_{m-2} + 3a_{m-1} + 3.$$

In fact,

$$3K\left(a_1, \ldots, a_m, -\sum a_i\right) =$$

$$(a_1 + \cdots + a_{m-2} + 3a_{m-1} + 3)$$

$$\cdot K_{\mathbf{no}\ \boldsymbol{e_{m-1,m}}}\left(a_1, \ldots, a_m, -\sum a_i\right).$$

Example and conjecture

Example.

$$K(a,b,c,d,-a-b-c-d) =$$

$$\frac{1}{360}(a+1)(a+2)(a+3)(a+b+3c+3)$$

$$\cdot(a^2+5ab+10b^2+9a+30b+20)$$

Example and conjecture

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Open: Are all coefficients of

$$K_S\left(a_1,\ldots,a_m,-\sum a_i\right)$$

nonnegative?

Matching polytopes (Ricky Liu)

$G = (V, E)$: a graph; $n = \#E$

M_G: **matching polytope** of G, i.e.,

$$M_G = \left\{ w : E \to \mathbb{R}_{\geq 0} \,\middle|\, \forall\, v \in V \sum_{v \in e} w(e) \leq 1 \right\} \subseteq \mathbb{R}^n.$$

Vertices of M_G

matching M: a set of vertex-disjoint edges

If $L \subseteq E$, define $\chi_L \in M_G$ by

$$\chi_L(e) = \begin{cases} 1, & e \in L \\ 0, & e \notin L. \end{cases}$$

Note. M_G has integer vertices if and only if G is bipartite. In that case, the vertices are χ_M, where M is a matching of G.

Vertices of M_G

matching $\boldsymbol{M}$: a set of vertex-disjoint edges

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Note. M_G has integer vertices if and only if G is bipartite. In that case, the vertices are $\boldsymbol{\chi_M}$, where M is a matching of G.

Corollary. G bipartite $\Rightarrow$

$$\boldsymbol{V(G)} := n! \cdot \mathrm{vol}(M_G) \in \mathbb{Z}$$

G, G_1, G_2

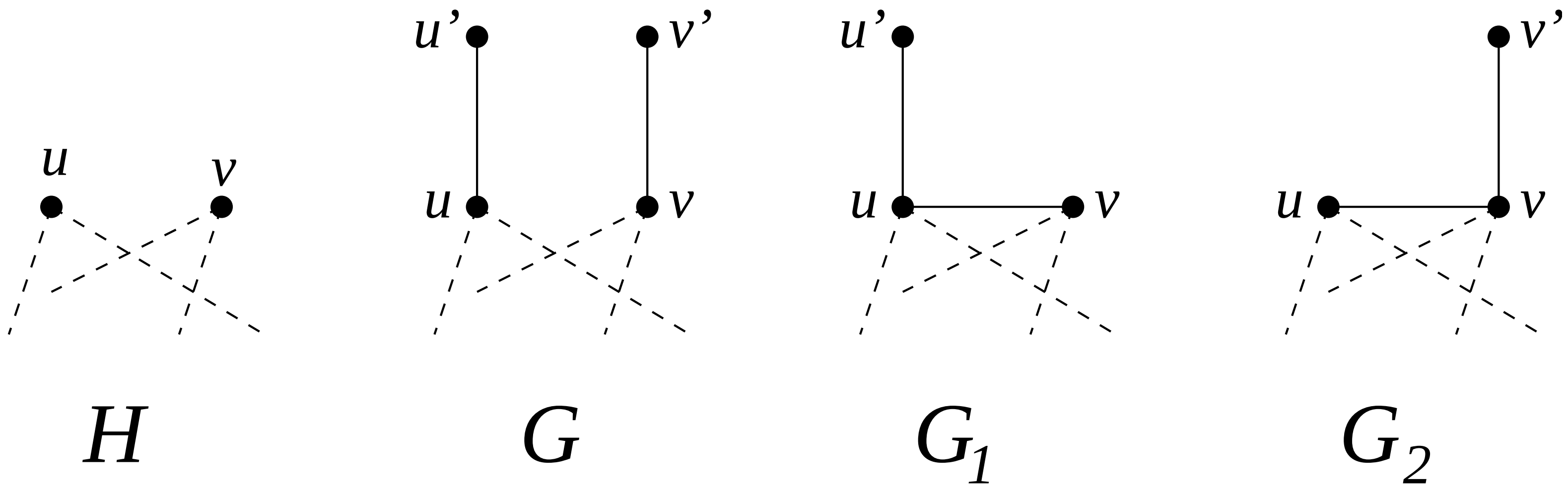
$\boldsymbol{H}$ = graph, $\boldsymbol{u}, \boldsymbol{v} \in V(H)$, $u \neq v$

G: adjoin pendant edges uu', vv' (so u', v' are endpoints)

G_1: adjoin pendant edge uu' and an edge uv

G_2: adjoin pendant edge vv' and an edge uv

Leaf recurrence



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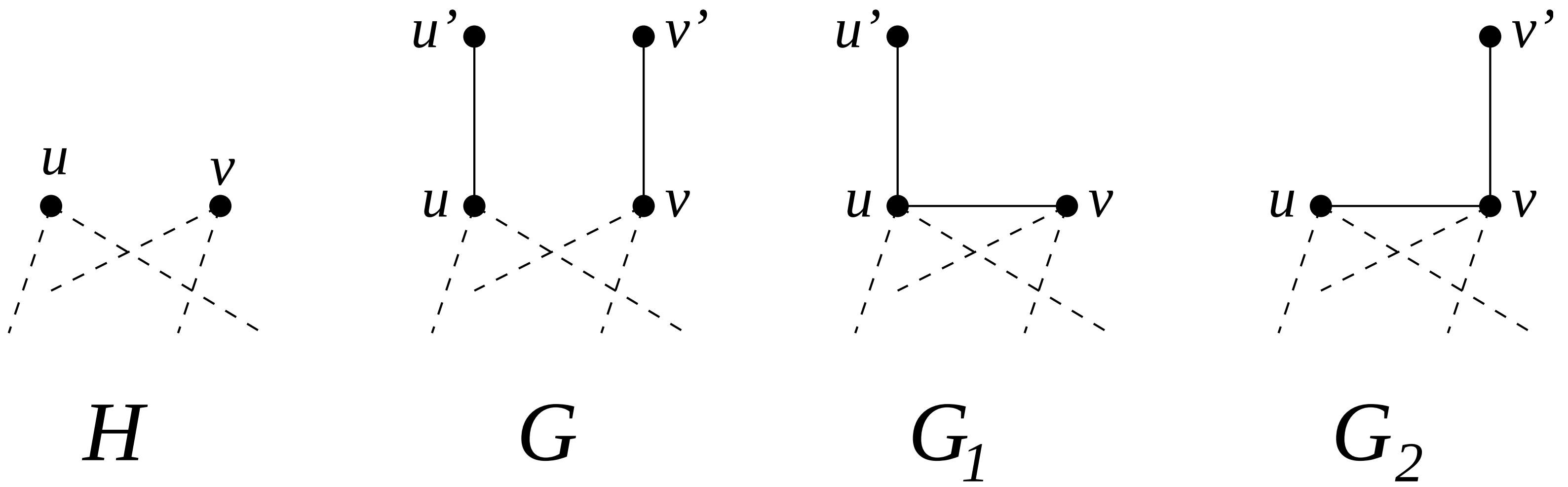

$\mathcal{F}$: set of all forests

$f : \mathcal{F} \to \mathbb{R}$ **satisfies the leaf recurrence** if

$$f(G) = f(G_1) + f(G_2).$$

Volume of M_G

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Then $f(G) = V(G)$*.*

Volume of M_G (continued)

Theorem. *The previous theorem can be used to compute* $V(F)$ *for any forest* F*.*

Diagrams and tableaux

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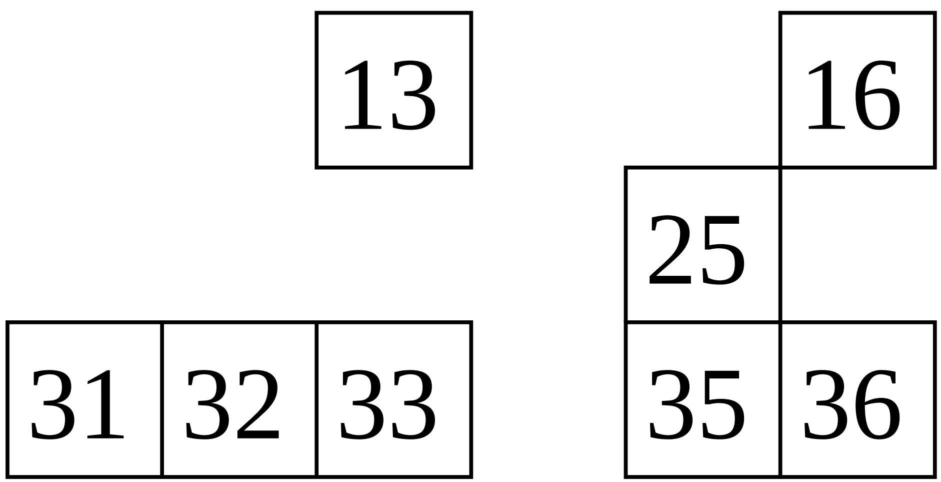

Row and column stabilizers

D: diagram with n boxes, ordered in some way

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$\boldsymbol{R_D}$ $(\boldsymbol{C_D})$: subgroup of $\mathfrak{S}_n$ stabilizing each row (column) of D

$$\boldsymbol{R(D)} = \sum_{w \in R_D} w, \qquad \boldsymbol{C(D)} = \sum_{w \in C_D} \operatorname{sgn}(w) w$$

The Specht module S^D

The **Specht module** S^D (over $\mathbb{C}$) is the left ideal

$$\boldsymbol{S^D} = \mathbb{C}[\mathfrak{S}_n]C(D)R(D)$$

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Note. S^D affords a representation of $\mathfrak{S}_n$ by left multiplication.

Irreducible Specht modules

Note. If D is the (Young) diagram of a partition λ of n, then S^D is irreducible. Conversely, if S^D is irreducible, then $S^D \cong S^{D'}$ for the diagram D' of some partition.

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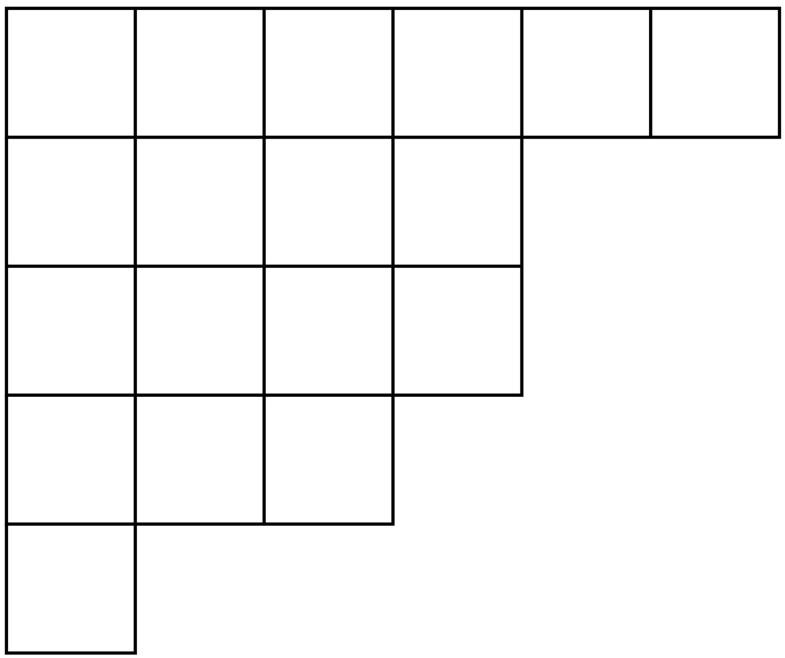

The diagram of a forest F

Let $V(F) = \boldsymbol{A} \cup \boldsymbol{B}$, so that all edges are between A and B. Label the A-vertices $1, \ldots, m$ and B-vertices $1, \ldots, n$. Let

$$\boldsymbol{D(F)} = \{(i,j) \;:\; ij \in E(F),\; i \in A,\; j \in B\}.$$

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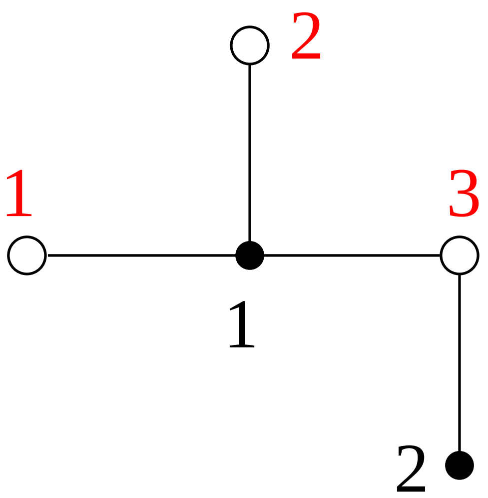

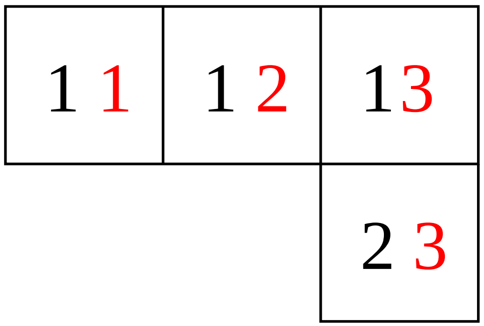

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Note for experts. The diagrams $D(F)$ are **not** %-avoiding diagrams in the sense of Reiner and Shimozono.

Decomposition of $S^{D(F)}$

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with initial conditions $f(K_{n,1}) = 1$.

Change the initial conditions to $f(K_{n,1}) = \boldsymbol{h_n}$, the complete homogeneous symmetric function (**generic leaf recurrence**).

Decomposition theorem

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In other words, if

$$f(F) = \sum_{\lambda \vdash n} \boldsymbol{c_\lambda} s_\lambda,$$

where s_λ is a Schur function, then c_λ is the multiplicity of the irreducible representation of $\mathfrak{S}_n$ indexed by λ in $S^{D(F)}$.

The Ehrhart polynomial of M_F

Open. What is the Ehrhart polynomial of M_F?

Does it have any representation-theoretic significance?


Darn!
That's
the
end...