Increasing and Decreasing Subsequences

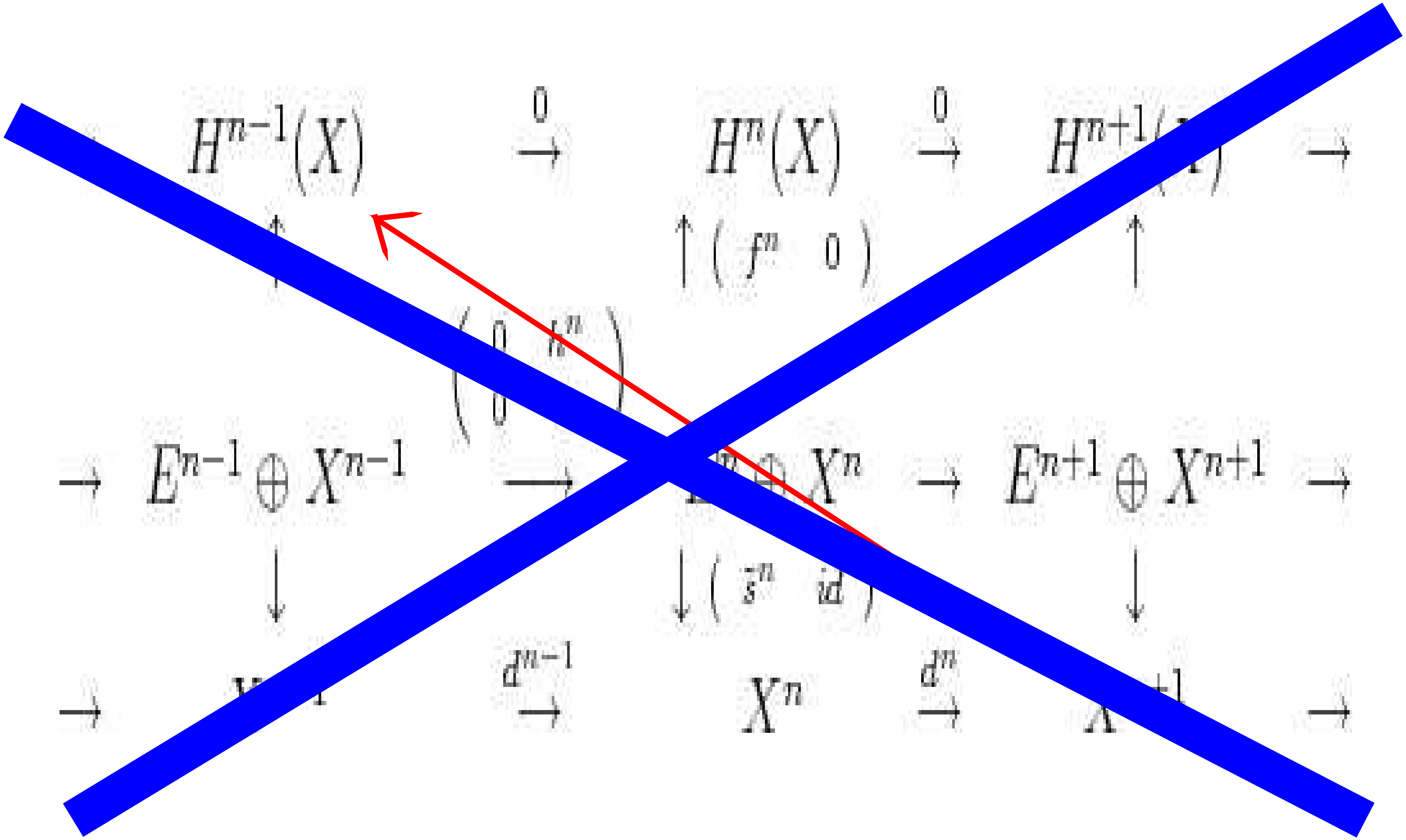
Richard P. Stanley

M.I.T.

$$
\begin{array}{ccccccccc}
\to & H^{n-1}(X) & \stackrel{0}{\to} & H^{n}(X) & \stackrel{0}{\to} & H^{n+1}(X) & \to \\
 & \uparrow & & \uparrow \begin{pmatrix} f^n & 0 \end{pmatrix} & & \uparrow & \\
\to & E^{n-1} \oplus X^{n-1} & \xrightarrow{\begin{pmatrix} 0 & h^n \\ 0 & 0 \end{pmatrix}} & E^{n} \oplus X^{n} & \to & E^{n+1} \oplus X^{n+1} & \to \\
 & \downarrow & & \downarrow \begin{pmatrix} s^n & id \end{pmatrix} & & \downarrow & \\
\to & X^{n-1} & \stackrel{d^{n-1}}{\to} & X^{n} & \stackrel{d^{n}}{\to} & X^{n+1} & \to
\end{array}
$$

$$
\begin{array}{ccccccc}
\to & H^{n-1}(X) & \stackrel{0}{\to} & H^{n}(X) & \stackrel{0}{\to} & H^{n+1}(X) & \to \\
 & \uparrow & & \uparrow \begin{pmatrix} f^n & 0 \end{pmatrix} & & \uparrow & \\
 & & \begin{pmatrix} 0 & h^n \\ 0 & 0 \end{pmatrix} & & & & \\
\to & E^{n-1} \oplus X^{n-1} & \longrightarrow & E^{n} \oplus X^{n} & \to & E^{n+1} \oplus X^{n+1} & \to \\
 & \downarrow & & \downarrow \begin{pmatrix} s^n & id \end{pmatrix} & & \downarrow & \\
\to & X^{n-1} & \stackrel{d^{n-1}}{\to} & X^{n} & \stackrel{d^{n}}{\to} & X^{n+1} & \to
\end{array}
$$

$$
\begin{array}{ccccccc}
\to & H^{n-1}(X) & \xrightarrow{0} & H^{n}(X) & \xrightarrow{0} & H^{n+1}(X) & \to \\
 & \uparrow & & \uparrow \begin{pmatrix} f^n & 0 \end{pmatrix} & & \uparrow & \\
\to & E^{n-1} \oplus X^{n-1} & \xrightarrow{\begin{pmatrix} 0 & h^n \\ 0 & 0 \end{pmatrix}} & E^{n} \oplus X^{n} & \to & E^{n+1} \oplus X^{n+1} & \to \\
 & \downarrow & & \downarrow \begin{pmatrix} s^n & id \end{pmatrix} & & \downarrow & \\
\to & X^{n-1} & \xrightarrow{d^{n-1}} & X^{n} & \xrightarrow{d^{n}} & X^{n+1} & \to
\end{array}
$$



Permutations

Permutations

First lecture: increasing and decreasing subsequences

Permutations

First lecture: increasing and decreasing subsequences

Second lecture: alternating permutations

Permutations

First lecture: increasing and decreasing subsequences

Second lecture: alternating permutations

Third lecture: reduced decompositions

Definitions

3 1 8 **4** 9 **6** **7** 2 5   (**increasing** subsequence)

3 1 **8** **4** 9 6 7 **2** 5   (**decreasing** subsequence)

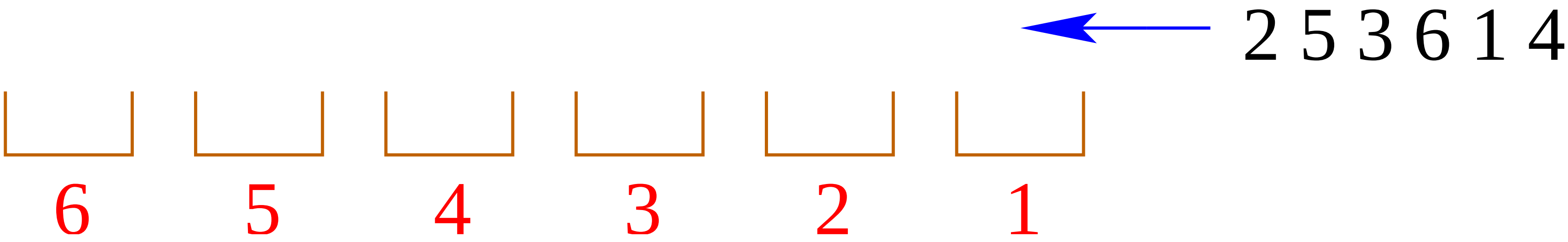
$$\mathbf{is}(w) = |\text{longest i.s.}| = 4$$

$$\mathbf{ds}(w) = |\text{longest d.s.}| = 3$$

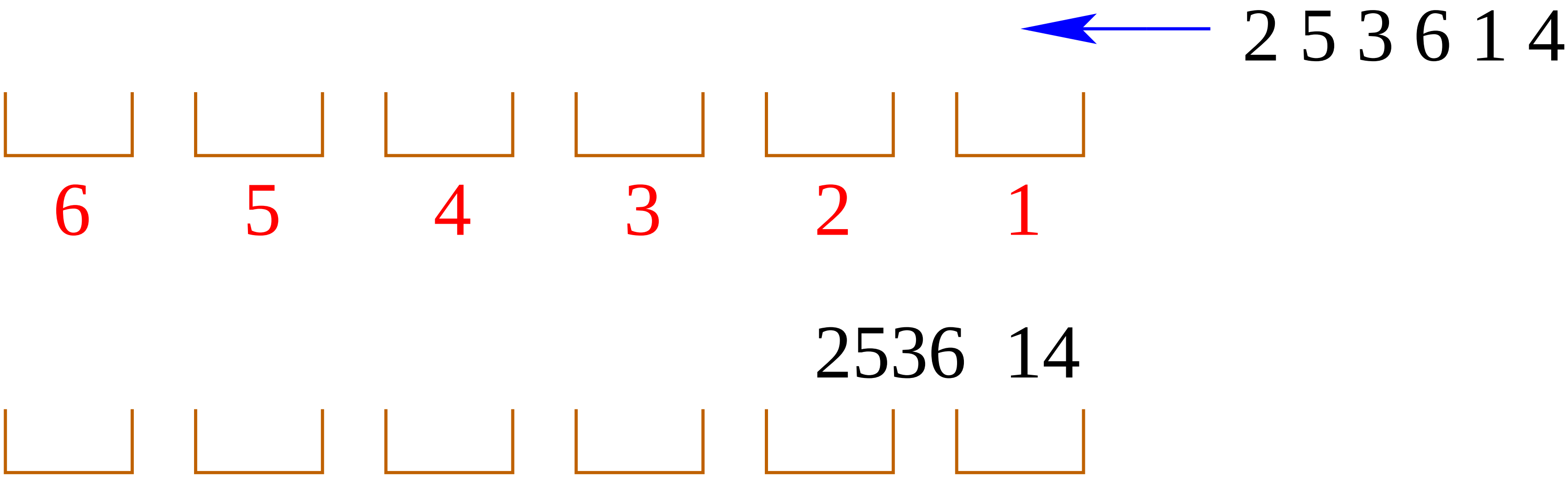
Application: airplane boarding

Naive model: passengers board in order $w = a_1a_2\cdots a_n$ for seats $1, 2, \ldots, n$. Each passenger takes one time unit to be seated after arriving at his seat.

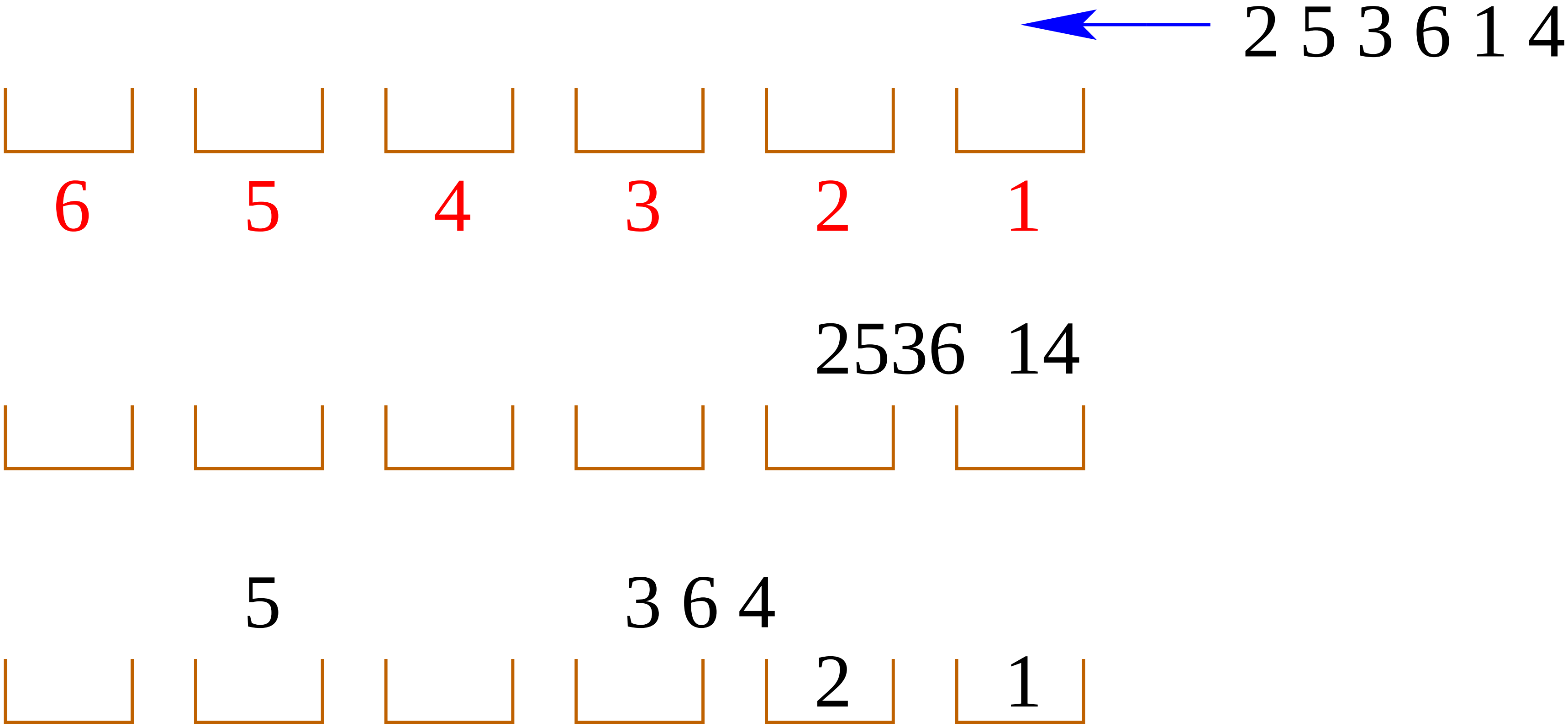
Boarding process



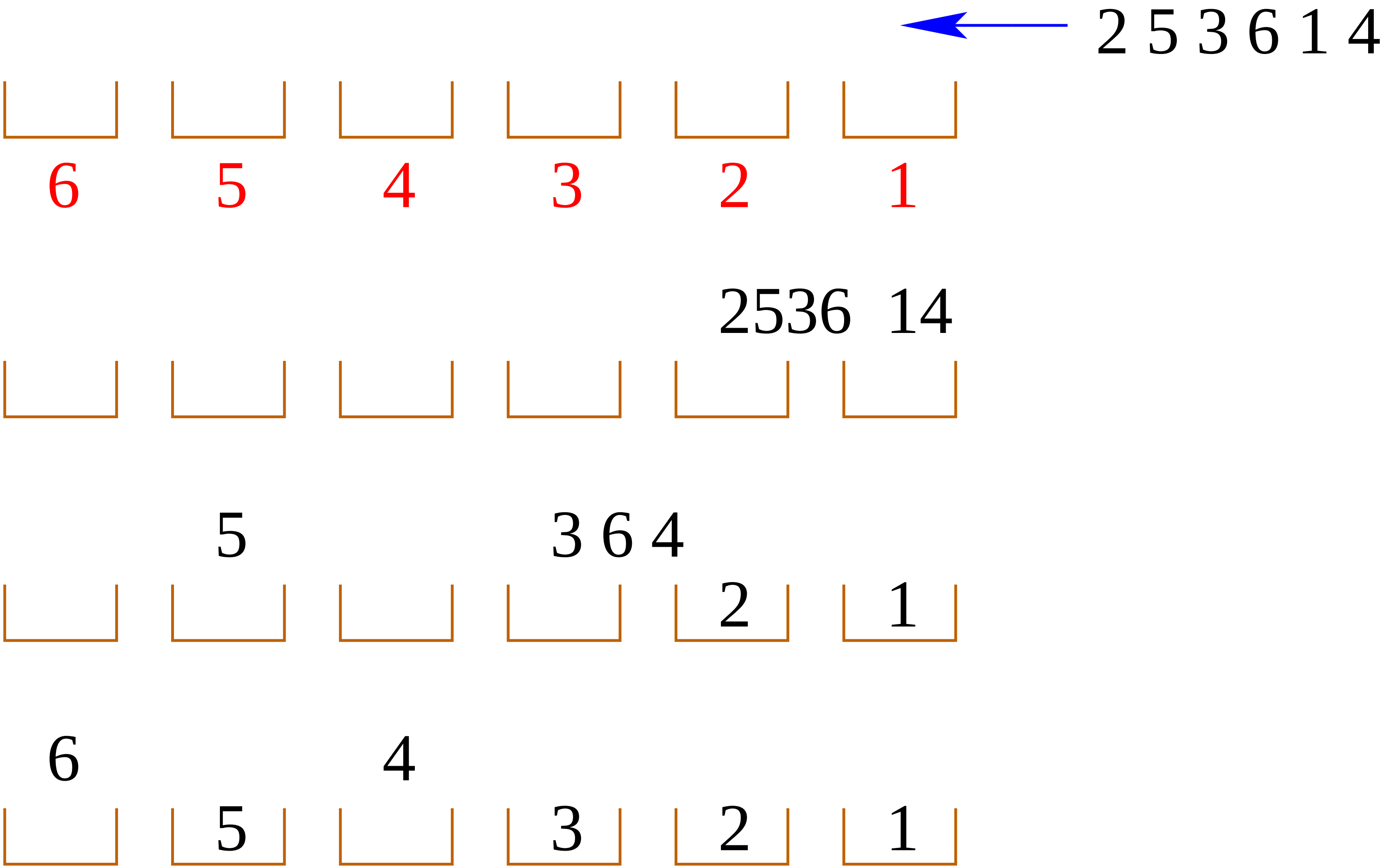
Boarding process



Boarding process



Boarding process



Results

Easy: Total waiting time $= \mathrm{is}(w)$.

Bachmat, et al.: more sophisticated model.

Results

Easy: Total waiting time $= \mathrm{is}(w)$.

Bachmat, et al.: more sophisticated model.

Two conclusions:

- Usual system (back-to-front) not much better than random.
- Better: first board window seats, then center, then aisle.

Partitions

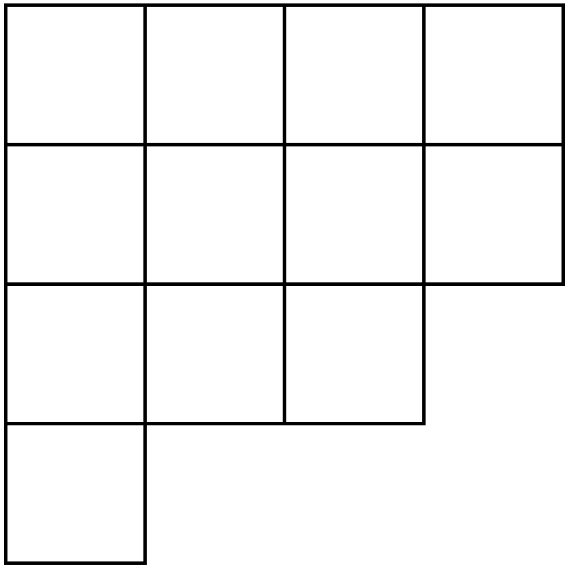
partition $\lambda \vdash n$: $\lambda = (\lambda_1, \lambda_2, \dots)$

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$$

$$\sum \lambda_i = n$$

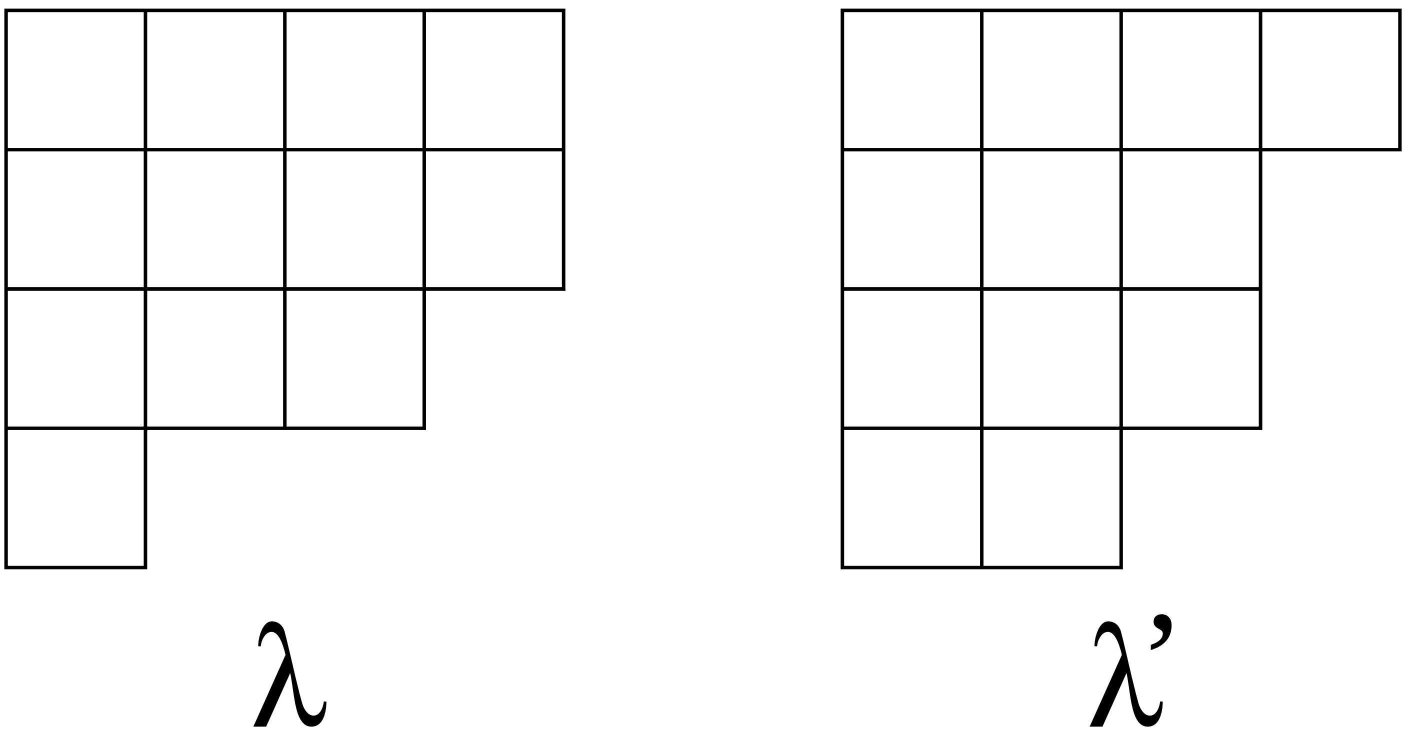
Young diagrams

(Young) diagram of $\lambda = (4, 4, 3, 1)$:



Conjugate partitions

$\lambda' = (4, 3, 3, 2)$, the **conjugate** partition to
$\lambda = (4, 4, 3, 2)$



Standard Young tableau

standard Young tableau (SYT) of shape $\lambda \vdash n$, e.g., $\lambda = (4, 4, 3, 1)$:

$<$

$\wedge$

1	2	7	10
3	5	8	12
4	6	11	
9			

$f^\lambda = \#$ of SYT of shape λ

E.g., $f^{(3,2)} = 5$:

1 2 3	1 2 4	1 2 5	1 3 4	1 3 5
4 5	3 5	3 4	2 5	2 4

f^λ

$f^\lambda = \#$ of SYT of shape λ

E.g., $f^{(3,2)} = 5$:

$1\,2\,3$	$1\,2\,4$	$1\,2\,5$	$1\,3\,4$	$1\,3\,5$
$4\,5$	$3\,5$	$3\,4$	$2\,5$	$2\,4$

$\exists$ simple formula for f^λ (Frame-Robinson-Thrall **hook-length formula**)

$f^\lambda = \#$ of SYT of shape λ

E.g., $f^{(3,2)} = 5$:

1 2 3	1 2 4	1 2 5	1 3 4	1 3 5
4 5	3 5	3 4	2 5	2 4

$\exists$ simple formula for f^λ (Frame-Robinson-Thrall **hook-length formula**)

Note. $f^\lambda = \dim(\text{irrep. of } \mathfrak{S}_n)$, where $\mathfrak{S}_n$ is the **symmetric group** of all permutations of $1, 2 \ldots, n$.

RSK algorithm

RSK algorithm: a bijection

$$w \xrightarrow{\text{rsk}} (P, Q),$$

where $w \in \mathfrak{S}_n$ and P, Q are SYT of the same shape $\lambda \vdash n$.

Write $\lambda = \mathbf{sh}(w)$, the **shape** of w.

RSK algorithm

RSK algorithm: a bijection

$$w \xrightarrow{\text{rsk}} (P, Q),$$

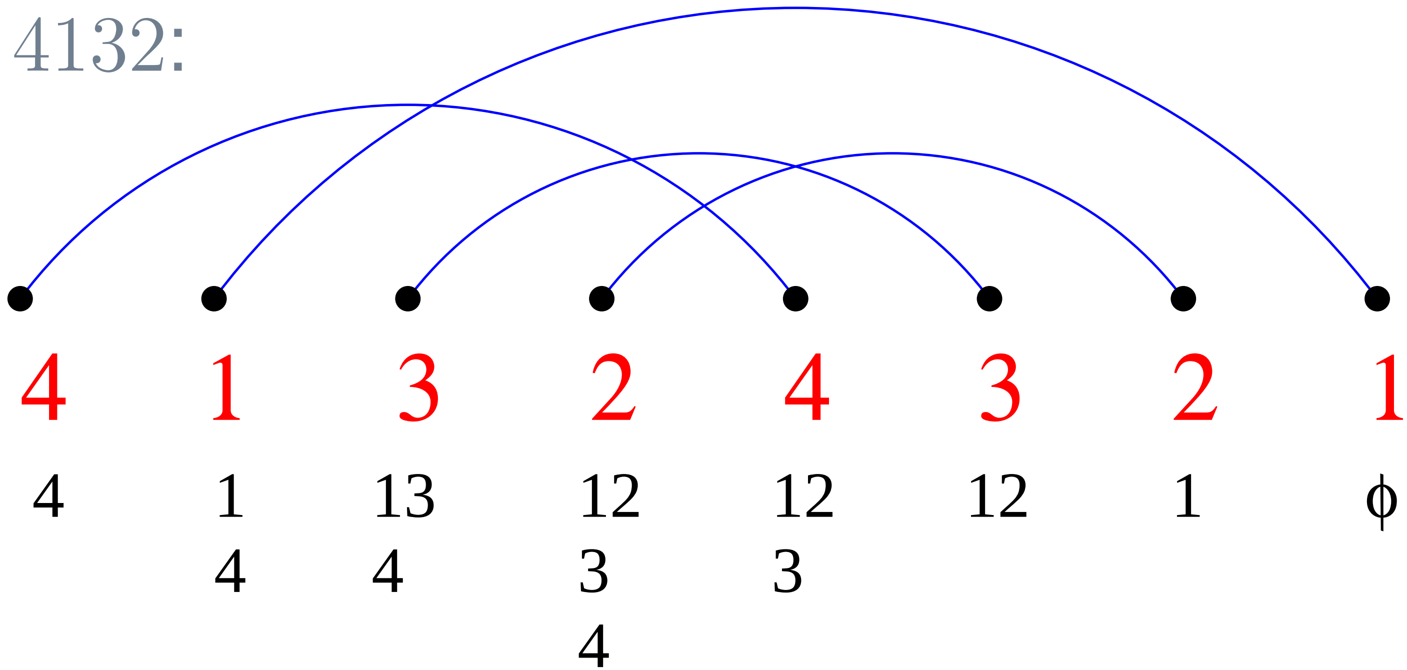
where $w \in \mathfrak{S}_n$ and P, Q are SYT of the same shape $\lambda \vdash n$.

Write $\lambda = \mathbf{sh}(w)$, the **shape** of w.

R = Gilbert de Beauregard Robinson
S = Craige Schensted (= Ea Ea)
K = Donald Ervin Knuth

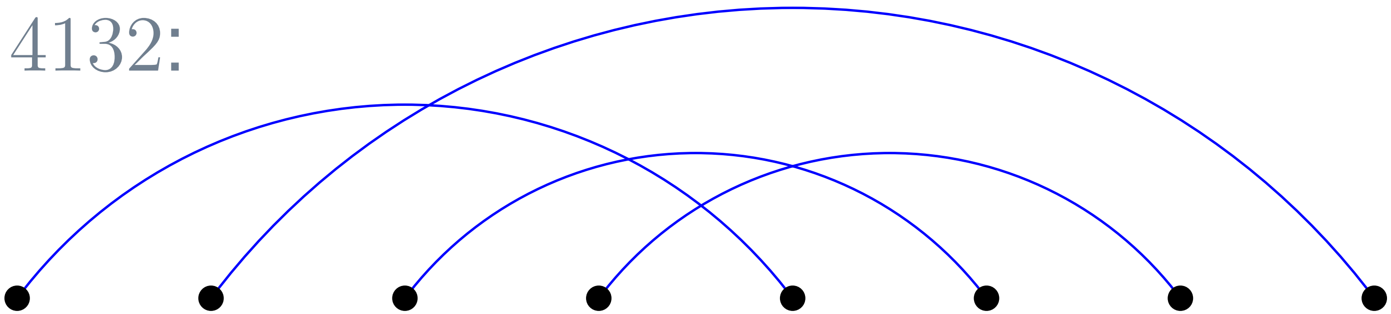
Example of RSK

$w = 4132$:



Example of RSK

$w = 4132$:



4	1	3	2	4	3	2	1
4	1	13	12	12	12	1	ϕ
	4	4	3	3			
			4				

$$(P, Q) = \left(\begin{array}{ll} 1\,2 \\ 3 \\ 4 \end{array} , \quad \begin{array}{ll} 1\,3 \\ 2 \\ 4 \end{array} \right)$$

Schensted's theorem

Theorem. *Let* $w \xrightarrow{\text{rsk}} (P, Q)$, *where* *sh*$(P) = \text{sh}(Q) = \lambda$. *Then*

$$\begin{aligned} \text{is}(w) &= \text{longest row length} = \lambda_1 \\ \text{ds}(w) &= \text{longest column length} = \lambda'_1. \end{aligned}$$

Schensted's theorem

Theorem. *Let* $w \xrightarrow{\text{rsk}} (P, Q)$*, where* *sh*$(P) = \text{sh}(Q) = \lambda$*. Then*

$$\begin{aligned} \text{is}(w) &= \text{longest row length} = \lambda_1 \\ \text{ds}(w) &= \text{longest column length} = \lambda'_1. \end{aligned}$$

Example. $4132 \xrightarrow{\text{rsk}} \begin{pmatrix} 1\,2 & & 1\,3 \\ 3 & , & 2 \\ 4 & & 4 \end{pmatrix}$

$$\text{is}(w) = 2, \quad \text{ds}(w) = 3.$$

Erdős-Szekeres theorem

Corollary (Erdős-Szekeres, Seidenberg). *Let* $w \in \mathfrak{S}_{pq+1}$. *Then either* $\mathrm{is}(w) > p$ *or* $\mathrm{ds}(w) > q$.

Erdős-Szekeres theorem

Corollary (Erdős-Szekeres, Seidenberg). *Let* $w \in \mathfrak{S}_{pq+1}$*. Then either* $\mathrm{is}(w) > p$ *or* $\mathrm{ds}(w) > q$*.*

Proof. Let $\lambda = \mathrm{sh}(w)$. If $\mathrm{is}(w) \leq p$ and $\mathrm{ds}(w) \leq q$ then $\lambda_1 \leq p$ and $\lambda'_1 \leq q$, so $\sum \lambda_i \leq pq$. $\square$

An extremal case

Corollary. *Say* $p \le q$*. Then*

$$\#\{w \in \mathfrak{S}_{pq} \;:\; \mathrm{is}(w) = p,\; \mathrm{ds}(w) = q\}$$

$$= \left(f^{(p^q)}\right)^2$$

An extremal case

Corollary. *Say $p \le q$. Then*

$$\#\{w \in \mathfrak{S}_{pq} \,:\, \mathrm{is}(w) = p,\ \mathrm{ds}(w) = q\}$$

$$= \left(f^{(p^q)}\right)^2$$

By hook-length formula, this is

$$\left(\frac{(pq)!}{1^1 2^2 \cdots p^p (p+1)^p \cdots q^p (q+1)^{p-1} \cdots (p+q-1)^1}\right)^2.$$

Romik's theorem

Romik: let

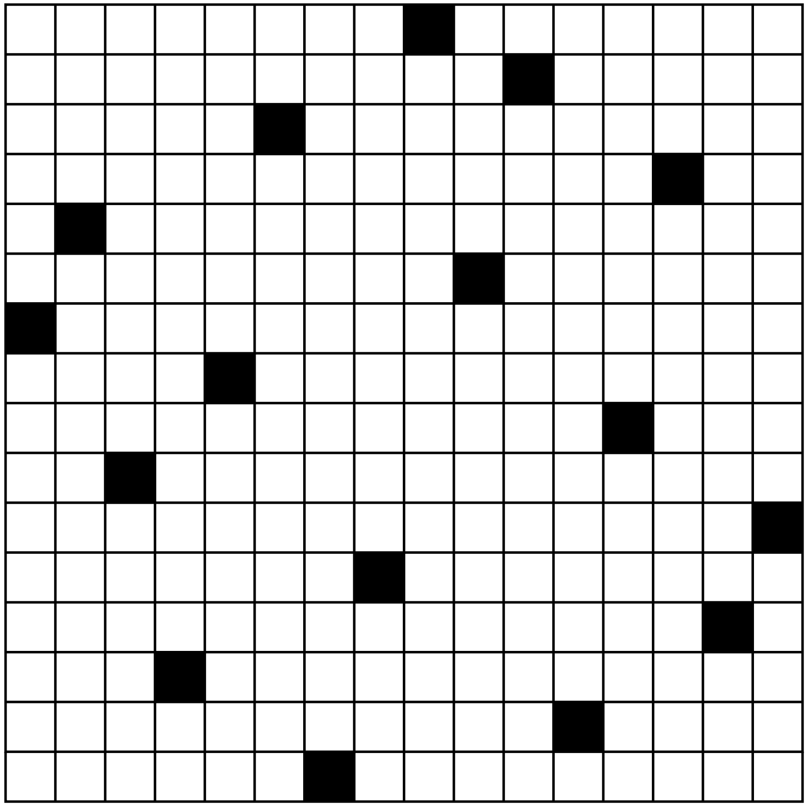
$$w \in \mathfrak{S}_{n^2},\ \operatorname{is}(w) = \operatorname{ds}(w) = n.$$

Let P_w be the permutation matrix of w with corners $(\pm 1, \pm 1)$. Then (informally) as $n \to \infty$ almost surely the 1's in P_w will become dense in the region bounded by the curve

$$(x^2 - y^2)^2 + 2(x^2 + y^2) = 3,$$

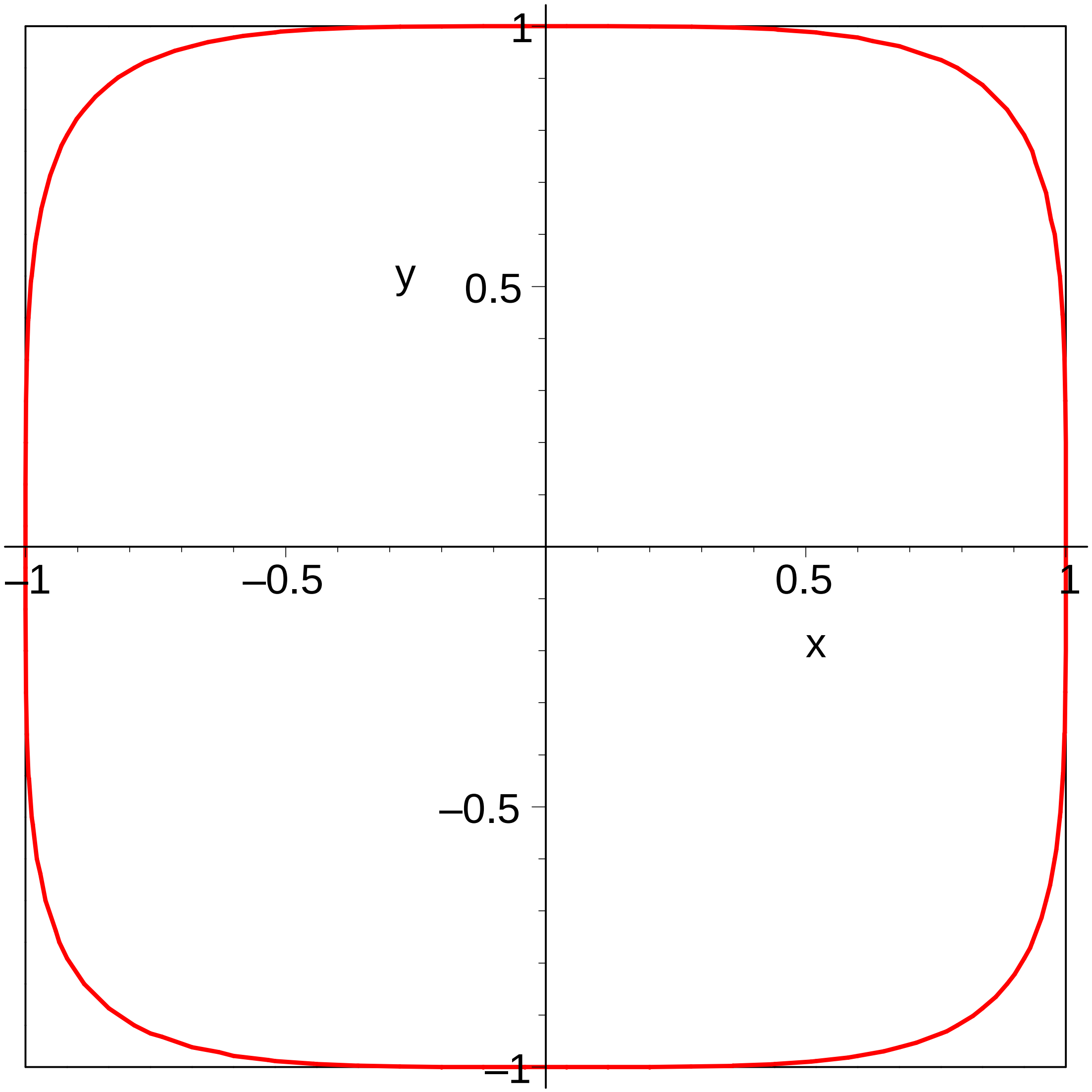
and will remain isolated outside this region.

An example



$$w = 9, 11, 6, 14, 2, 10, 1, 5, 13, 3, 16, 8, 15, 4, 12, 17$$

$(x^2 - y^2)^2 + 2(x^2 + y^2) = 3$



Area enclosed by curve

$$\begin{aligned}\alpha &= 8\int_0^1 \frac{1}{\sqrt{(1-t^2)(1-(t/3)^2)}}dt \\ &\quad -6\int_0^1 \sqrt{\frac{1-(t/3)^2}{1-t^2}}dt \\ &= 4(0.9454596 2\cdots)\end{aligned}$$

Expectation of is(w)

$$\boldsymbol{E(n)} = \text{expectation of } \mathrm{is}(w),\ w \in \mathfrak{S}_n$$

$$= \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \mathrm{is}(w)$$

$$= \frac{1}{n!} \sum_{\lambda \vdash n} \lambda_1 \left(f^\lambda\right)^2$$

Expectation of $\mathrm{is}(w)$

$$\begin{aligned} E(n) &= \text{expectation of } \mathrm{is}(w),\ w \in \mathfrak{S}_n \\ &= \frac{1}{n!}\sum_{w\in\mathfrak{S}_n} \mathrm{is}(w) \\ &= \frac{1}{n!}\sum_{\lambda\vdash n} \lambda_1 \left(f^\lambda\right)^2 \end{aligned}$$

Ulam: what is distribution of $\mathrm{is}(w)$? rate of growth of $E(n)$?

Work of Hammersley

Hammersley (1972):

$$\exists\ c = \lim_{n\to\infty} n^{-1/2} E(n),$$

and

$$\frac{\pi}{2} \leq c \leq e.$$

Work of Hammersley

Hammersley (1972):

$$\exists\ c = \lim_{n\to\infty} n^{-1/2} E(n),$$

and

$$\frac{\pi}{2} \leq c \leq e.$$

Conjectured $c = 2$.

$c = 2$

Logan-Shepp, Vershik-Kerov (1977): $c = 2$

$c = 2$

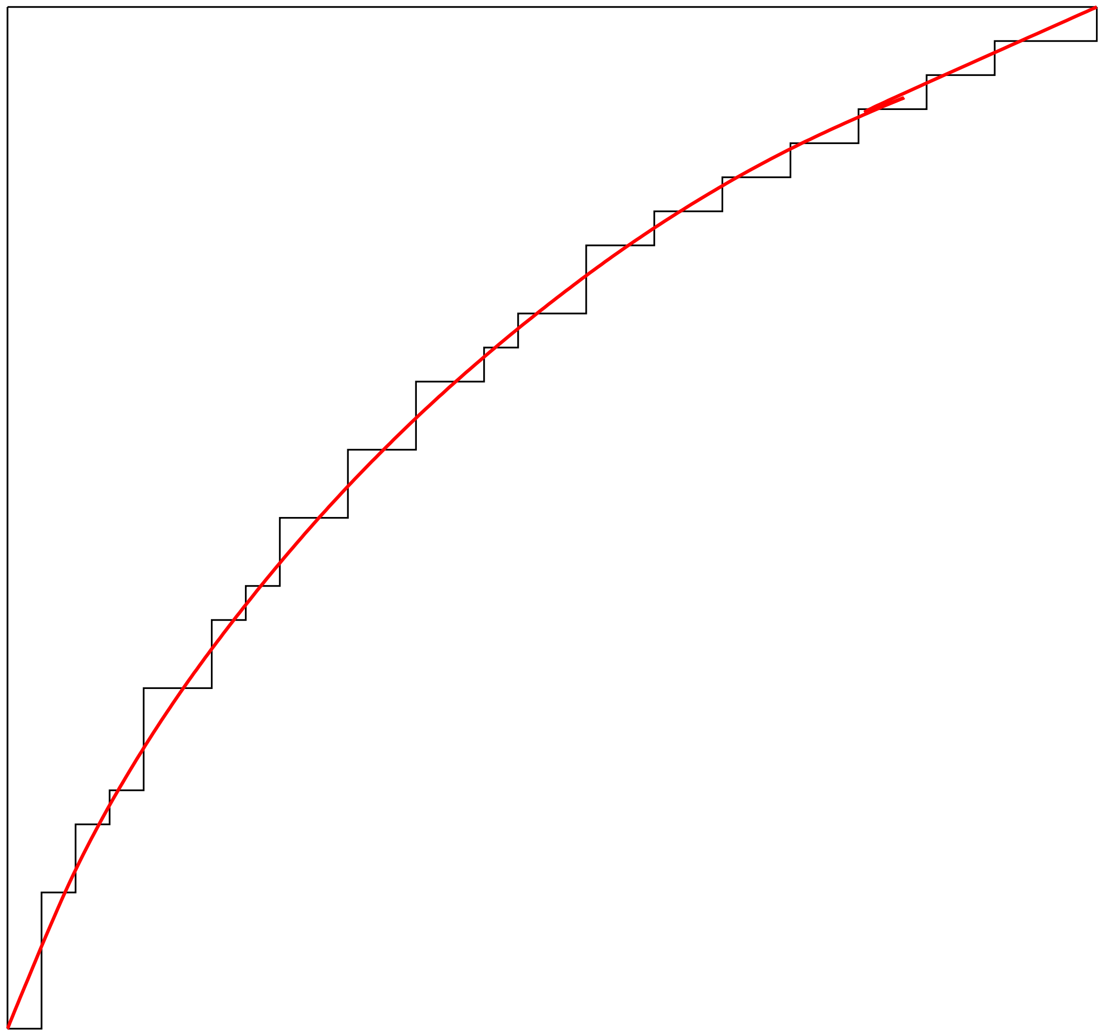
Logan-Shepp, Vershik-Kerov (1977): $c = 2$

Idea of proof.

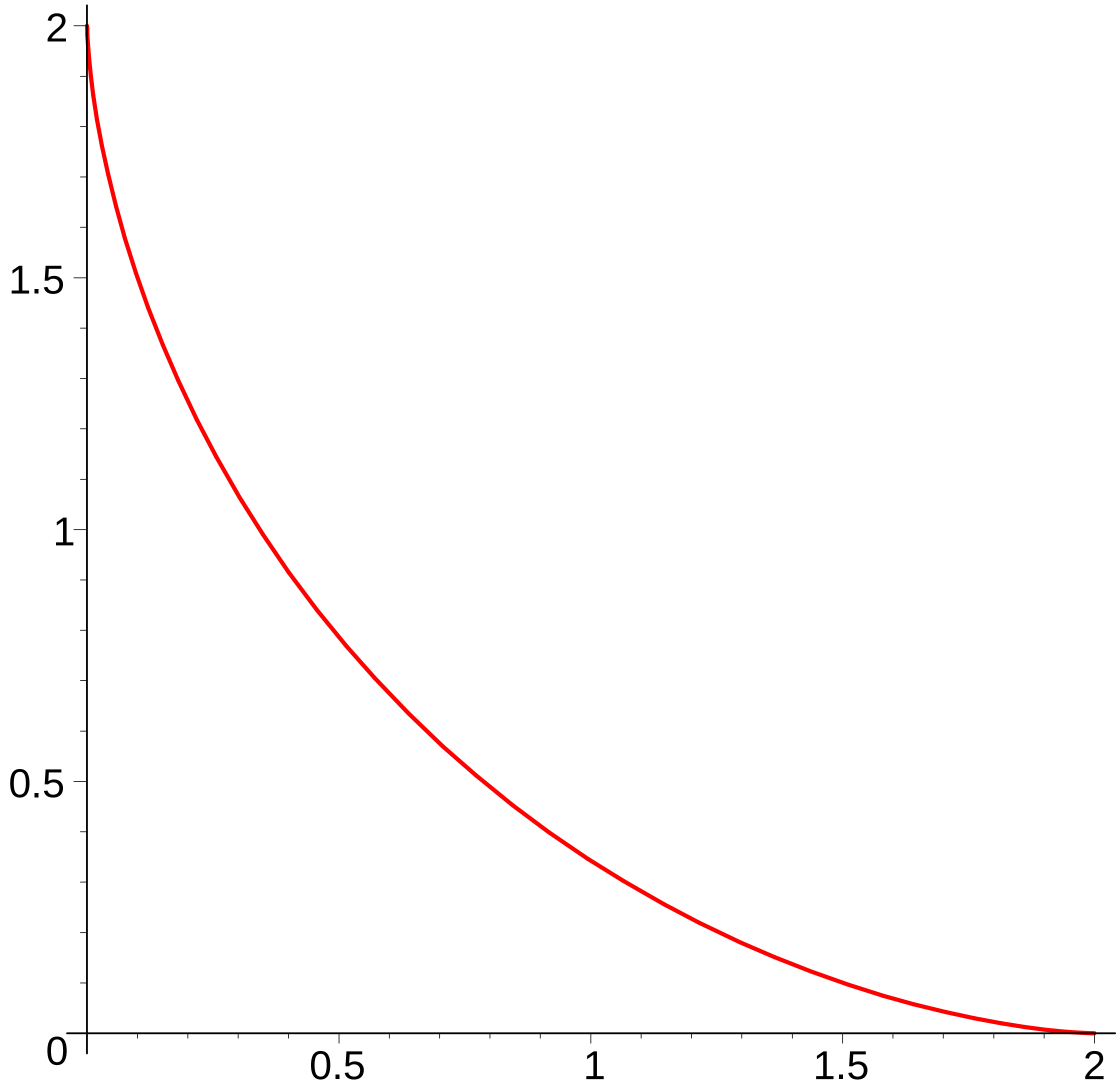
$$\begin{aligned} E(n) &= \frac{1}{n!}\sum_{\lambda \vdash n} \lambda_1 \left(f^\lambda\right)^2 \\ &\approx \frac{1}{n!}\max_{\lambda \vdash n} \lambda_1 \left(f^\lambda\right)^2 . \end{aligned}$$

Find "limiting shape" of $\lambda \vdash n$ maximizing λ as $n \to \infty$ using hook-length formula.

A “big” partition



The limiting curve



Equation of limiting curve

$$\begin{aligned} x &= y + 2\cos\theta \\ y &= \frac{2}{\pi}(\sin\theta - \theta\cos\theta) \end{aligned}$$

$$0 \leq \theta \leq \pi$$

$\mathbf{is}(w) \leq 2$

$$u_k(n) := \#\{w \in \mathfrak{S}_n \ : \ \mathrm{is}_n(w) \leq k\}.$$

$\mathbf{is}(w) \leq 2$

$$u_k(n) := \#\{w \in \mathfrak{S}_n \ : \ \mathrm{is}_n(w) \leq k\}.$$

J. M. Hammersley (1972):

$$u_2(n) = C_n = \frac{1}{n+1}\binom{2n}{n},$$

a **Catalan number**.

$\mathrm{is}(w) \leq 2$

$$u_k(n) := \#\{w \in \mathfrak{S}_n \ : \ \mathrm{is}_n(w) \leq k\}.$$

J. M. Hammersley (1972):

$$u_2(n) = C_n = \frac{1}{n+1}\binom{2n}{n},$$

a **Catalan number**.

For >170 combinatorial interpretations of C_n, see

`www-math.mit.edu/~rstan/ec`

Gessel's theorem

I. Gessel (1990):

$$\sum_{n\geq 0} u_k(n)\frac{x^{2n}}{n!^2} = \det\left[I_{|i-j|}(2x)\right]_{i,j=1}^{k},$$

where

$$I_m(2x) = \sum_{j\geq 0} \frac{x^{m+2j}}{j!(m+j)!},$$

a **hyperbolic Bessel function** of the first kind of order m.

The case $k = 2$

Example.
$$\sum_{n\geq 0} u_2(n)\frac{x^{2n}}{n!^2} = I_0(2x)^2 - I_1(2x)^2 = \sum_{n\geq 0} C_n \frac{x^{2n}}{n!^2}.$$

Painlevé II equation

Baik-Deift-Johansson:

Define $u(x)$ by

$$\frac{d^2}{dx^2}u(x) = 2u(x)^3 + xu(x) \quad (*),$$

with certain initial conditions.

Painlevé II equation

Baik-Deift-Johansson:

Define $\boldsymbol{u(x)}$ by

$$\frac{d^2}{dx^2}u(x) = 2u(x)^3 + xu(x) \quad (*),$$

with certain initial conditions.

$(*)$ is the **Painlevé II** equation (roughly, the branch points and essential singularities are independent of the initial conditions).

Paul Painlevé

1863: born in Paris.

Paul Painlevé

1863: born in Paris.

1890: Grand Prix des Sciences Mathématiques

Paul Painlevé

1863: born in Paris.

1890: Grand Prix des Sciences Mathématiques

1908: first passenger of Wilbur Wright; set flight duration record of one hour, 10 minutes.

Paul Painlevé

1863: born in Paris.

1890: Grand Prix des Sciences Mathématiques

1908: first passenger of Wilbur Wright; set flight duration record of one hour, 10 minutes.

1917, 1925: Prime Minister of France.

Paul Painlevé

1863: born in Paris.

1890: Grand Prix des Sciences Mathématiques

1908: first passenger of Wilbur Wright; set flight duration record of one hour, 10 minutes.

1917, 1925: Prime Minister of France.

1933: died in Paris.

The Tracy-Widom distribution

$$\boldsymbol{F(t)} = \exp\left(-\int_t^{\infty}(x-t)u(x)^2\,dx\right)$$

where $u(x)$ is the Painlevé II function.

The Baik-Deift-Johansson theorem

Let χ be a random variable with distribution F, and let χ_n be the random variable on $\mathfrak{S}_n$:

$$\chi_n(w) = \frac{\mathrm{is}_n(w) - 2\sqrt{n}}{n^{1/6}}.$$

The Baik-Deift-Johansson theorem

Let χ be a random variable with distribution F, and let χ_n be the random variable on $\mathfrak{S}_n$:

$$\chi_n(w) = \frac{\mathrm{is}_n(w) - 2\sqrt{n}}{n^{1/6}}.$$

Theorem. *As* $n \to \infty$,

$$\chi_n \to \chi \quad \text{in distribution,}$$

i.e.,

$$\lim_{n\to\infty} \mathrm{Prob}(\chi_n \le t) = F(t).$$

Expectation redux

Recall $E(n) \sim 2\sqrt{n}$.

Expectation redux

Recall $E(n) \sim 2\sqrt{n}$.

Corollary to BDJ theorem.

$$\begin{aligned} E(n) &= 2\sqrt{n} + \left(\int t\, dF(t)\right) n^{1/6} + o(n^{1/6}) \\ &= 2\sqrt{n} - (1.7711\cdots)n^{1/6} + o(n^{1/6}) \end{aligned}$$

Proof of BDJ theorem

Gessel's theorem reduces the problem to "just" analysis, viz., the **Riemann-Hilbert problem** in the theory of integrable systems, and the **method of steepest descent** to analyze the asymptotic behavior of integrable systems.

Origin of Tracy-Widom distribution

Where did the Tracy-Widom distribution $F(t)$ come from?

$$F(t) = \exp\left(-\int_t^\infty (x-t)u(x)^2\,dx\right)$$

$$\frac{d^2}{dx^2}u(x) = 2u(x)^3 + xu(x)$$

Gaussian Unitary Ensemble (GUE)

Analogue of normal distribution for $n \times n$ hermitian matrices $M = (M_{ij})$:

Gaussian Unitary Ensemble (GUE)

Analogue of normal distribution for $n \times n$ hermitian matrices $M = (M_{ij})$:

$$Z_n^{-1} e^{-\mathrm{tr}(M^2)} dM,$$

$$dM = \prod_i dM_{ii} \cdot \prod_{i<j} d(\Re M_{ij}) d(\Im M_{ij}),$$

where Z_n is a normalization constant.

Tracy-Widom theorem

Tracy-Widom (1994): let $\boldsymbol{\alpha_1}$ denote the largest eigenvalue of M. Then

$$\lim_{n\to\infty} \operatorname{Prob}\left(\left(\alpha_1 - \sqrt{2n}\right)\sqrt{2}n^{1/6} \le t\right) = F(t).$$

Random topologies

Is the connection between is(w) and GUE a coincidence?

Random topologies

Is the connection between is(w) and GUE a coincidence?

Okounkov provides a connection, via the theory of **random topologies on surfaces**. Very briefly, a surface can be described in two ways:

- Gluing polygons along their edges, connected to random matrices via quantum gravity.
- Ramified covering of a sphere, which can be formulated in terms of permutations.

Two variations

1. Matchings
2. Pattern avoidance

Matching collaborators

Joint with:

Bill Chen 陈永川

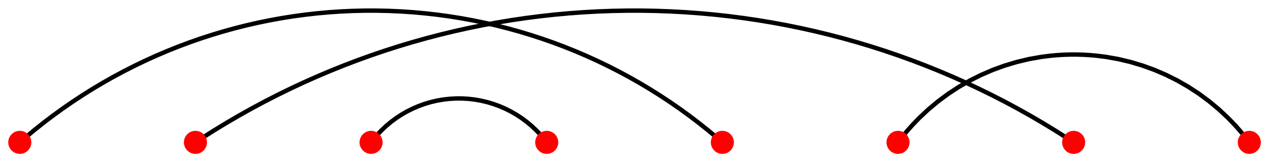
Eva Deng 邓玉平

Rosena Du 杜若霞

Catherine Yan 颜华菲

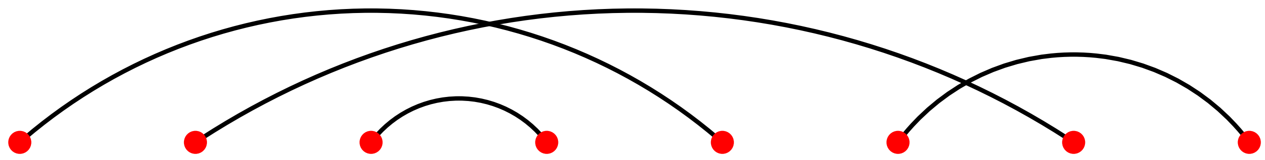
Complete matchings

(complete) matching:



Complete matchings

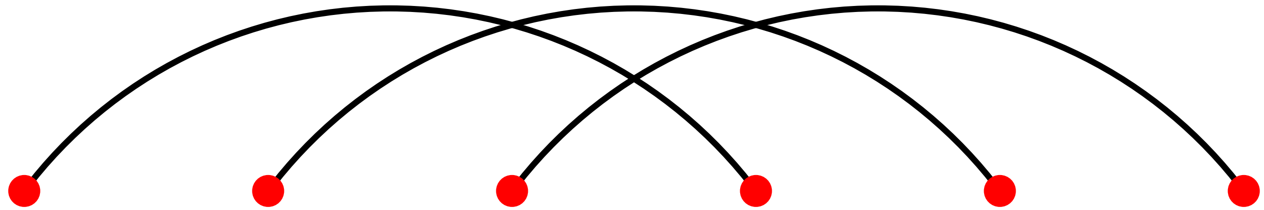
(complete) matching:



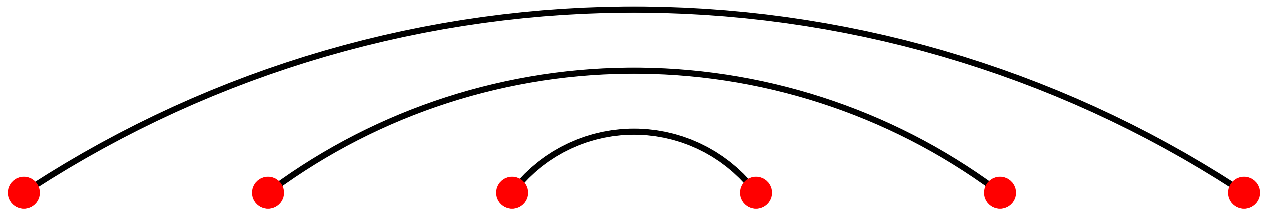
total number of matchings on $[2n] := \{1, 2, \ldots, 2n\}$ is

$$\boldsymbol{(2n-1)!!} := 1 \cdot 3 \cdot 5 \cdots (2n-1).$$

Crossings and nestings



3–crossing



3–nesting

Crossing and nesting number

$$\begin{aligned} \mathbf{M} &= \text{matching} \\ \mathbf{cr}(M) &= \max\{k : \exists\ k\text{-crossing}\} \\ \mathbf{ne}(M) &= \max\{k : \exists\ k\text{-nesting}\} \end{aligned}$$

Crossing and nesting number

$$\begin{aligned} M &= \text{matching} \\ \mathbf{cr}(M) &= \max\{k : \exists\ k\text{-crossing}\} \\ \mathbf{ne}(M) &= \max\{k : \exists\ k\text{-nesting}\} \end{aligned}$$

Theorem. *The number of matchings on* $[2n]$ *with no crossings (or with no nestings) is*

$$C_n := \frac{1}{n+1}\binom{2n}{n}.$$

Main result on matchings

Theorem. *Let* $f_n(i,j) = \#$ *matchings* M *on* $[2n]$ *with* $\mathrm{cr}(M) = i$ *and* $\mathrm{ne}(M) = j$. *Then*

$$\boldsymbol{f_n(i,j) = f_n(j,i)}.$$

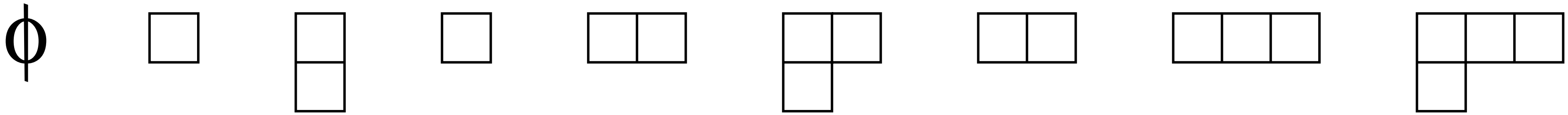
Main result on matchings

Theorem. *Let* $f_n(i,j) = \#$ *matchings* M *on* $[2n]$ *with* $\mathrm{cr}(M) = i$ *and* $\mathrm{ne}(M) = j$*. Then*

$$\boldsymbol{f_n(i,j) = f_n(j,i)}.$$

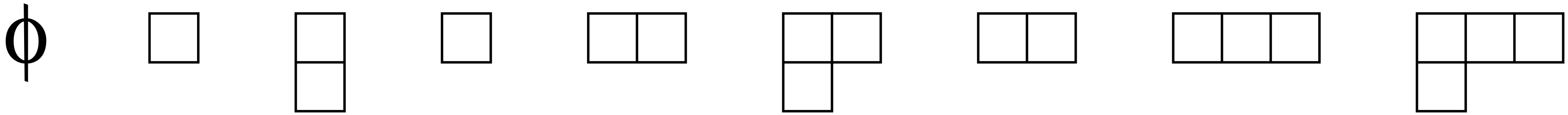
Corollary. # *matchings* M *on* $[2n]$ *with* $\mathrm{cr}(M) = k$ *equals* # *matchings* M *on* $[2n]$ *with* $\mathrm{ne}(M) = k$*.*

Oscillating tableaux

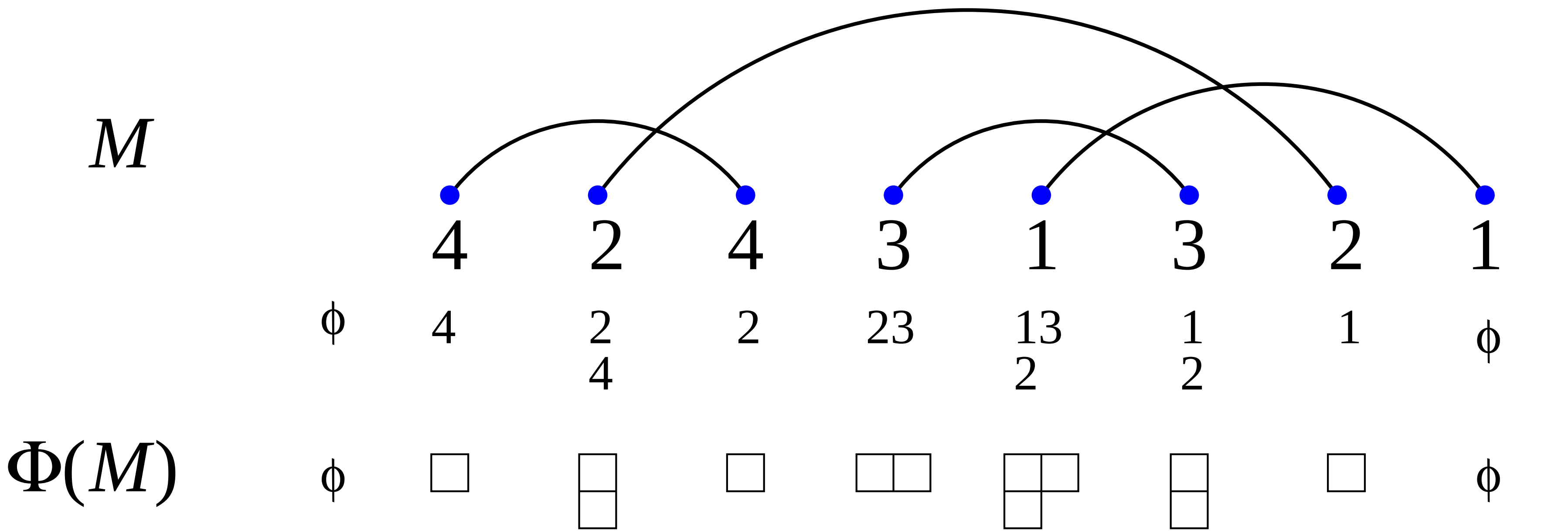


shape $(3, 1)$, length 8

Oscillating tableaux



shape $(3, 1)$, length 8



Proof sketch

Φ is a bijection from matchings on $1, 2, \ldots, 2n$ to oscillating tableaux of length $2n$, shape $\emptyset$.

Corollary. *Number of oscillating tableaux of length $2n$, shape $\emptyset$, is $(2n-1)!!$.*

Proof sketch

Φ is a bijection from matchings on $1, 2, \ldots, 2n$ to oscillating tableaux of length $2n$, shape $\emptyset$.

Corollary. *Number of oscillating tableaux of length $2n$, shape $\emptyset$, is $(2n-1)!!$.*

(related to **Brauer algebra** of dimension $(2n-1)!!$).

Schensted for matchings

Schensted's theorem for matchings. *Let*

$$\Phi(M) = (\emptyset = \lambda^0, \lambda^1, \ldots, \lambda^{2n} = \emptyset).$$

Then

$$\begin{aligned} \mathrm{cr}(M) &= \max\{(\lambda^i)'_1 : 0 \leq i \leq n\} \\ \mathrm{ne}(M) &= \max\{\lambda^i_1 : 0 \leq i \leq n\}. \end{aligned}$$

Schensted for matchings

Schensted's theorem for matchings. *Let*

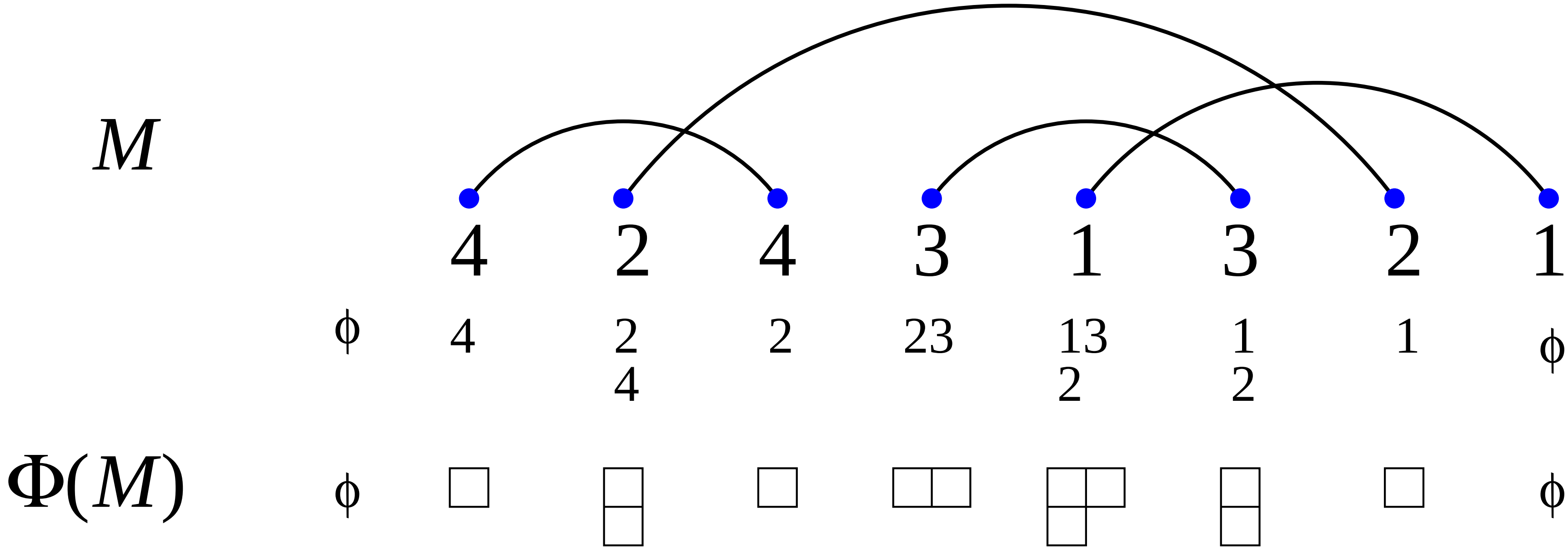
$$\Phi(M) = (\emptyset = \lambda^0, \lambda^1, \ldots, \lambda^{2n} = \emptyset).$$

Then

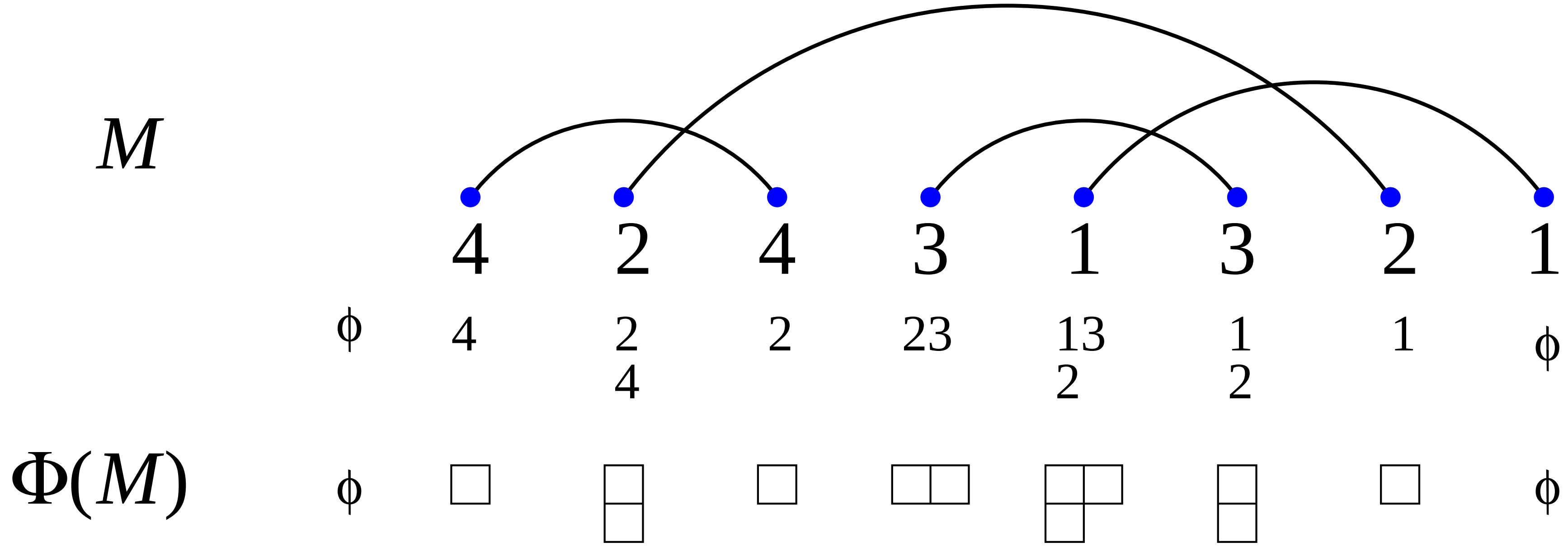
$$\begin{aligned} \mathrm{cr}(M) &= \max\{(\lambda^i)'_1 : 0 \le i \le n\} \\ \mathrm{ne}(M) &= \max\{\lambda^i_1 : 0 \le i \le n\}. \end{aligned}$$

Proof. Reduce to ordinary RSK.

An example



An example



$$\mathrm{cr}(M) = 2, \quad \mathrm{ne}(M) = 2$$

M'

Now let $\mathrm{cr}(M) = i$, $\mathrm{ne}(M) = j$, and

$$\Phi(M) = (\emptyset = \lambda^0, \lambda^1, \ldots, \lambda^{2n} = \emptyset).$$

Define M' by

$$\Phi(M') = (\emptyset = (\lambda^0)', (\lambda^1)', \ldots, (\lambda^{2n})' = \emptyset).$$

By Schensted's theorem for matchings,

$$\mathrm{cr}(M') = j, \quad \mathrm{ne}(M') = i.$$

Conclusion of proof

Thus $M \mapsto M'$ is an involution on matchings of $[2n]$ interchanging cr and ne.

$\Rightarrow$ **Theorem.** *Let* $f_n(i,j) =$ *# matchings* M *on* $[2n]$ *with* $\mathrm{cr}(M) = i$ *and* $\mathrm{ne}(M) = j$. *Then* $\boldsymbol{f_n(i,j) = f_n(j,i)}$.

Simple description?

Open: simple description of $M \mapsto M'$, the analogue of

$$a_1 a_2 \cdots a_n \mapsto a_n \cdots a_2 a_1,$$

which interchanges is and ds.

$g_k(n)$

$$g_k(n) = \text{number of matching } M \text{ on } [2n] \text{ with } \mathrm{cro}(M) \leq k$$

(matching analogue of $u_k(n)$)

Grabiner-Magyar theorem

Theorem. *Define*

$$\boldsymbol{H_k(x)} = \sum_n g_k(n) \frac{x^{2n}}{(2n)!}.$$

Then

$$H_k(x) = \det \left[I_{|i-j|}(2x) - I_{i+j}(2x) \right]_{i,j=1}^{k}$$

where

$$I_m(2x) = \sum_{j \geq 0} \frac{x^{m+2j}}{j!(m+j)!}$$

as before.

Gessel's theorem redux

Compare:

I. Gessel (1990):

$$\sum_{n\geq 0} u_k(n)\frac{x^{2n}}{n!^2} = \det\left[I_{|i-j|}(2x)\right]_{i,j=1}^{k}.$$

Noncrossing example

Example. $k = 1$ (noncrossing matchings):

$$\begin{aligned} H_1(x) &= I_0(2x) - I_2(2x) \\ &= \sum_{j \geq 0} C_j \frac{x^{2j}}{(2j)!}. \end{aligned}$$

Baik-Rains theorem

Baik-Rains (implicitly):

$$\lim_{n\to\infty} \mathrm{Prob}\left(\frac{\mathrm{cr}_n(M)-\sqrt{2n}}{(2n)^{1/6}} \le \frac{t}{2}\right) = F_1(t),$$

where

$$\boldsymbol{F_1(t)} = \sqrt{F(t)}\exp\left(\frac{1}{2}\int_t^\infty u(s)ds\right),$$

where $F(t)$ is the Tracy-Widom distribution and $u(t)$ the Painlevé II function.

Bounding $\mathrm{cr}(M)$ and $\mathrm{ne}(N)$

$$\boldsymbol{g_{j,k}(n)} := \#\{\text{matchings } M \text{ on } [2n],$$
$$\mathrm{cr}(M) \le j,\ \mathrm{ne}(M) \le k\}$$

Bounding $\mathrm{cr}(M)$ and $\mathrm{ne}(N)$

$$\boldsymbol{g_{j,k}(n)} := \#\{\text{matchings } M \text{ on } [2n],$$

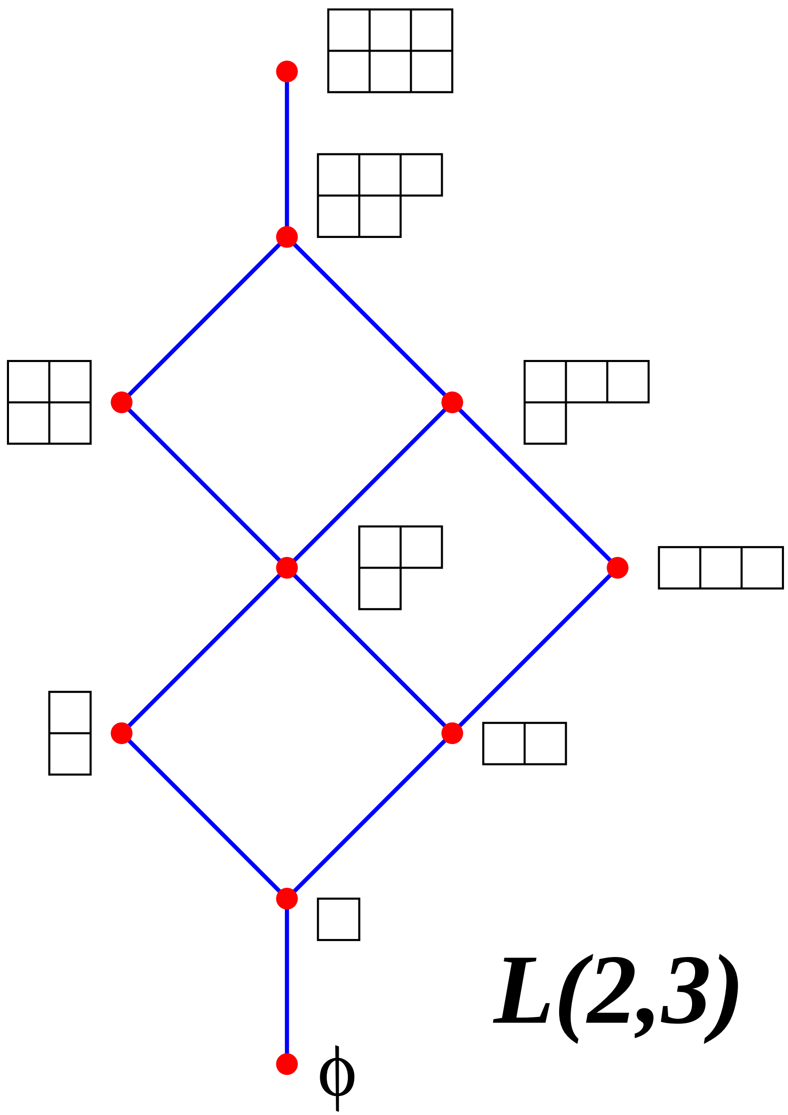
$$\mathrm{cr}(M) \leq j,\ \mathrm{ne}(M) \leq k\}$$

$$g_{j,k}(n) = \#\{(\emptyset = \lambda^0, \lambda^1, \ldots, \lambda^{2n} = \emptyset)\ :$$

$$\lambda^{i+1} = \lambda^i \pm \square,\ \lambda^i \subseteq j \times k \text{ rectangle}\},$$

a walk on a graph $\boldsymbol{L(j,k)}$.

$L(2,3)$



Transfer matrix generating function

$$\begin{aligned} \boldsymbol{A} &= \text{adjacency matrix of } \mathcal{H}(j,k) \\ \boldsymbol{A_0} &= \text{adjacency matrix of } \mathcal{H}(j,k) - \{\emptyset\}. \end{aligned}$$

Transfer matrix generating function

$$\boldsymbol{A} = \text{adjacency matrix of } \mathcal{H}(j,k)$$
$$\boldsymbol{A_0} = \text{adjacency matrix of } \mathcal{H}(j,k) - \{\emptyset\}.$$

Transfer-matrix method $\Rightarrow$

$$\sum_{n\geq 0} g_{j,k}(n)x^{2n} = \frac{\det(I - xA_0)}{\det(I - xA)}.$$

Zeros of $\det(I - xA)$

Theorem (**Grabiner**, implicitly) Every zero of $\det(I - xA)$ has the form

$$2(\cos(\pi r_1/m) + \cdots + \cos(\pi r_j/m)),$$

where each $r_i \in \mathbb{Z}$ and $m = j + k + 1$.

Zeros of $\det(I - xA)$

Theorem (**Grabiner**, implicitly) Every zero of $\det(I - xA)$ has the form

$$2(\cos(\pi r_1/m) + \cdots + \cos(\pi r_j/m)),$$

where each $r_i \in \mathbb{Z}$ and $m = j + k + 1$.

Corollary. *Every irreducible factor of* $\det(I - xA)$ *over* $\mathbb{Q}$ *has degree dividing*

$$\frac{1}{2}\phi(2(j + k + 1)),$$

where ϕ *is the Euler phi-function.*

An example

Example.

$j = 2$, $k = 5$, $\frac{1}{2}\phi(16) = 4$:

$$\det(I - xA) = (1 - 2x^2)(1 - 4x^2 + 2x^4)$$

$$(1 - 8x^2 + 8x^4)(1 - 8x^2 + 8x^3 - 2x^4)$$

$$(1 - 8x^2 - 8x^3 - 2x^4)$$

Another example

$j = k = 3$, $\frac{1}{2}\phi(14) = 3$:

$$\det(I - xA) = (1-x)(1+x)(1+x-9x^2-x^3)$$
$$(1-x-9x^2+x^3)(1-x-2x^2+x^3)^2$$
$$(1+x-2x^2-x^3)^2$$

An open problem

$\text{rank}(A) = ?$

Or even: when is A invertible?

An open problem

rank$(A) = ?$

Or even: when is A invertible?

Eigenvalues are known (and A is symmetric)

An open problem

rank$(A) = ?$

Or even: when is A invertible?

Eigenvalues are known (and A is symmetric)

Cannot tell from the trigonometric expression for the eigenvalues when they are 0.

Pattern avoidance

$$\begin{aligned} \boldsymbol{v} &= b_1 \cdots b_k \in \mathfrak{S}_k \\ \boldsymbol{w} &= a_1 \cdots a_n \in \mathfrak{S}_n \end{aligned}$$

w **avoids** v if no subsequence $a_{i_1} \cdots a_{i_k}$ of w is in the same relative order as v.

Pattern avoidance

$$\boldsymbol{v} = b_1 \cdots b_k \in \mathfrak{S}_k$$
$$\boldsymbol{w} = a_1 \cdots a_n \in \mathfrak{S}_n$$

w **avoids** v if no subsequence $a_{i_1} \cdots a_{i_k}$ of w is in the same relative order as v.

$3\ \mathbf{5}\ \mathbf{2}\ 9\ 6\ \mathbf{8}\ 1\ \mathbf{4}\ 7$ does **not** avoid 3142.

Pattern avoidance

$$\begin{aligned} \boldsymbol{v} &= b_1 \cdots b_k \in \mathfrak{S}_k \\ \boldsymbol{w} &= a_1 \cdots a_n \in \mathfrak{S}_n \end{aligned}$$

w **avoids** v if no subsequence $a_{i_1} \cdots a_{i_k}$ of w is in the same relative order as v.

$3\,\mathbf{5}\,\mathbf{2}\,9\,6\,\mathbf{8}\,1\,\mathbf{4}\,7$ does **not** avoid 3142.

w has no increasing (decreasing) subsequence of length $k \Leftrightarrow w$ avoids $12\cdots k$ ($k \cdots 21$).

The case $k = 3$

Let $v \in \mathfrak{S}_k$. Define

$$\begin{aligned} \mathfrak{S}_n(v) &= \{w \in \mathfrak{S}_n \ : \ w \text{ avoids } v\} \\ s_n(v) &= \#\mathfrak{S}_n(v). \end{aligned}$$

The case $k = 3$

Let $v \in \mathfrak{S}_k$. Define

$$\begin{aligned} \mathfrak{S}_n(v) &= \{w \in \mathfrak{S}_n \ : \ w \text{ avoids } v\} \\ s_n(v) &= \#\mathfrak{S}_n(v). \end{aligned}$$

Hammersley-Knuth-Rotem:

$$s_n(123) = s_n(321) = C_n.$$

The case $k = 3$

Let $v \in \mathfrak{S}_k$. Define

$$\begin{aligned} \mathfrak{S}_n(v) &= \{w \in \mathfrak{S}_n \ : \ w \text{ avoids } v\} \\ s_n(v) &= \#\mathfrak{S}_n(v). \end{aligned}$$

Hammersley-Knuth-Rotem:

$$s_n(123) = s_n(321) = C_n.$$

Knuth:

$$s_n(132) = s_n(213) = s_n(231) = s_n(312) = C_n.$$

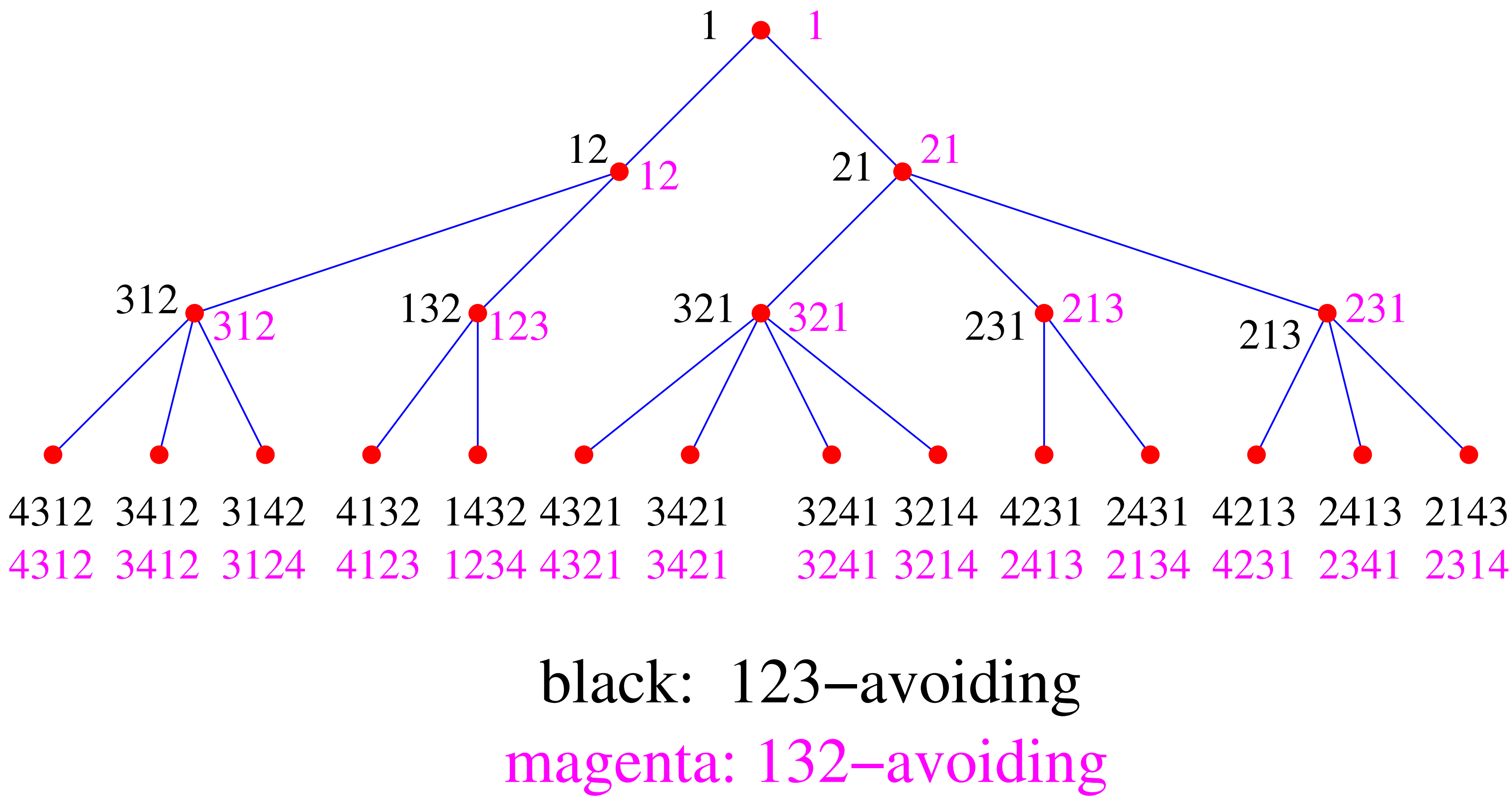
Generating trees

Chung-Graham-Hoggatt-Kleiman, West:

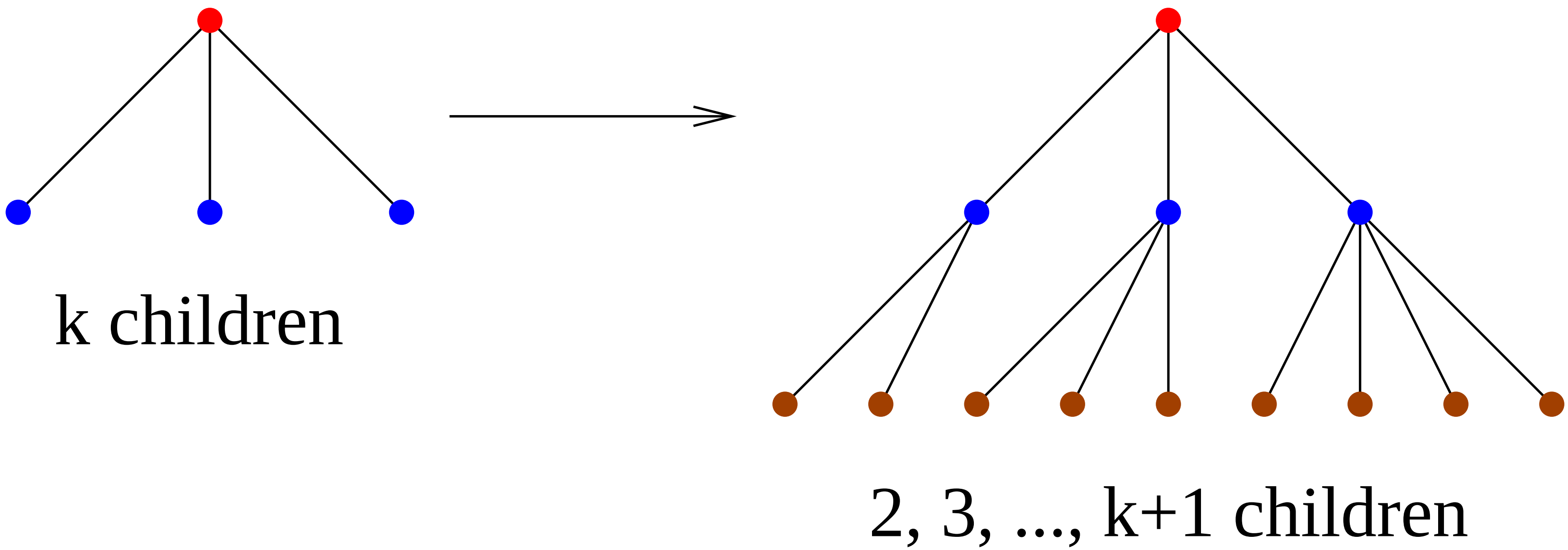
define $u \leq v$ if u is a **subsequence** of v.

$$3142 \leq 8\mathbf{3}5\mathbf{1}96\mathbf{4}\mathbf{2}7$$

123 and 132-avoiding trees



Structure of the tree



Wilf equivalence

Define $u \sim v$ if $s_n(u) = s_n(v)$ for all n.

Wilf equivalence

Define $u \sim v$ if $s_n(u) = s_n(v)$ for all n.

One equivalence class for $k = 3$.

Wilf equivalence

Define $u{\sim}v$ if $s_n(u) = s_n(v)$ for all n.

One equivalence class for $k = 3$.

Three equivalence classes for $k = 4$.

The three classes for $k = 4$

Gessel: $s_n(1234) =$

$$\frac{1}{(n+1)^2(n+2)}\sum_{j=0}^{n}\binom{2j}{j}\binom{n+1}{j+1}\binom{n+2}{j+2}$$

The three classes for $k = 4$

Gessel: $s_n(1234) =$

$$\frac{1}{(n+1)^2(n+2)} \sum_{j=0}^{n} \binom{2j}{j} \binom{n+1}{j+1} \binom{n+2}{j+2}$$

Bóna:

$$\sum_{n \geq 0} s_n(1342) x^n = \frac{32x}{1 + 20x - 8x^2 - (1-8x)^{3/2}},$$

The three classes for $k = 4$

Gessel: $s_n(1234) =$

$$\frac{1}{(n+1)^2(n+2)} \sum_{j=0}^{n} \binom{2j}{j} \binom{n+1}{j+1} \binom{n+2}{j+2}$$

Bóna:

$$\sum_{n \geq 0} s_n(1342) x^n = \frac{32x}{1 + 20x - 8x^2 - (1-8x)^{3/2}},$$

Open: $s_n(1324)$

Typical application

Ryan, Lakshmibai-Sandhya, Haiman:

Let $w \in \mathfrak{S}_n$. The Schubert variety Ω_w in the complete flag variety GL$(n, \mathbb{C})$ is smooth if and only if w avoids 4231 and 3412.

Typical application

Ryan, Lakshmibai-Sandhya, Haiman:

Let $w \in \mathfrak{S}_n$. The Schubert variety Ω_w in the complete flag variety GL$(n, \mathbb{C})$ is smooth if and only if w avoids 4231 and 3412.

$$\sum_{n \geq 0} f(n)x^n = \frac{1}{1 - x - \frac{x^2}{1-x}\left(\frac{2x}{1+x-(1-x)C(x)} - 1\right)},$$

where

$$C(x) = \sum_{n \geq 0} C_n x^n = \frac{1 - \sqrt{1-4x}}{2x}.$$


THE END