The Erdős-Moser Conjecture

An Application of Linear Algebra to Combinatorics

Richard P. Stanley

M.I.T.

The function $f(S, \alpha)$

Let $S \subset \mathbb{R}$, $\#S < \infty$, $\alpha \in \mathbb{R}$.

$$f(S, \alpha) = \#\{T \subseteq S : \sum_{i \in T} i = \alpha\}$$

NOTE. $\sum_{i \in \emptyset} i = 0$

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Example. $f(\{1,2,4,5,7,10\},7) = 3$:

$$7 = 2 + 5 = 1 + 2 + 4$$

The conjecture for $S \subset \mathbb{R}$

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Erdős-Moser Conjecture.

$$\#S = 2n + 1$$
$$\Rightarrow f(S, \alpha) \leq f(\{-n, -n+1, \ldots, n\}, 0)$$

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Weak Erdős-Moser Conjecture.

$$S \subset \mathbb{R}^+, \; \#S = n$$

$$\Rightarrow f(S, \alpha) \leq f\left(\{1, 2, \ldots, n\}, \left\lfloor \frac{1}{2}\binom{n+1}{2} \right\rfloor\right)$$

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NOTE. $\frac{1}{2}\binom{n+1}{2} = \frac{1}{2}(1 + 2 + \cdots + n)$

Posets

A **poset** (partially ordered set) is a set $\boldsymbol{P}$ with a binary relation $\leq$ satisfying:

- **Reflexivity:** $t \leq t$
- **Antisymmetry:** $s \leq t,\ t \leq s \Rightarrow s = t$
- **Transitivity:** $s \leq t,\ t \leq u \Rightarrow s \leq u$

Graded posets

chain: $u_1 < u_2 < \cdots < u_k$

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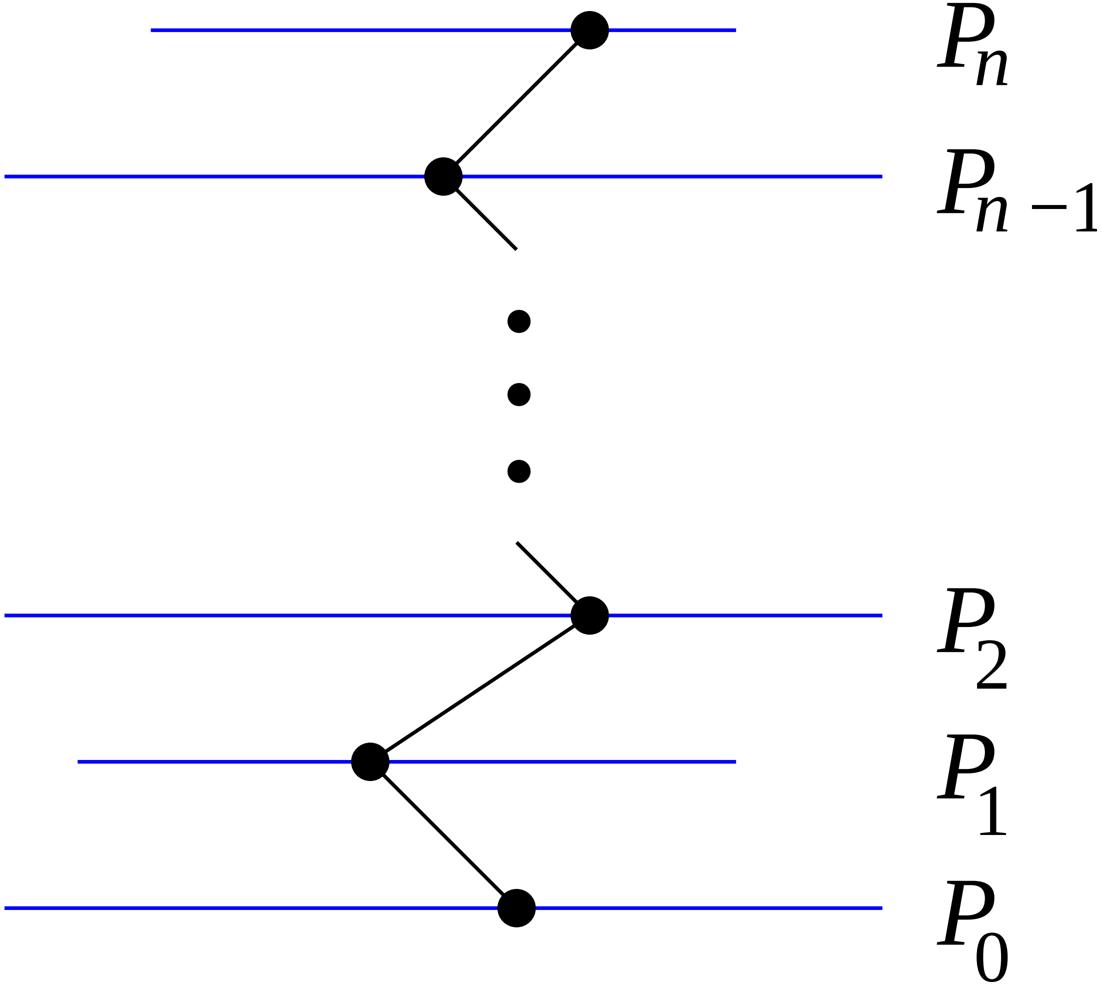
Assume P is **finite**. P is **graded of rank n** if

$$P = P_0 \cup P_1 \cup \cdots \cup P_n,$$

such that every maximal chain has the form

$$t_0 < t_1 < \cdots < t_n, \quad t_i \in P_i.$$

Diagram of a graded poset



Rank-symmetry and unimodality

Let $p_i = \#P_i$.

Rank-generating function: $F_P(q) = \sum_{i=0}^{n} p_i q^i$

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rank-unimodal and rank-symmetric $\Rightarrow j = \lfloor n/2 \rfloor$

The Sperner property

antichain $A \subseteq P$:

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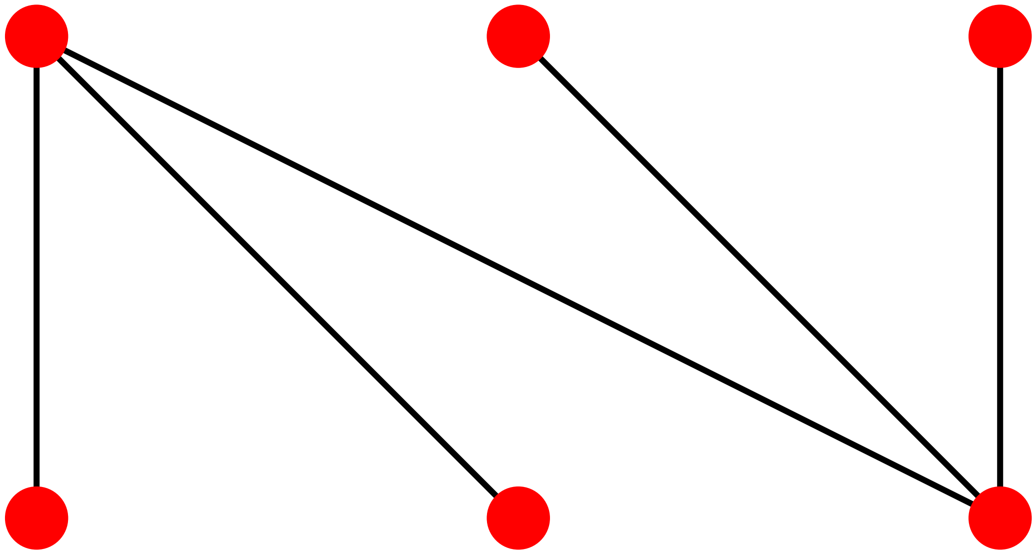
P is **Sperner** (or has the **Sperner property**) if

$$\max_A \#A = \max_i p_i$$

An example



rank-symmetric, rank-unimodal, $F_P(q) = 3 + 3q$

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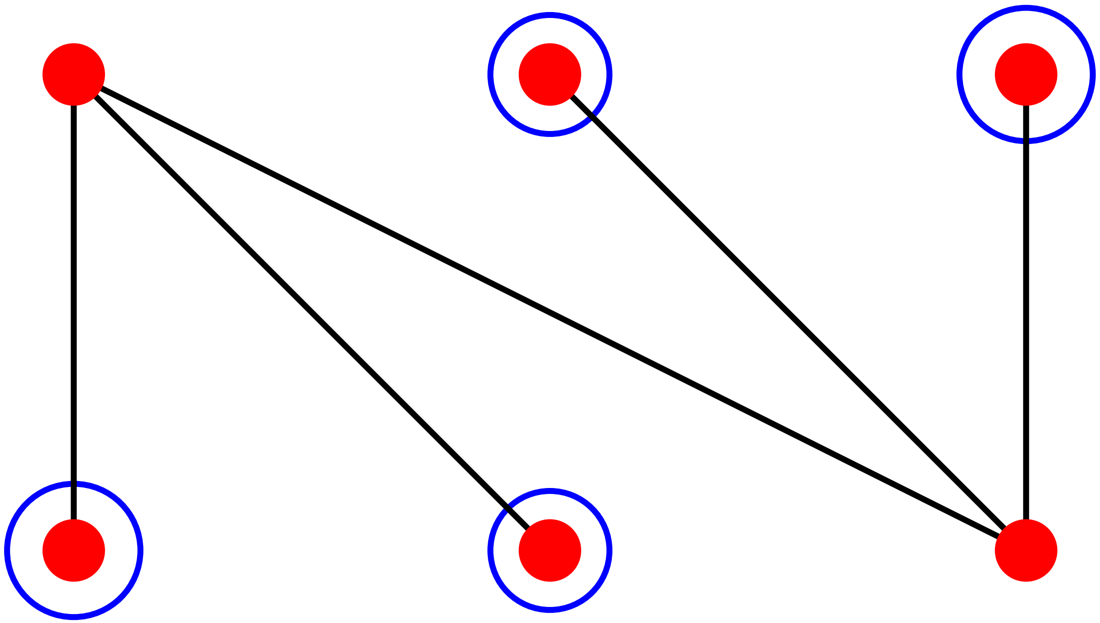

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not Sperner

The boolean algebra

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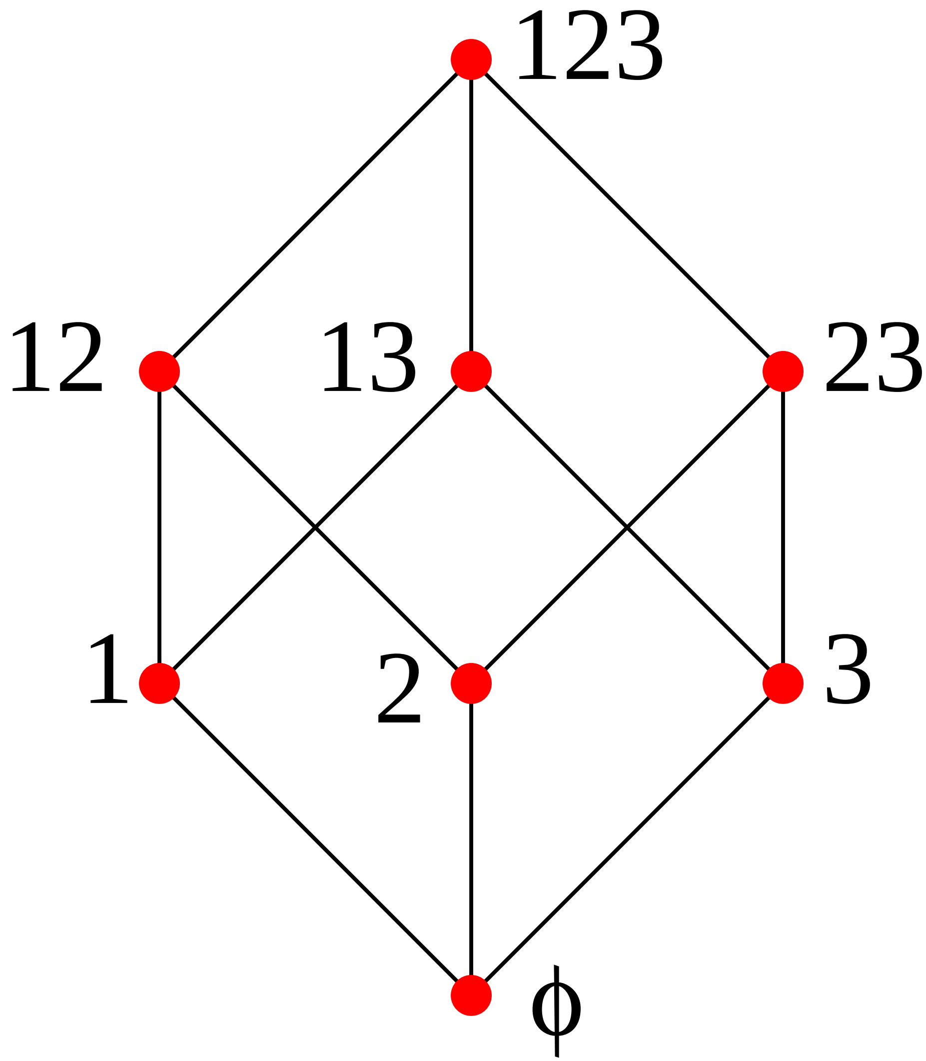
$$p_i = \binom{n}{i}, \quad F_{B_n}(q) = (1+q)^n$$

rank-symmetric, rank-unimodal

Diagram of B_3



Sperner's theorem, 1927

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Emanuel Sperner
9 December 1905 – 31 January 1980

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Let

$$n \geq a_1 > a_2 > \cdots > a_k \geq 1$$

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$$S = \{a_1, \ldots, a_k\}, \ \ T = \{b_1, \ldots, b_j\}$$

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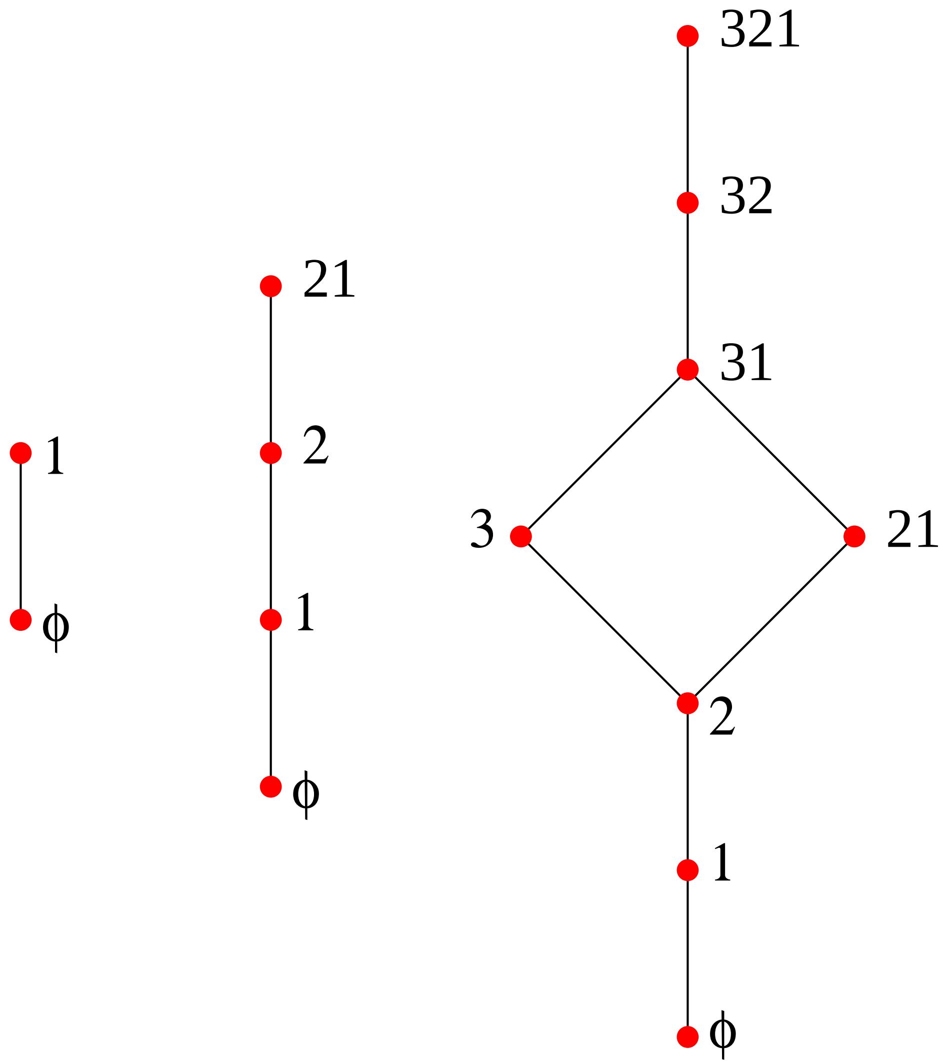
$$S = \{a_1, \ldots, a_k\}, \;\; T = \{b_1, \ldots, b_j\}$$

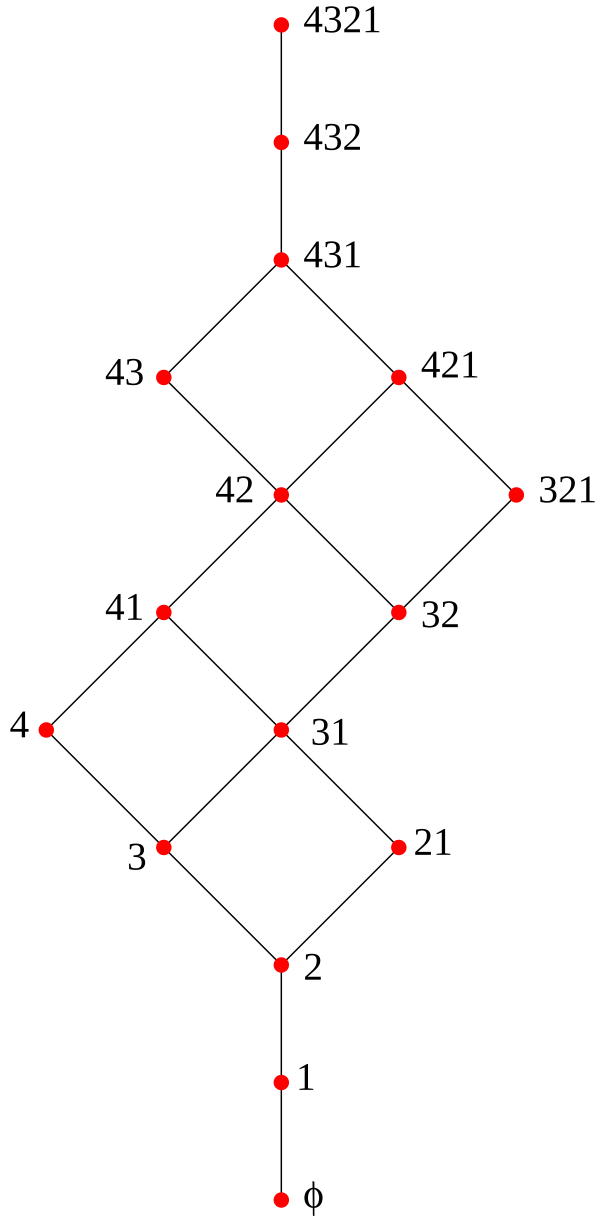
Define $S \geq T$ if $k \geq j$ and

$$a_1 \geq b_1, \;\; a_2 \geq b_2, \ldots, \;\; a_j \geq b_j.$$

$M(1), M(2), M(3)$



$M(4)$



Rank function of $M(n)$

Easy:

$$M(n)_k = \{S \subseteq \{1, \ldots, n\} : \sum_{i \in S} i = k\}$$

$$\#M(n)_k = f(\{1, \ldots, n\}, k)$$

$$\Rightarrow F_P(q) = (1+q)(1+q^2)\cdots(1+q^n)$$

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Rank-unimodality is **unclear** (no combinatorial proof known).

Lindström's observation

Theorem. *If $M(n)$ is Sperner, then the weak Erdős-Moser conjecture holds for $\#S = n$.*

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Weak Erdős-Moser Conjecture.

$$S \subset \mathbb{R}^+, \ \#S = n$$

$$\Rightarrow f(S, \alpha) \leq f\left((\{1, 2, \ldots, n\}, \left\lfloor \frac{1}{2} \binom{n+1}{2} \right\rfloor \right)$$

Lindström's observation

Theorem. *If $M(n)$ is Sperner, then the weak Erdős-Moser conjecture holds for $\#S = n$.*

Proof. Suppose $S = \{a_1, \ldots, a_k\}$, $a_1 > \cdots > a_k$. Let

$$a_{i_1} + \cdots + a_{i_r} = a_{j_1} + \cdots + a_{j_s},$$

where $i_1 > \cdots > i_r$, $j_1 > \cdots > j_s$.

Conclusion of proof

Now $\{i_1, \ldots, i_r\} \geq \{j_1, \ldots, j_s\}$ in $M(n)$

$$\Rightarrow \quad r \geq s,\ i_1 \geq j_1, \ldots, i_s \geq j_s$$
$$\Rightarrow \quad a_{i_1} \geq b_{j_1},\ \ldots, a_{i_s} \geq b_{j_s}$$
$$\Rightarrow \quad r = s,\ a_{i_k} = b_{i_k}\ \forall k.$$

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$$\Rightarrow \quad r = s, \; a_{i_k} = b_{i_k} \; \forall k.$$

Thus $a_{i_1} + \cdots + a_{i_r} = b_{j_1} + \cdots + b_{j_s}$

$\Rightarrow \{i_1, \ldots, i_r\}$ and $\{j_1, \ldots, j_s\}$ are incomparable or equal in $M(n)$

$$\Rightarrow \#S \leq \max_A \#A = f\left(\{1, \ldots, n\}, \left\lfloor \frac{1}{2}\binom{n+1}{2} \right\rfloor\right) \square$$

Linear algebra to the rescue!

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$$\boldsymbol{P} = P_0 \cup \cdots \cup P_m : \quad \text{graded poset}$$

$$\mathbb{Q}\boldsymbol{P_i} : \quad \text{vector space with basis } \mathbb{Q}$$

$U : \mathbb{Q}P_i \to \mathbb{Q}P_{i+1}$ is **order-raising** if

$$U(s) \in \text{span}_{\mathbb{Q}}\{t \in P_{i+1} \ : \ s < t\}$$

Order-matchings

Order matching: $\mu\colon P_i \longrightarrow P_{i+1}$: injective and $\mu(t) < t$

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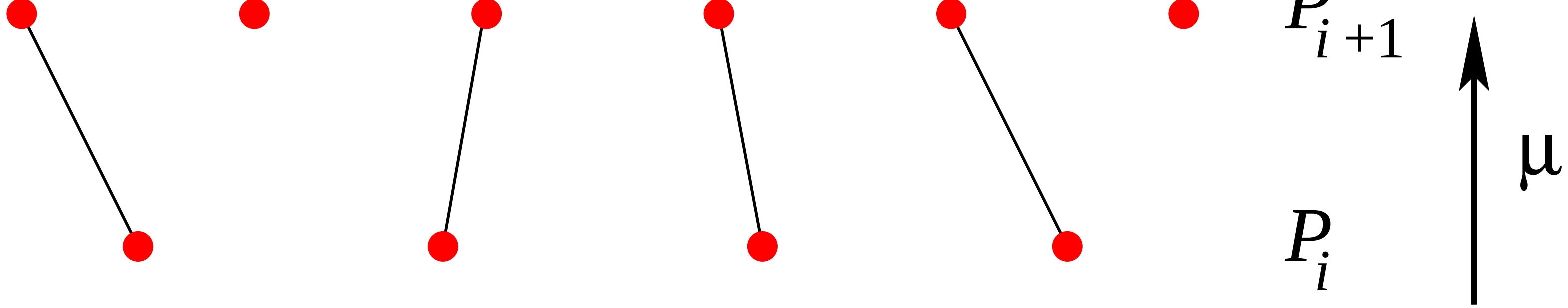

Order-raising and order-matchings

Key Lemma. *If* $U\colon \mathbb{Q}P_i \longrightarrow \mathbb{Q}P_{i+1}$ *is injective and order-raising, then there exists an order-matching* $\mu\colon P_i \longrightarrow P_{i+1}$*.*

Order-raising and order-matchings

Key Lemma. *If* $U: \mathbb{Q}P_i \to \mathbb{Q}P_{i+1}$ *is injective and order-raising, then there exists an order-matching* $\mu\colon P_i \to P_{i+1}$.

Proof. Consider the matrix of U with respect to the bases P_i and P_{i+1}.

Key lemma proof

$$\overbrace{t_1 \quad \cdots \quad t_m \quad \cdots \quad t_n}^{P_{i+1}}$$

$$P_i \left\{ \begin{matrix} s_1 \\ \vdots \\ s_m \end{matrix} \right. \begin{bmatrix} \neq 0 & & & | & * \\ & \ddots & & | & * \\ & & \neq 0 & | & * \end{bmatrix}$$

$$\mathbf{det} \neq \mathbf{0}$$

Key lemma proof

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$$\mathbf{det \neq 0}$$

$$\Rightarrow s_1 < t_1, \ldots, s_m < t_m \qquad \square$$

Minor variant

Similarly if there exists **surjective** order-raising $U: \mathbb{Q}P_i \longrightarrow \mathbb{Q}P_{i+1}$, then there exists an order-matching $\mu: P_{i+1} \longrightarrow P_i$.

A criterion for Spernicity

$$\boldsymbol{P} = P_0 \cup \cdots \cup P_n : \text{ finite graded poset}$$

Proposition. *If for some j there exist order-raising operators*

$$\mathbb{Q}P_0 \overset{\text{inj.}}{\longrightarrow} \mathbb{Q}P_1 \overset{\text{inj.}}{\longrightarrow} \cdots \overset{\text{inj.}}{\longrightarrow} \mathbb{Q}P_j \overset{\text{surj.}}{\longrightarrow} \mathbb{Q}P_{j+1} \overset{\text{surj.}}{\longrightarrow} \cdots \overset{\text{surj.}}{\longrightarrow} \mathbb{Q}P_n,$$

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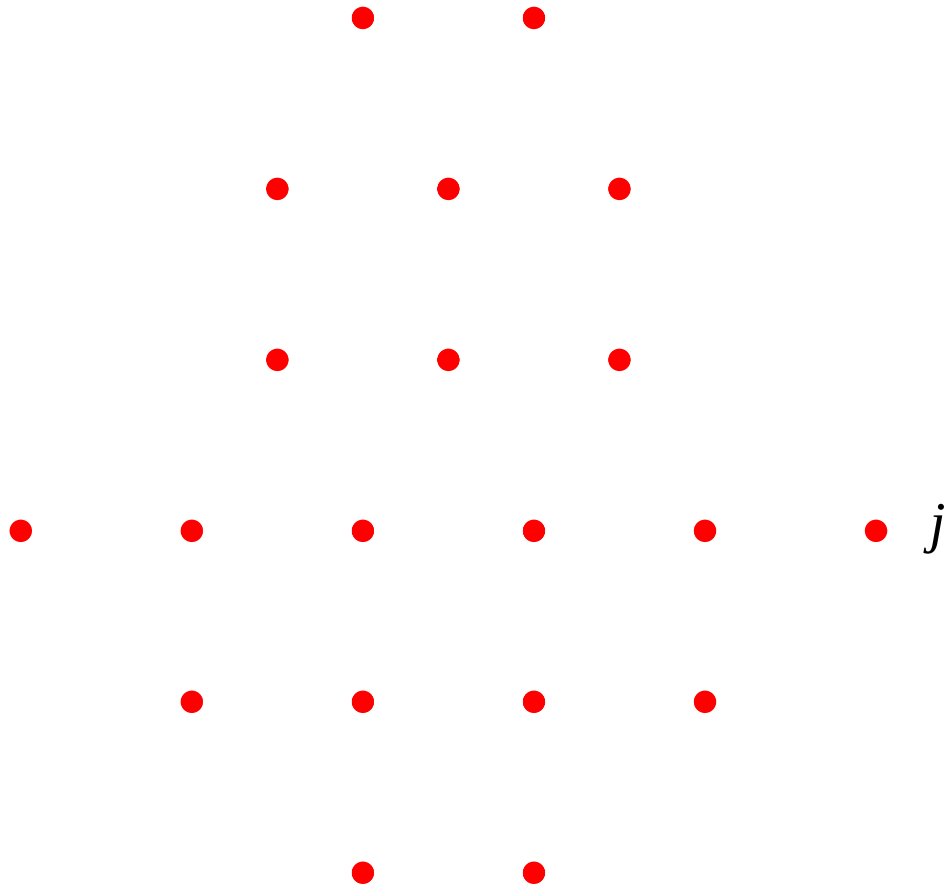
Proof. “Glue together” the order-matchings.

Gluing example



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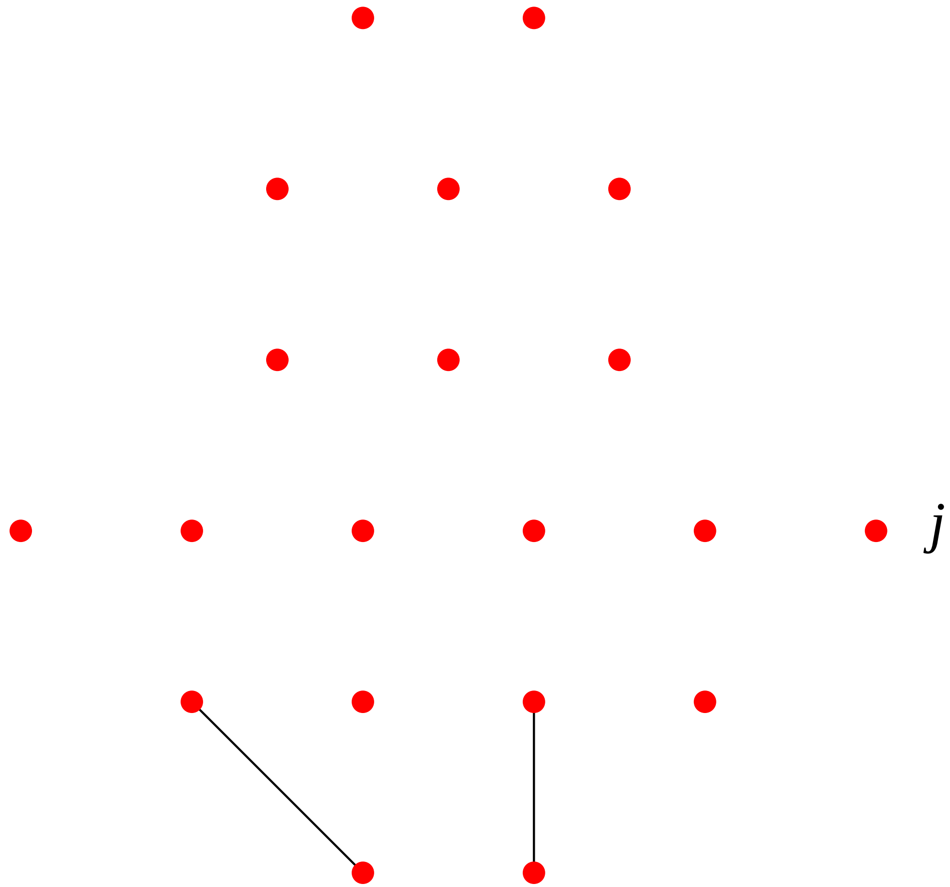

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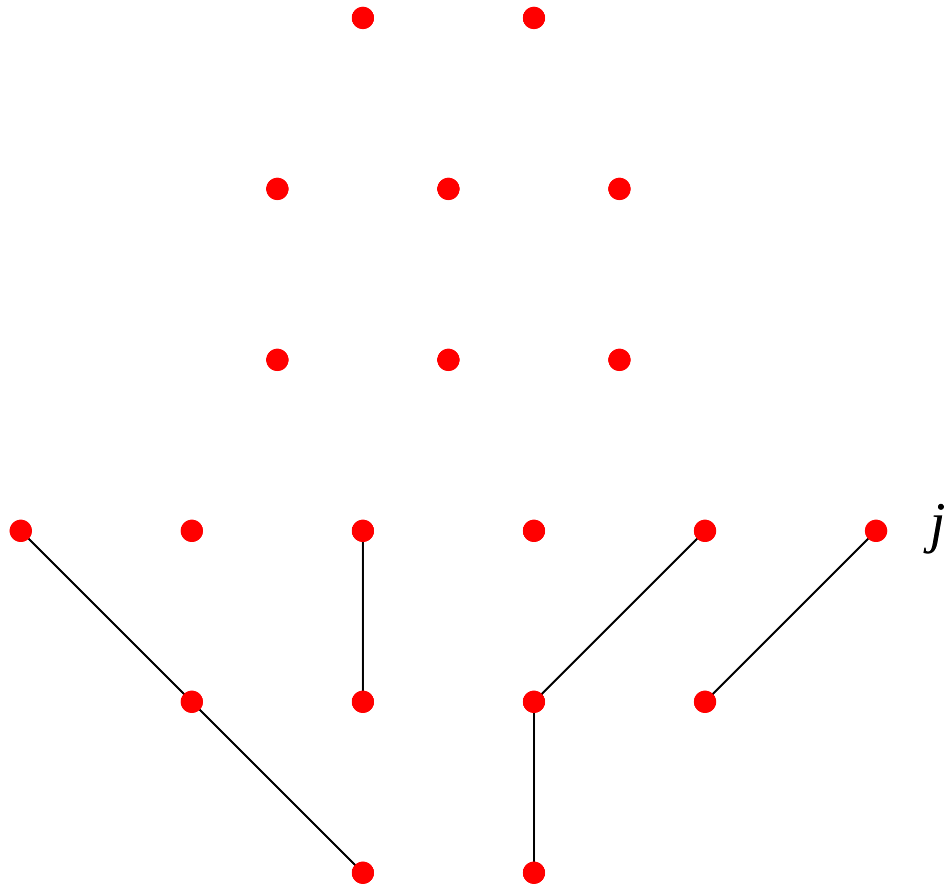

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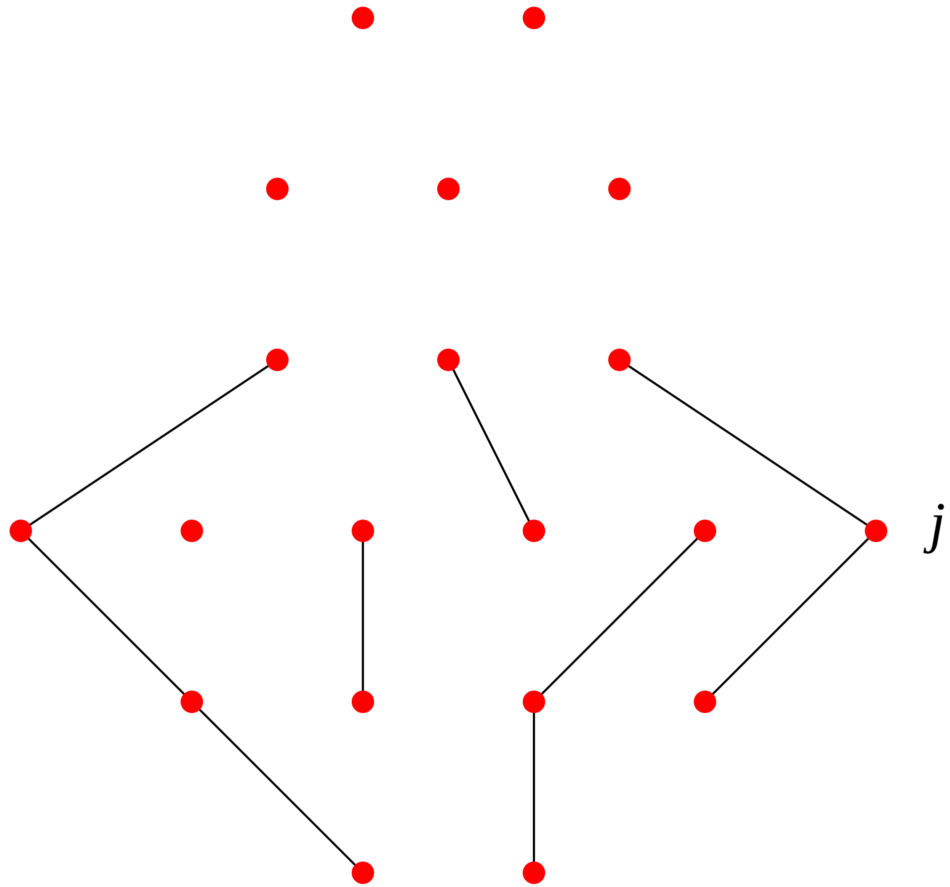

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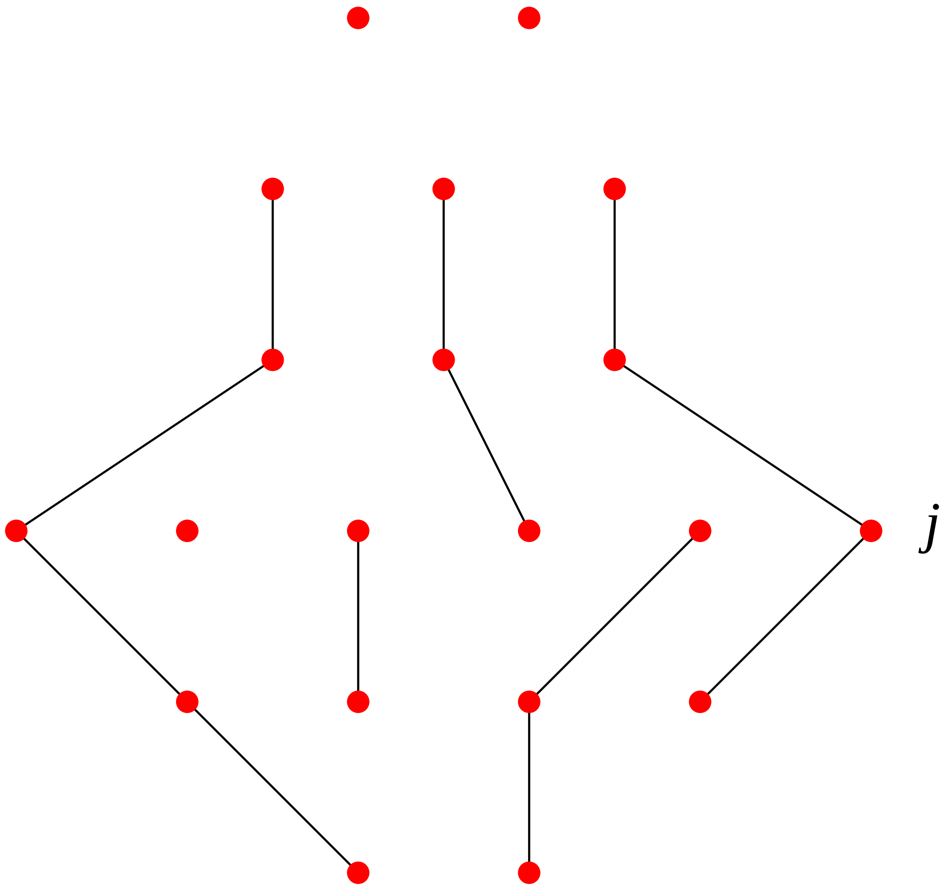

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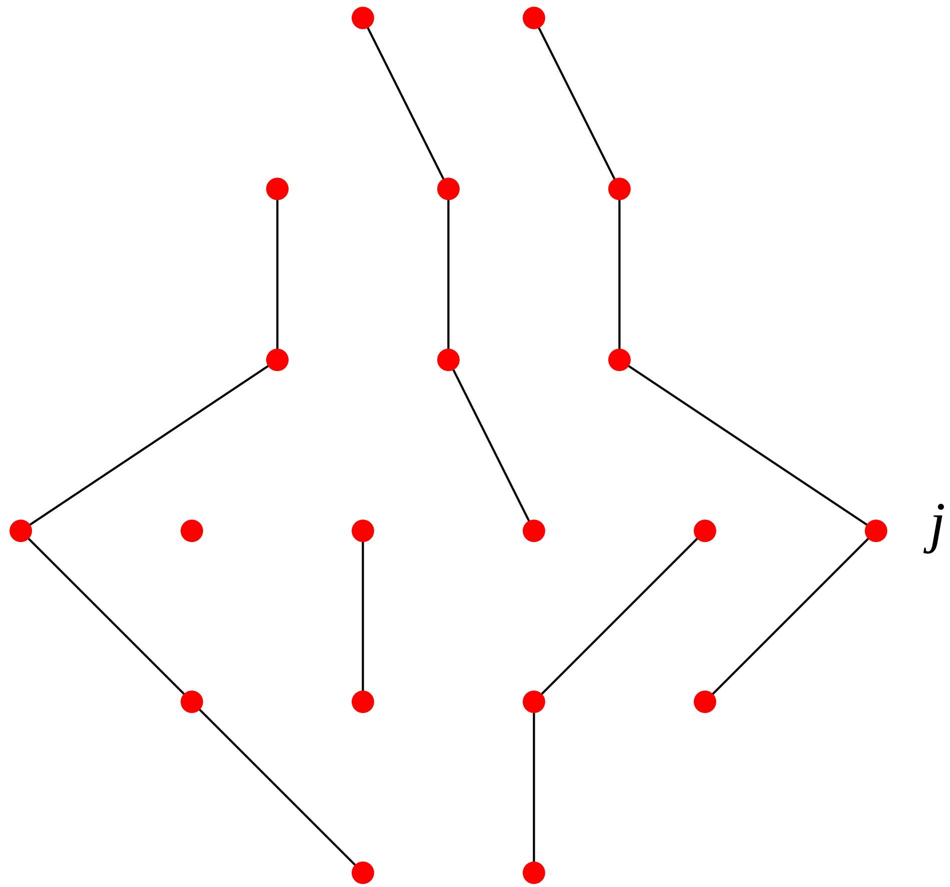

A chain decomposition

$$P = C_1 \cup \cdots \cup C_{p_j} \quad \text{(chains)}$$

$$A = \text{ antichain}, C = \text{ chain} \Rightarrow \#(A \cap C) \leq 1$$

$$\Rightarrow \#A \leq p_j. \qquad \square$$

Back to $M(n)$

Since $M(n)$ has rank $\binom{n+1}{2}$ and is self-dual, suffices to find injective order-raising operators

$$U : \mathbb{Q}M(n)_i \to \mathbb{Q}M(n)_{i+1}, \quad i < \left\lfloor \binom{n+1}{2} \right\rfloor$$

How to find U?

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We don't know how to choose $\mu(s)$, so we make all possible choices at once (a "quantum" matching).

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A lowering operator

Define $\boldsymbol{D} \colon \mathbb{Q}P_i \to \mathbb{Q}P_{i-1}$ by

$$\boldsymbol{D(t)} = \sum_{\substack{s \in M(n)_{i-1} \\ s < t}} c(s,t)s,$$

where $c(s,t)$ is given as follows.

$$\begin{aligned} s &= \{a_1 > \cdots > a_j\} \subseteq \{1, \ldots, n\} \\ t &= \{b_1 > \cdots > b_k\} \subseteq \{1, \ldots, n\} \end{aligned}$$

There is a unique r for which $a_r = b_r - 1$ (and $a_m = b_m$ for all other m). In the case $b_r = 1$ we set $a_r = 0$.

Two examples

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Definition of $c(s,t)$

$$c(s,t) = \begin{cases} \binom{n+1}{2}, & \text{if } a_r = 0 \\ (n-a_r)(n+a_r+1), & \text{if } a_r > 0. \end{cases}$$

$$\boldsymbol{D(t)} = \sum_{\substack{s \in M(n)_{i-1} \\ s<t}} c(s,t)s, \quad t \in M(n)_i$$

Why this choice of U and D?

Lemma.

$$D_{i+1}U_i - U_{i-1}D_i = \left(\binom{n+1}{2} - 2i\right) I_i.$$

(Subscripts denote level at which operator acts.)

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Proof. Straightforward computation.

Injectivity

Claim: $D_{i+1}U_i \colon \mathbb{Q}M(n)_i \longrightarrow \mathbb{Q}M(n)_i$ has positive eigenvalues for $i < \left\lfloor \frac{1}{2}\binom{n+1}{2} \right\rfloor$.

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Recall from linear algebra:

$\boldsymbol{V}, \boldsymbol{W}$: finite-dimensional vector spaces

$\boldsymbol{A} : V \to W$, $\boldsymbol{B} : W \to V$: linear transformations

$$\Rightarrow x^{\dim W} \det(I - xBA) = x^{\dim V} \det(I - xAB)$$

Eigenvalues of $D_{i+1}I_i$

Thus AB and BA have same nonzero eigenvalues.

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- Thus $U_{i-1}D_i$ has **nonnegative** eigenvalues.
- Eigenvalues of $D_{i+1}U_i$ exceed those of $U_{i-1}D_i$ by $\binom{n+1}{2} - 2i > 0$. $\square$

Completion of proof

We showed: $D_{i+1}U_i$ has positive eigenvalues, $i < \left\lfloor \frac{1}{2}\binom{n+1}{2} \right\rfloor$.

Thus: $D_{i+1}U_i$ is invertible.

If $v \in \ker(U_i)$ then $v \in \ker(D_{i+1}U_i)$, so U_i is injective.

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The original conjecture

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Original conjecture can be proved in several ways:

- Combinatorial argument using the weak conjecture (Kleitman)
- Spernicity of $M(n) \times M(n)^*$ using same methods, where $\boldsymbol{M(n)^*}$ is the dual of $M(n)$. Note: $M(n) \cong M(n)^*$
- Use results on preservation of Spernicity under product.

How were U and D found?

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