Lattice Points in Polytopes

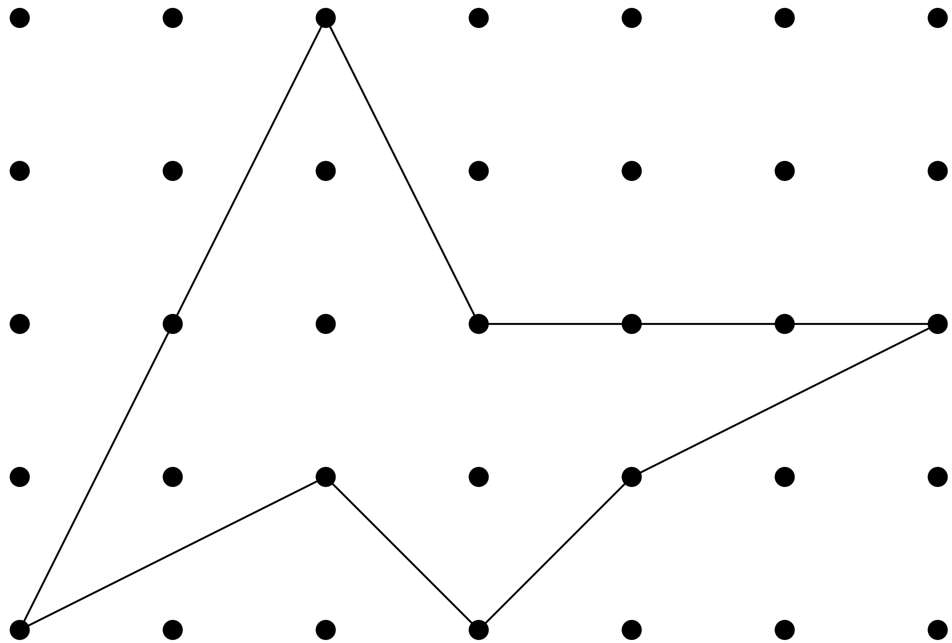
Richard P. Stanley

M.I.T.

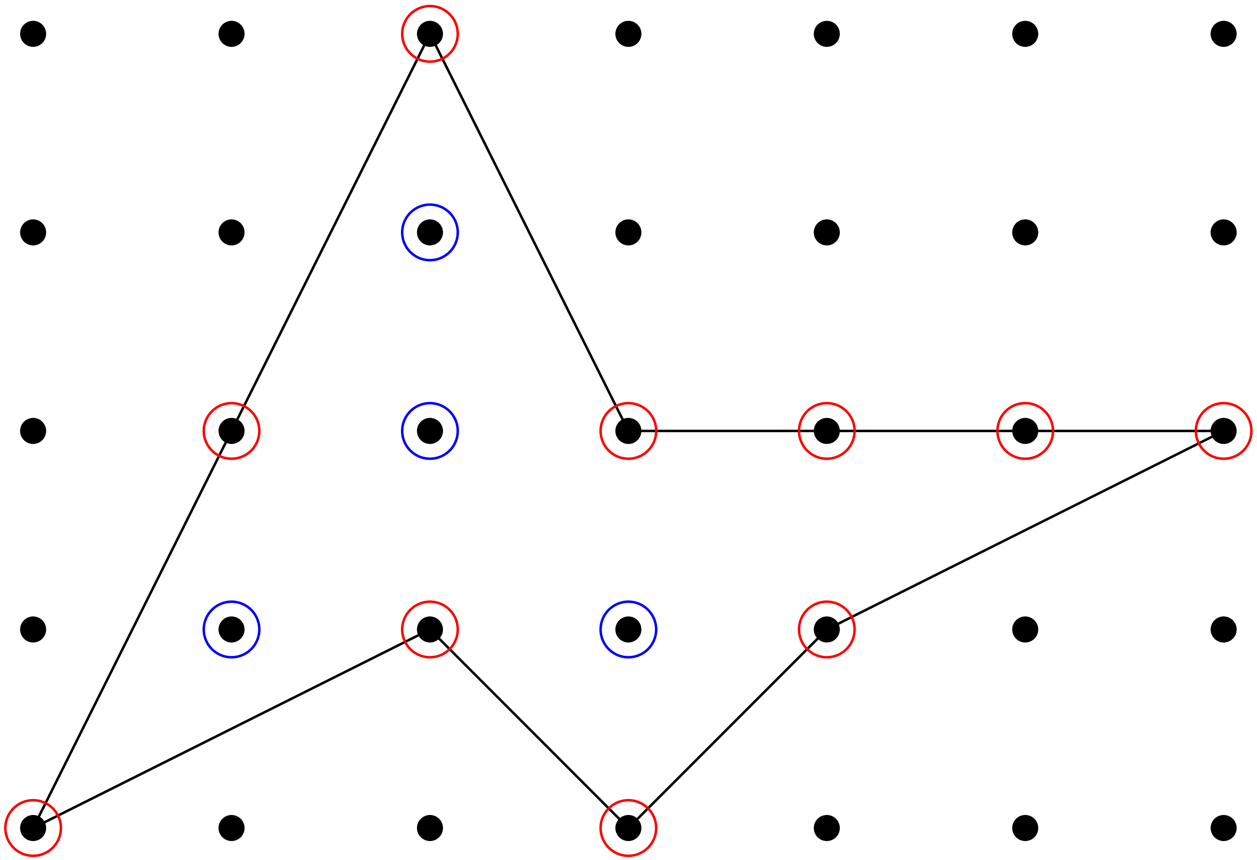
A lattice polygon

Georg Alexander Pick (1859–1942)

P: lattice polygon in $\mathbb{R}^2$
(vertices $\in \mathbb{Z}^2$, no self-intersections)



Boundary and interior lattice points



Pick's theorem

$$\begin{aligned} \boldsymbol{A} &= \text{area of } P \\ \boldsymbol{I} &= \#\text{ interior points of } P\ (=4) \\ \boldsymbol{B} &= \#\text{boundary points of } P\ (=10) \end{aligned}$$

Then

$$\boldsymbol{A} = \frac{2\boldsymbol{I} + \boldsymbol{B} - 2}{2}.$$

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Example on previous slide:

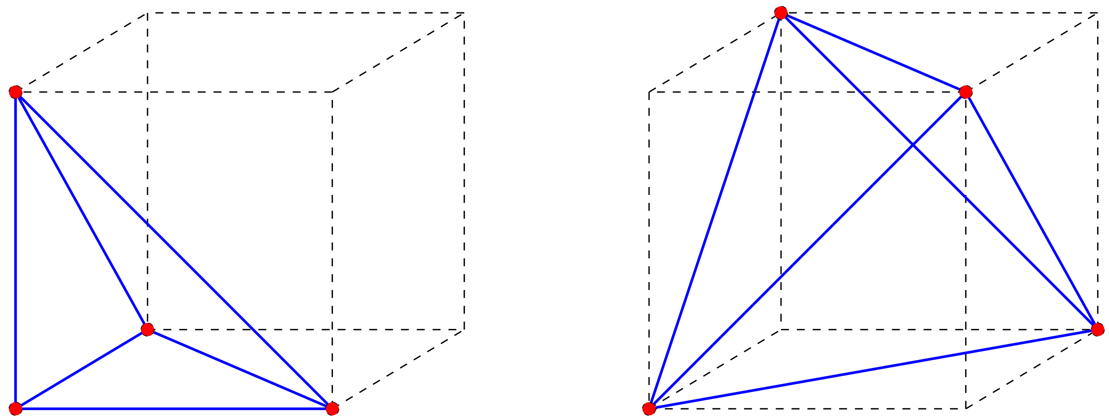
$$\frac{2 \cdot \mathbf{4} + \mathbf{10} - 2}{2} = 9.$$

Two tetrahedra

Pick's theorem (seemingly) fails in higher dimensions. For example, let T_1 and T_2 be the tetrahedra with vertices

$$\begin{aligned} v(T_1) &= \{(0,0,0),(1,0,0),(0,1,0),(0,0,1)\} \\ v(T_2) &= \{(0,0,0),(1,1,0),(1,0,1),(0,1,1)\}. \end{aligned}$$

Failure of Pick's theorem in dim 3



Then

$$I(T_1) = I(T_2) = 0$$

$$B(T_1) = B(T_2) = 4$$

$$A(T_1) = 1/6, \quad A(T_2) = 1/3.$$

Polytope dilation

Let $\mathcal{P}$ be a convex polytope (convex hull of a finite set of points) in $\mathbb{R}^d$. For $n \geq 1$, let

$$\boldsymbol{n\mathcal{P}} = \{n\alpha \, : \, \alpha \in \mathcal{P}\}.$$

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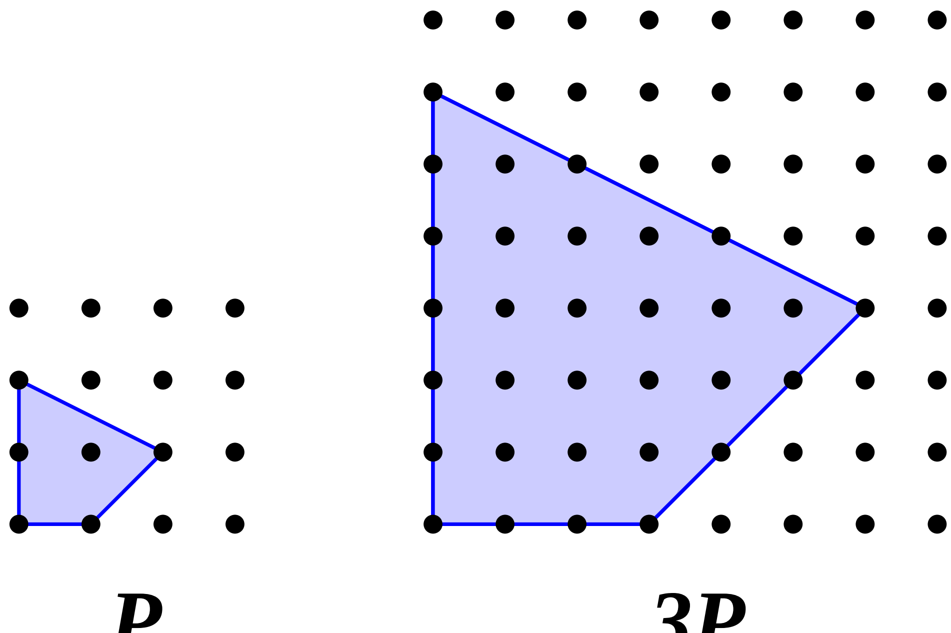

$i(\mathcal{P}, n)$

Let

$$\begin{aligned} i(\mathcal{P}, n) &= \#(n\mathcal{P} \cap \mathbb{Z}^d) \\ &= \#\{\alpha \in \mathcal{P} \,:\, n\alpha \in \mathbb{Z}^d\}, \end{aligned}$$

the number of lattice points in $n\mathcal{P}$.

$\bar{i}(\mathcal{P}, n)$

Similarly let

$$\mathcal{P}^\circ = \text{interior of } \mathcal{P} = \mathcal{P} - \partial\mathcal{P}$$

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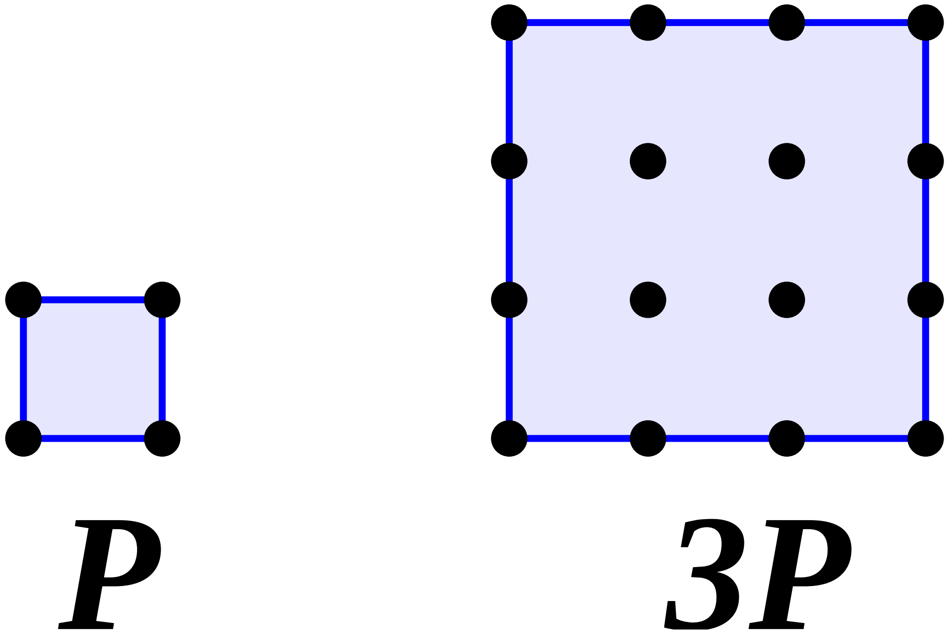
Note. Could use any lattice L instead of $\mathbb{Z}^d$.

An example



$$i(\mathcal{P}, n) \;=\; (n+1)^2$$

$$\bar{i}(\mathcal{P}, n) \;=\; (n-1)^2 = i(\mathcal{P}, -n).$$

Reeve's theorem

lattice polytope: polytope with integer vertices

Theorem (Reeve, 1957). *Let $\mathcal{P}$ be a three-dimensional lattice polytope. Then the volume $V(\mathcal{P})$ is a certain (explicit) function of $i(\mathcal{P},1)$, $\bar{i}(\mathcal{P},1)$, and $i(\mathcal{P},2)$.*

The main result

Theorem (Ehrhart 1962, Macdonald 1963). *Let*

$$\mathcal{P} = \text{lattice polytope in } \mathbb{R}^N, \ \dim \mathcal{P} = d.$$

Then $i(\mathcal{P}, n)$ is a polynomial (the ***Ehrhart polynomial*** *of $\mathcal{P}$) in n of degree d.*

Reciprocity and volume

Moreover,

$$\begin{aligned} i(\mathcal{P}, 0) &= 1 \\ \bar{i}(\mathcal{P}, n) &= (-1)^d i(\mathcal{P}, -n), \; n > 0 \end{aligned}$$

(reciprocity).

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(reciprocity).

If $d = N$ *then*

$$i(\mathcal{P},n) = V(\mathcal{P})n^d + \text{ lower order terms,}$$

where $\boldsymbol{V(\mathcal{P})}$ *is the volume of* $\mathcal{P}$*.*

Eugène Ehrhart

- April 29, 1906: born in Guebwiller, France
- 1932: begins teaching career in lycées
- 1959: Prize of French Sciences Academy
- 1963: begins work on Ph.D. thesis
- 1966: obtains Ph.D. thesis from Univ. of Strasbourg
- 1971: retires from teaching career
- January 17, 2000: dies

Photo of Ehrhart

Self-portrait

Generalized Pick's theorem

Corollary. *Let* $\mathcal{P} \subset \mathbb{R}^d$ *and* $\dim \mathcal{P} = d$*. Knowing any* d *of* $i(\mathcal{P}, n)$ *or* $\bar{i}(\mathcal{P}, n)$ *for* $n > 0$ *determines* $V(\mathcal{P})$*.*

Generalized Pick's theorem

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Proof. Together with $i(\mathcal{P}, 0) = 1$, this data determines $d + 1$ values of the polynomial $i(\mathcal{P}, n)$ of degree d. This uniquely determines $i(\mathcal{P}, n)$ and hence its leading coefficient $V(\mathcal{P})$. $\square$

An example: Reeve's theorem

Example. When $d = 3$, $V(\mathcal{P})$ is determined by

$$\begin{aligned} i(\mathcal{P},1) &= \#(\mathcal{P}\cap\mathbb{Z}^3) \\ i(\mathcal{P},2) &= \#(2\mathcal{P}\cap\mathbb{Z}^3) \\ \bar{i}(\mathcal{P},1) &= \#(\mathcal{P}^\circ\cap\mathbb{Z}^3), \end{aligned}$$

which gives Reeve's theorem.

Birkhoff polytope

Example. Let $\mathcal{B}_M \subset \mathbb{R}^{M\times M}$ be the **Birkhoff polytope** of all $M \times M$ **doubly-stochastic** matrices $A = (a_{ij})$, i.e.,

$$\begin{aligned} a_{ij} &\geq 0 \\ \sum_i a_{ij} &= 1 \text{ (column sums 1)} \\ \sum_j a_{ij} &= 1 \text{ (row sums 1).} \end{aligned}$$

(Weak) magic squares

Note. $B = (b_{ij}) \in n\mathcal{B}_M \cap \mathbb{Z}^{M\times M}$ if and only if

$$b_{ij} \ \in \ \mathbb{N} = \{0, 1, 2, \ldots\}$$

$$\sum_i b_{ij} \ = \ n$$

$$\sum_j b_{ij} \ = \ n.$$

Example of a magic square

$$\begin{bmatrix} 2 & 1 & 0 & 4 \\ 3 & 1 & 1 & 2 \\ 1 & 3 & 2 & 1 \\ 1 & 2 & 4 & 0 \end{bmatrix} \qquad (M = 4,\ n = 7)$$

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$$\in 7\mathcal{B}_4$$

$H_M(n)$

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$$\begin{aligned} H_1(n) &= 1 \\ H_2(n) &= n+1 \end{aligned}$$

$$\begin{bmatrix} a & n-a \\ n-a & a \end{bmatrix}, \quad 0 \le a \le n.$$

The case $M = 3$

$$H_3(n) = \binom{n+2}{4} + \binom{n+3}{4} + \binom{n+4}{4}$$

(MacMahon)

Values for small n

$$H_M(0) = ??$$

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$$\sum_{M \geq 0} H_M(2) \frac{x^M}{M!^2} = ??$$

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Anand-Dumir-Gupta, 1966:

$$\sum_{M \geq 0} H_M(2) \frac{x^M}{M!^2} = \frac{e^{x/2}}{\sqrt{1-x}}$$

Anand-Dumir-Gupta conjecture

Theorem (Birkhoff-von Neumann). *The vertices of $\mathcal{B}_M$ consist of the $M!$ $M \times M$ permutation matrices. Hence $\mathcal{B}_M$ is a lattice polytope.*

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Corollary (Anand-Dumir-Gupta conjecture). $H_M(n)$ *is a polynomial in* n *(of degree* $(M-1)^2$*).*

$H_4(n)$

Example. $H_4(n) = \frac{1}{11340}\left(11n^9 + 198n^8 + 1596n^7\right.$

$+7560n^6 + 23289n^5 + 48762n^5 + 70234n^4 + 68220n^2$

$\left.+40950n + 11340\right).$

Reciprocity for magic squares

Reciprocity $\Rightarrow \pm H_M(-n) =$

$\#\{M \times M$ matrices B of **positive** integers, line sum $n\}$

But every such B can be obtained from an $M \times M$ matrix A of **nonnegative** integers by adding 1 to each entry.

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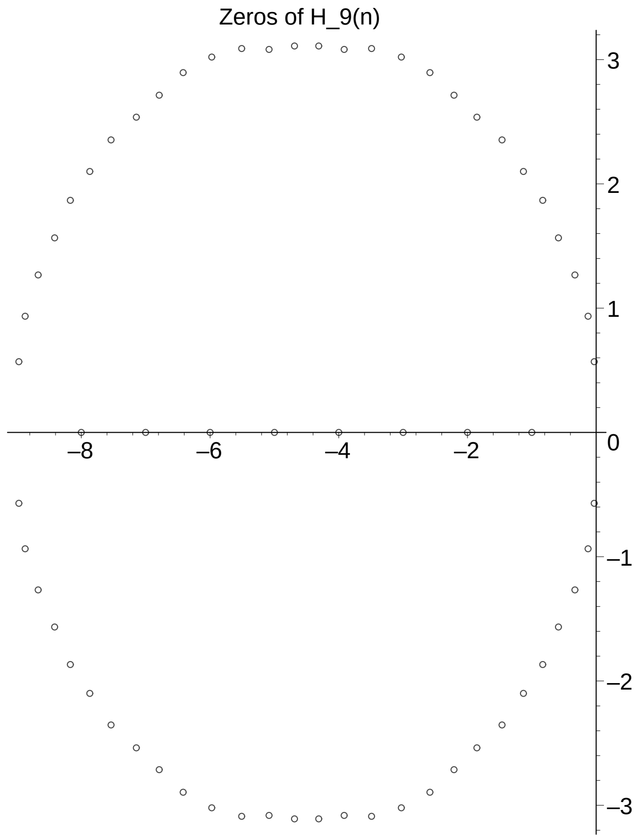
Corollary.
$H_M(-1) = H_M(-2) = \cdots = H_M(-M+1) = 0$

$$H_M(-M-n) = (-1)^{M-1} H_M(n)$$

Two remarks

- Reciprocity greatly reduces computation.
- Applications of magic squares, e.g., to statistics (contingency tables).

Zeros of $H_9(n)$ in complex plane



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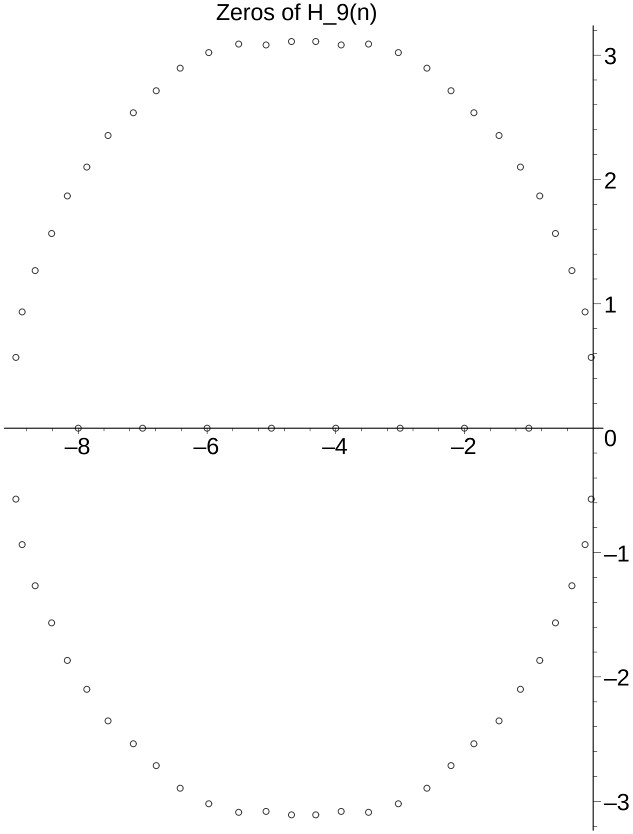

No explanation known.

Zonotopes

Let $\boldsymbol{v_1, \ldots, v_k} \in \mathbb{R}^d$. The **zonotope** $Z(v_1, \ldots, v_k)$ generated by $v_1, \ldots, v_k$:

$$\boldsymbol{Z(v_1, \ldots, v_k)} = \{\lambda_1 v_1 + \cdots + \lambda_k v_k \ : \ 0 \le \lambda_i \le 1\}$$

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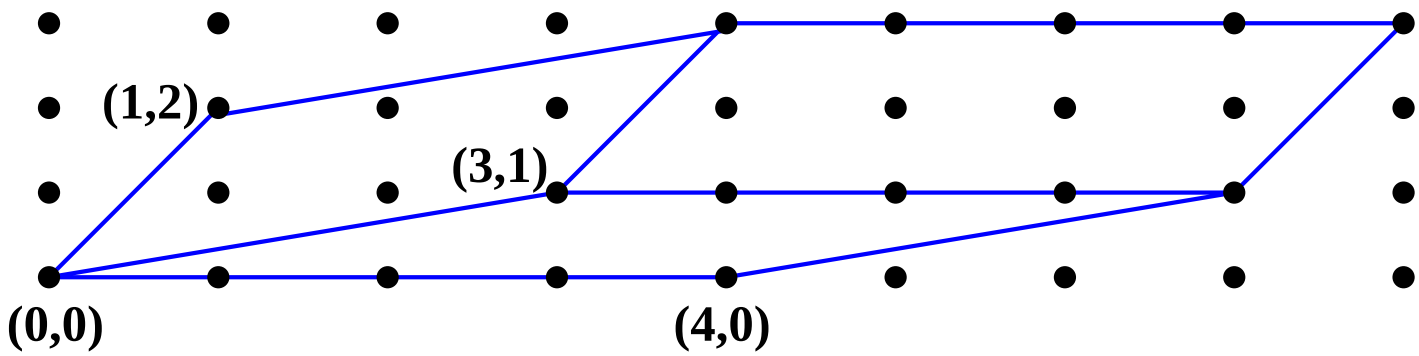
Example. $v_1 = (4, 0)$, $v_2 = (3, 1)$, $v_3 = (1, 2)$



Lattice points in a zonotope

Theorem. *Let*

$$Z = Z(v_1, \ldots, v_k) \subset \mathbb{R}^d,$$

where $v_i \in \mathbb{Z}^d$*. Then*

$$i(Z, 1) = \sum_X h(X),$$

where X *ranges over all linearly independent subsets of* $\{v_1, \ldots, v_k\}$*, and* $h(X)$ *is the gcd of all* $j \times j$ *minors (*$j = \#X$*) of the matrix whose rows are the elements of* X*.*

An example

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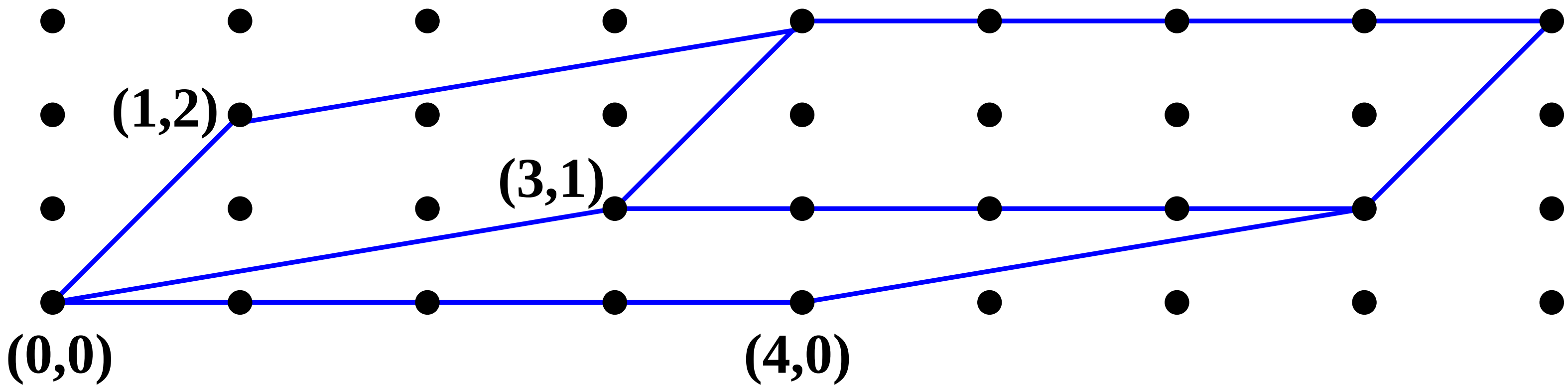

Computation of $i(Z, 1)$

$$\begin{aligned} i(Z,1) &= \begin{vmatrix} 4 & 0 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 4 & 0 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} \\ &\quad + \gcd(4,0) + \gcd(3,1) \\ &\quad + \gcd(1,2) + \det(\emptyset) \\ &= 4 + 8 + 5 + 4 + 1 + 1 + 1 \\ &= 24. \end{aligned}$$

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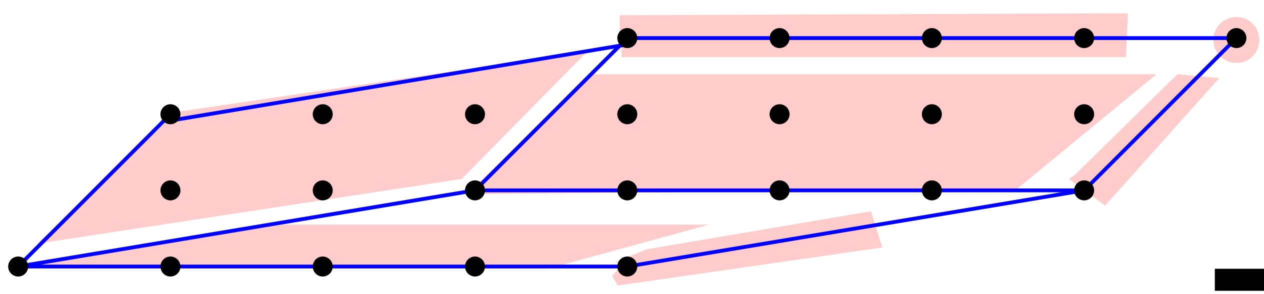

Corollaries

Corollary. *If* Z *is an integer zonotope generated by integer vectors, then the coefficients of* $i(Z, n)$ *are nonnegative integers.*

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Neither property is true for general integer polytopes. There are numerous conjectures concerning special cases.

The permutohedron

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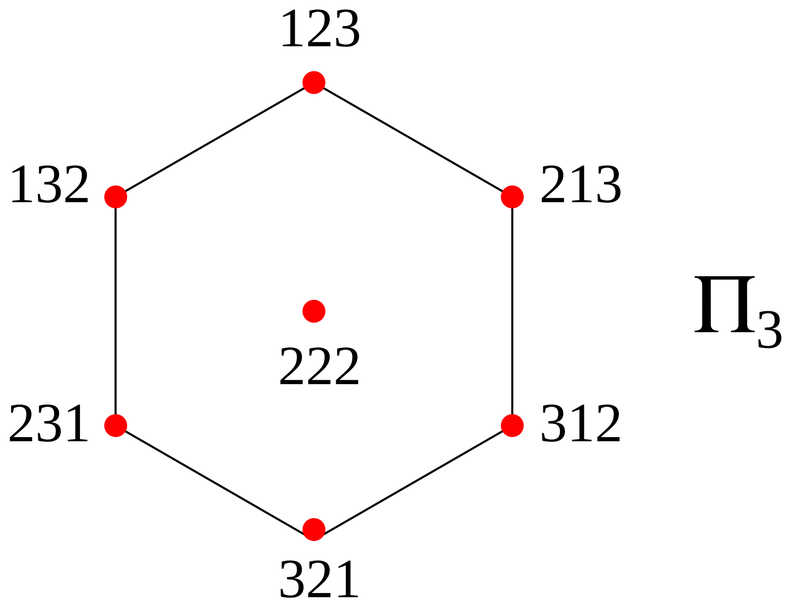
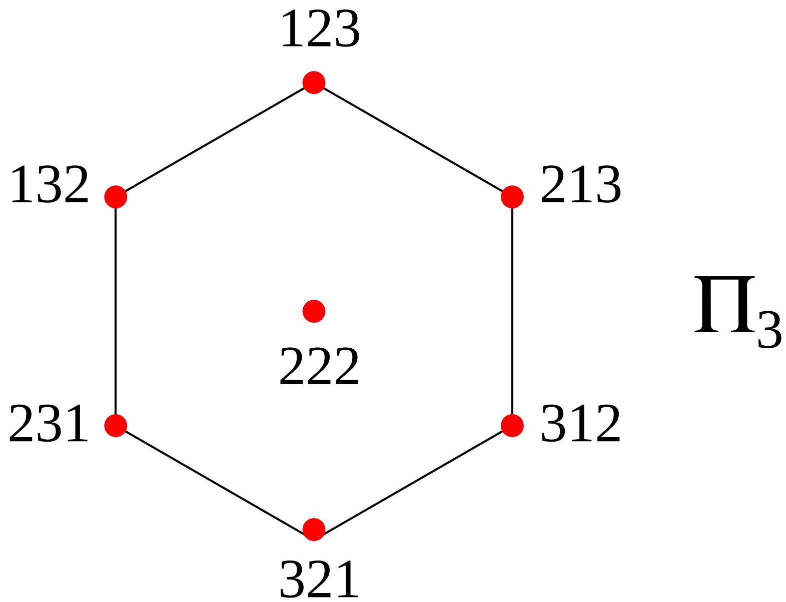
$$\dim \Pi_d = d - 1, \text{ since } \sum w(i) = \binom{d+1}{2}$$

$$\Pi_d \approx Z(e_i - e_j : 1 \leq i < j \leq d)$$

Π_3

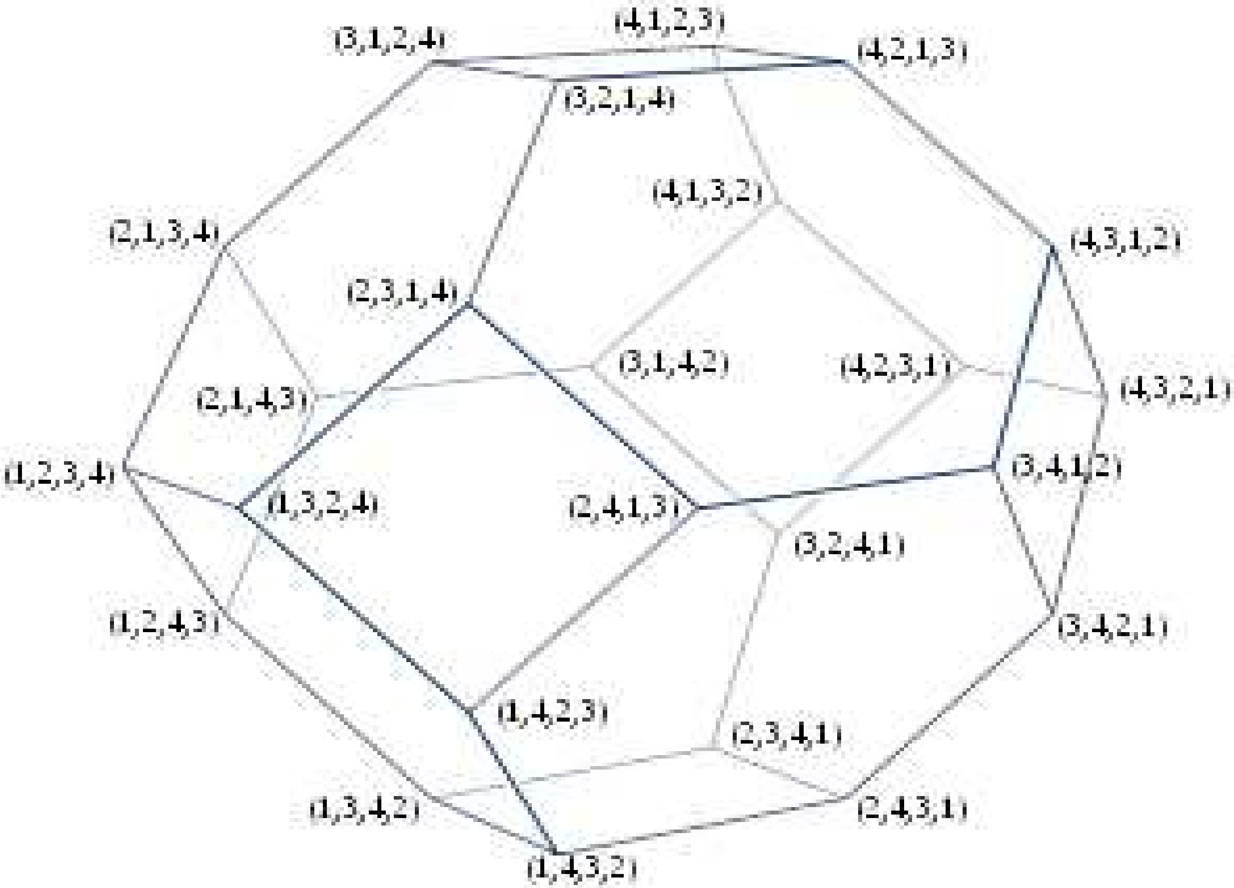


Π_3



$$i(\Pi_3, n) \;=\; 3n^2 + 3n + 1$$

Π_4



(truncated octahedron)

$i(\Pi_d, n)$

Theorem. $i(\Pi_d, n) = \sum_{k=0}^{d-1} f_k(d) x^k$, *where*

$f_k(d) = \#\{$forests with k edges on vertices $1, \ldots, d\}$

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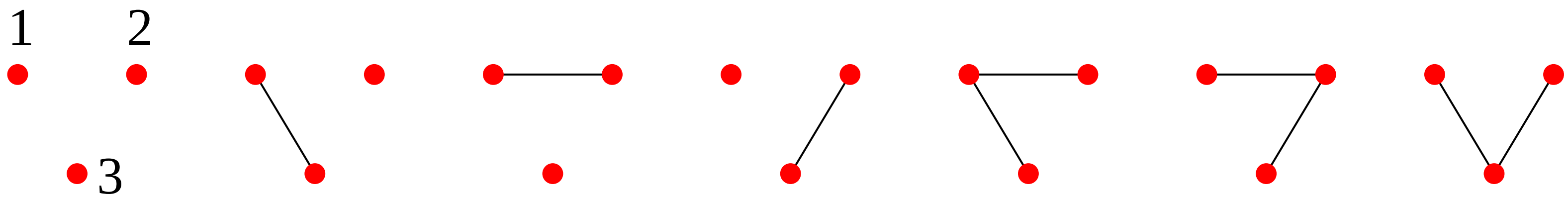

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Application to graph theory

Let G be a graph (with no loops or multiple edges) on the vertex set $V(G) = \{1, 2, \ldots, n\}$. Let

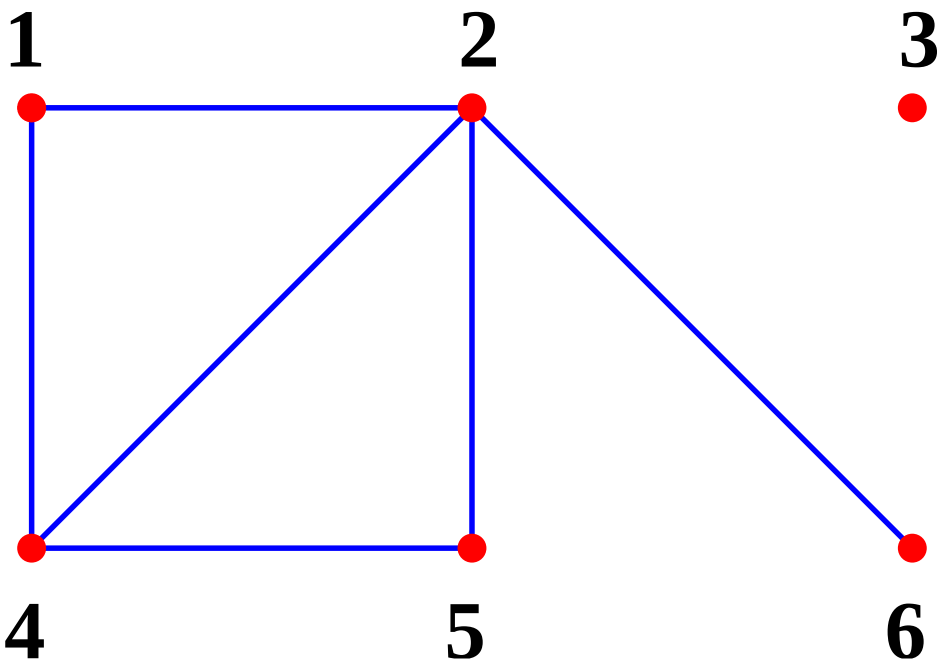
$$d_i = \text{degree (\# incident edges) of vertex } i.$$

Define the **ordered degree sequence** $d(G)$ of G by

$$d(G) = (d_1, \ldots, d_n).$$

Example of $d(G)$

Example. $d(G) = (2, 4, 0, 3, 2, 1)$

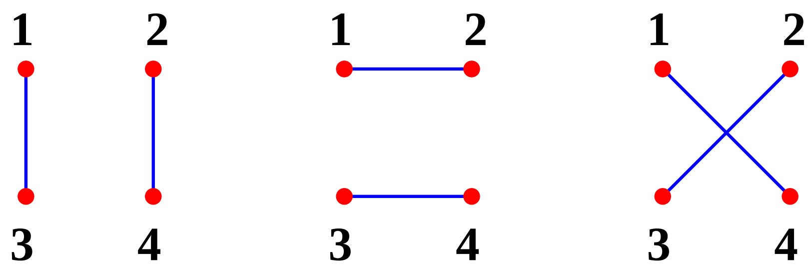


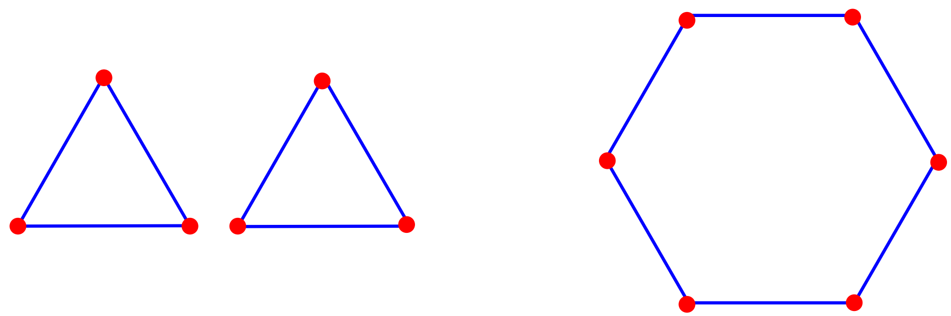
ordered degree sequences

Let $\boldsymbol{f(n)}$ be the number of distinct $d(G)$, where $V(G) = \{1, 2, \ldots, n\}$.

$f(n)$ for $n \le 4$

Example. If $n \le 3$, all $d(G)$ are distinct, so $f(1) = 1$, $f(2) = 2^1 = 2$, $f(3) = 2^3 = 8$. For $n \ge 4$ we can have $G \ne H$ but $d(G) = d(H)$, e.g.,





In fact, $f(4) = 54 < 2^6 = 64$.

The polytope of degree sequences

Let **conv** denote convex hull, and

$$\mathcal{D}_n = \mathrm{conv}\{d(G) : V(G) = \{1, \ldots, n\}\} \subset \mathbb{R}^n,$$

the **polytope of degree sequences** (Perles, Koren).

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Easy fact. Let e_i be the ith unit coordinate vector in $\mathbb{R}^n$. E.g., if $n = 5$ then $e_2 = (0, 1, 0, 0, 0)$. Then

$$\mathcal{D}_n = Z(e_i + e_j : 1 \leq i < j \leq n).$$

The Erdős-Gallai theorem

Theorem. *Let*

$$\boldsymbol{\alpha} = (a_1, \ldots, a_n) \in \mathbb{Z}^n.$$

Then $\alpha = d(G)$ *for some* G *if and only if*

- $\alpha \in \mathcal{D}_n$
- $a_1 + a_2 + \cdots + a_n$ *is even.*

A generating function

Enumerative techniques leads to:

Theorem. *Let*

$$F(x) = \sum_{n\geq 0} f(n)\frac{x^n}{n!}$$
$$= 1 + x + 2\frac{x^2}{2!} + 8\frac{x^3}{3!} + 54\frac{x^4}{4!} + \cdots .$$

Then:

A formula for $F(x)$

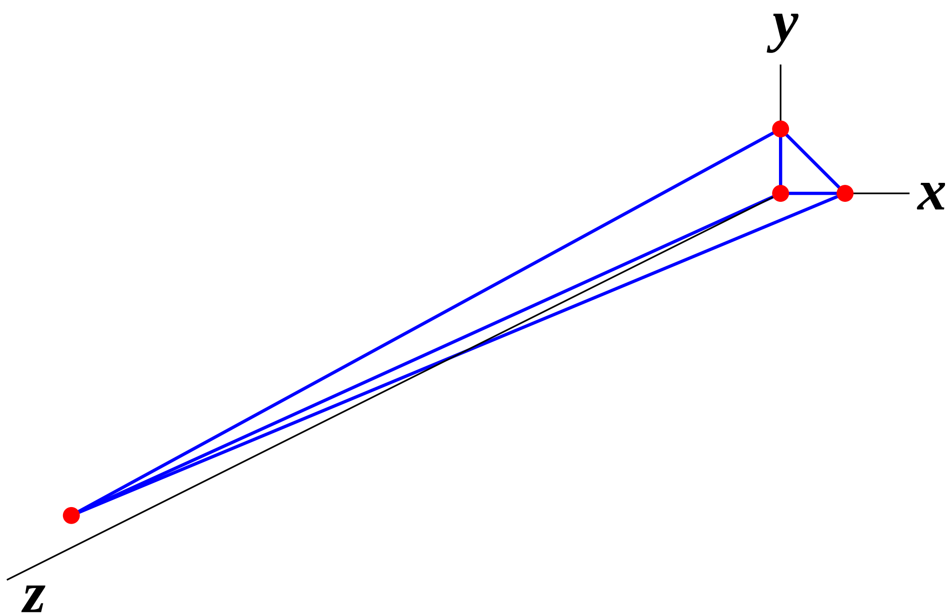
$$F(x) = \frac{1}{2}\left[\left(1 + 2\sum_{n\geq 1} n^n \frac{x^n}{n!}\right)^{1/2} \times \left(1 - \sum_{n\geq 1}(n-1)^{n-1}\frac{x^n}{n!}\right) + 1\right] \times \exp \sum_{n\geq 1} n^{n-2}\frac{x^n}{n!} \qquad (0^0 = 1)$$

Coefficients of $i(\mathcal{P}, n)$

Let $\mathcal{P}$ denote the tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(1,1,13)$. Then

$$i(\mathcal{P}, n) = \frac{13}{6}n^3 + n^2 - \frac{1}{6}n + 1.$$

The “bad” tetrahedron



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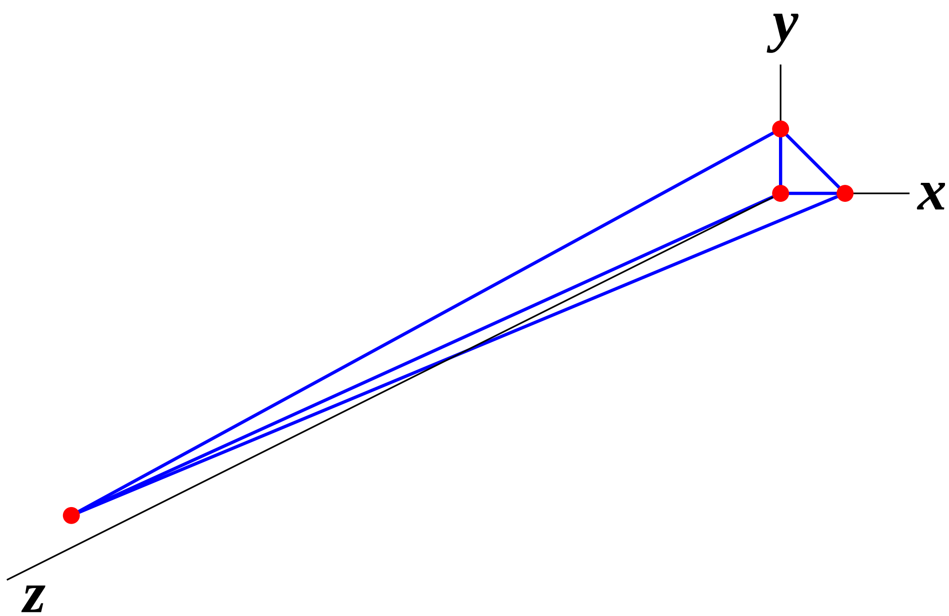

Thus in general the coefficients of Ehrhart polynomials are not “nice.” Is there a “better” basis?

The h^*-vector of $i(\mathcal{P}, n)$

Let $\mathcal{P}$ be a lattice polytope of dimension d. Since $i(\mathcal{P}, n)$ is a polynomial of degree d, $\exists$ $\boldsymbol{h_i} \in \mathbb{Z}$ such that

$$\sum_{n \geq 0} i(\mathcal{P}, n) x^n = \frac{h_0 + h_1 x + \cdots + h_d x^d}{(1-x)^{d+1}}.$$

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Definition. Define

$$h(\mathcal{P}) = (h_0, h_1, \ldots, h_d),$$

the **h^*-vector** of $\mathcal{P}$.

Example of an h^*-vector

Example. Recall

$$i(\mathcal{B}_4, n) = \frac{1}{11340}(11n^9$$

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Then

$$h^*(\mathcal{B}_4) = (1, 14, 87, 148, 87, 14, 1, 0, 0, 0).$$

Elementary properties of $h^*(\mathcal{P})$

- $h_0^* = 1$
- $h_d^* = (-1)^{\dim \mathcal{P}} i(\mathcal{P}, -1) = I(\mathcal{P})$
- $\max\{i \;:\; h_i^* \neq 0\} = \min\{j \geq 0 \;:$

  $i(\mathcal{P}, -1) = i(\mathcal{P}, -2) = \cdots = i(\mathcal{P}, -(d-j)) = 0\}$

  E.g., $h^*(\mathcal{P}) = (h_0^*, \ldots, h_{d-2}^*, 0, 0) \Leftrightarrow i(\mathcal{P}, -1) = i(\mathcal{P}, -2) = 0$.

Another property

- $i(\mathcal{P}, -n-k) = (-1)^d \, i(\mathcal{P}, n) \; \forall n \Leftrightarrow$

$$h_i^* = h_{d+1-k-i}^* \; \forall i, \text{ and}$$

$$h_{d+2-k-i}^* = h_{d+3-k-i}^* = \cdots = h_d^* = 0$$

Back to $\mathcal{B}_4$

Recall:

$$h^*(\mathcal{B}_4) = (1, 14, 87, 148, 87, 14, 1, 0, 0, 0).$$

Thus

$$i(\mathcal{B}_4, -1) = i(\mathcal{B}_4, -2) = i(\mathcal{B}_4, -3) = 0$$

$$i(\mathcal{B}_4, -n-4) = -i(\mathcal{B}_4, n).$$

Main properties of $h^*(\mathcal{P})$

Theorem A (nonnegativity). (McMullen, RS)
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$$h_i^*(\mathcal{Q}) \leq h_i^*(\mathcal{P}) \ \forall i.$$

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Theorem A (nonnegativity). (McMullen, RS) $h_i^* \geq 0$.

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$$h_i^*(\mathcal{Q}) \leq h_i^*(\mathcal{P}) \ \forall i.$$

B $\Rightarrow$ A: take $\mathcal{Q} = \emptyset$.

Proofs

Both theorems can be proved geometrically.

There are also elegant algebraic proofs based on **commutative algebra**.

Further directions

I. Zeros of Ehrhart polynomials

Sample theorem (de Loera, Develin, Pfeifle, RS). *Let $\mathcal{P}$ be a lattice d-polytope. Then*

$$i(\mathcal{P}, \alpha) = 0,\ \alpha \in \mathbb{R} \Rightarrow -d \leq \alpha \leq \lfloor d/2 \rfloor.$$

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$$i(\mathcal{P}, \alpha) = 0, \ \alpha \in \mathbb{R} \Rightarrow -d \leq \alpha \leq \lfloor d/2 \rfloor.$$

Theorem. *Let d be odd. There exists a $0/1$ d-polytope $\mathcal{P}_d$ and a real zero α_d of $i(\mathcal{P}_d, n)$ such that*

$$\lim_{\substack{d \to \infty \\ d \text{ odd}}} \frac{\alpha_d}{d} = \frac{1}{2\pi e} = 0.0585 \cdots .$$

An open problem

Open. Is the set of all complex zeros of all Ehrhart polynomials of lattice polytopes dense in $\mathbb{C}$? (True for chromatic polynomials of graphs.)

II. Brion's theorem

Example. Let $\mathcal{P}$ be the polytope $[2,5]$ in $\mathbb{R}$, so $\mathcal{P}$ is defined by

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II. Brion's theorem

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$$\textbf{(1)}\ x \geq 2, \qquad \textbf{(2)}\ x \leq 5.$$

Let

$$\begin{aligned} F_1(t) &= \sum_{\substack{n \geq 2 \\ n \in \mathbb{Z}}} t^n = \frac{t^2}{1 - t} \\ F_2(t) &= \sum_{\substack{n \leq 5 \\ n \in \mathbb{Z}}} t^n = \frac{t^5}{1 - \frac{1}{t}}. \end{aligned}$$

$F_1(t) + F_2(t)$

$$\begin{aligned} F_1(t) + F_2(t) &= \frac{t^2}{1-t} + \frac{t^5}{1-\frac{1}{t}} \\ &= t^2 + t^3 + t^4 + t^5 \\ &= \sum_{m \in \mathcal{P} \cap \mathbb{Z}} t^m . \end{aligned}$$

Cone at a vertex

$\boldsymbol{\mathcal{P}}$: $\mathbb{Z}$-polytope in $\mathbb{R}^N$ with vertices $\boldsymbol{v_1}, \ldots, \boldsymbol{v_k}$

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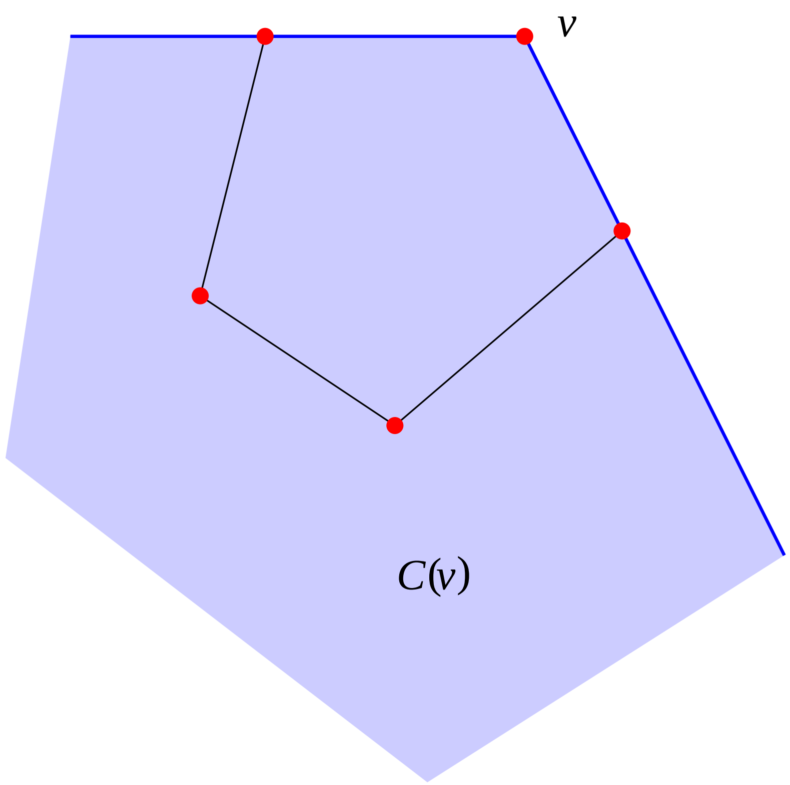

The general result

Let $\boldsymbol{F_i}(t_1, \ldots, t_N) = \sum_{(m_1,\ldots,m_N) \in \mathcal{C}_i \cap \mathbb{Z}^N} t_1^{m_1} \cdots t_N^{m_N}$.

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Let $\boldsymbol{F_i}(t_1, \ldots, t_N) = \sum_{(m_1,\ldots,m_N)\in\mathcal{C}_i\cap\mathbb{Z}^N} t_1^{m_1}\cdots t_N^{m_N}$.

Theorem (Brion). *Each F_i is a rational function of $t_1, \ldots, t_N$, and*

$$\sum_{i=1}^{k} F_i(t_1, \ldots, t_N) = \sum_{(m_1,\ldots,m_N)\in\mathcal{P}\cap\mathbb{Z}^N} t_1^{m_1}\cdots t_N^{m_N}$$

(as rational functions).

III. Toric varieties

Given an integer polytope $\mathcal{P}$, can define a projective algebraic variety $\boldsymbol{X_{\mathcal{P}}}$, a **toric variety**.

Leads to deep connections with toric geometry, including new formulas for $i(\mathcal{P}, n)$.

IV. Complexity

Computing $i(\mathcal{P}, n)$, or even $i(\mathcal{P}, 1)$ is **$\#P$-complete**. Thus an "efficient" (polynomial time) algorithm is extremely unlikely. However:

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Theorem (A. Barvinok, 1994). *For **fixed** $\dim \mathcal{P}$, $\exists$ polynomial-time algorithm for computing $i(\mathcal{P}, n)$.*

V. Fractional lattice polytopes

Example. Let $S_M(n)$ denote the number of **symmetric** $M \times M$ matrices of nonnegative integers, every row and column sum n. Then

$$S_3(n) = \begin{cases} \frac{1}{8}(2n^3 + 9n^2 + 14n + 8), & n \text{ even} \\ \frac{1}{8}(2n^3 + 9n^2 + 14n + 7), & n \text{ odd} \end{cases}$$

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Why a different polynomial depending on n modulo 2?

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Fact: vertices of $\mathcal{T}_M$ have the form $\frac{1}{2}(P + P^t)$, where P is a permutation matrix.

Thus if v is a vertex of $\mathcal{T}_M$ then $\mathbf{2}v \in \mathbb{Z}^{M\times M}$.

$S_M(n)$ in general

Theorem. *There exist polynomials* $P_M(n)$ *and* $Q_M(n)$ *for which*

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Moreover, $\deg P_M(n) = \binom{M}{2}$.

Difficult result (W. Dahmen and C. A. Micchelli, 1988):

$$\deg Q_M(n) = \begin{cases} \binom{n-1}{2} - 1, & n \text{ odd} \\ \binom{n-2}{2} - 1, & n \text{ even.} \end{cases}$$

The last slide

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