$(n+1)^{n-1}$

April 5, 2020

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Variant (will occur later): number of cosets of the subgroup $H = \langle (1, 1, \ldots, 1) \rangle$ in $G = (\mathbb{Z}/(n+1)\mathbb{Z})^n$, since

$$\#G = (n+1)^n, \qquad \#H = n+1$$

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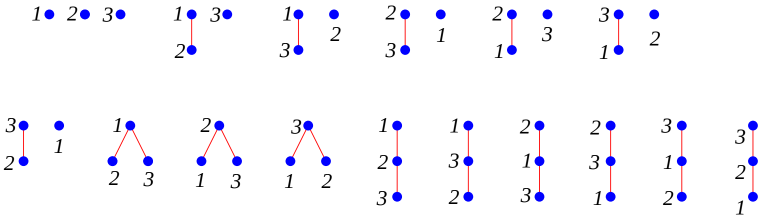
- first proof by **Borchardt**, 1860
- often attributed to **Cayley**, 1889

The case $n = 3$



Three proofs.

Proof #1 (**Prüfer**, 1918). Remove largest leaf from T and record its neighbor p_1. Continue until only two vertices remain, obtaining the **Prüfer sequence** $p(T) = (p_1, p_2, \ldots, p_{n-2})$.

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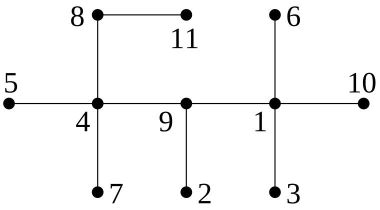

Prüfer sequence: $(8, 1, 4, 4, 1, 4, 9, 1, 9)$.

Prüfer sequence proof

Theorem. *The map* $T \mapsto p(T)$ *is a bijection from trees* T *on the vertex set* $[n]$ *to sequences* $(p_1, \ldots, p_{n-2}) \in [n]^{n-2}$.

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Proof (sketch). *Crucial fact:* the first vertex v_1 to be removed from T is the largest vertex w_1 of T missing from $p(T)$.

Thus v_1 and w_1 are adjacent in T. Now remove p_1 from $p(T)$ and continue recursively, adding one new edge each time. At the end of this procedure we have $n-2$ edges, and the remaining two unremoved vertices form the final edge. $\square$

Joyal's proof

Proof #2 (**Joyal**, 1981). **Doubly rooted tree:** a tree on the vertex set $[n]$ with one vertex labelled S (start) and one vertex (possibly the same) labelled E (end).

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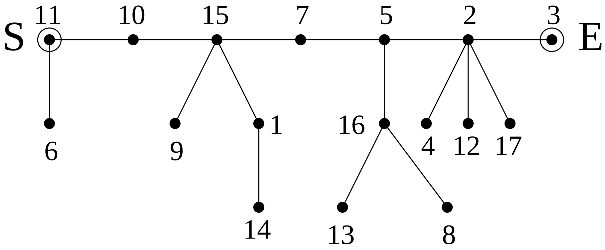
Given a doubly rooted tree, let $S = b_1, b_2, \ldots, b_k = E$ be the unique path from S to E.

Continuation of proof



$$(b_1, \ldots, b_7) = 11, 10, 15, 7, 5, 2, 3$$

Cycle form

$$(b_1, \ldots, b_7) = 11, 10, 15, 7, 5, 2, 3$$

Regard $b_1, \ldots, b_k$ as a permutation w of its elements in increasing order.

$$\begin{pmatrix} 2 & 3 & 5 & 7 & 10 & 11 & 15 \\ 11 & 10 & 15 & 7 & 5 & 2 & 3 \end{pmatrix},$$

i.e., $2 \to 11$, $3 \to 10$, etc.

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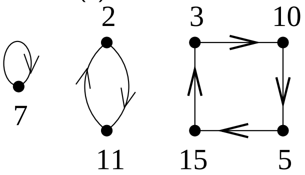
Digraph D_w of w: $i \to w(i)$



A new digraph D_T

Attach to each vertex v of D_w the same subgraph $\boldsymbol{T_v}$ that was attached "below" v in T, and direct the edges of T_v toward v, obtaining a digraph $\boldsymbol{D_T}$.

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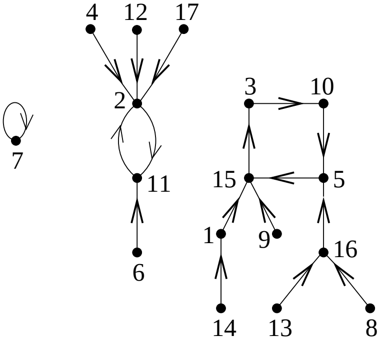

Functional digraphs

Crucial property of D_T: every vertex has outdegree one, i.e., D_T is the digraph of a function $f : [n] \to [n]$ (with edges $i \to f(i)$).

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Since $d(n) = n^2 t(n)$, we get $\boxed{t(n) = n^{n-2}}$.

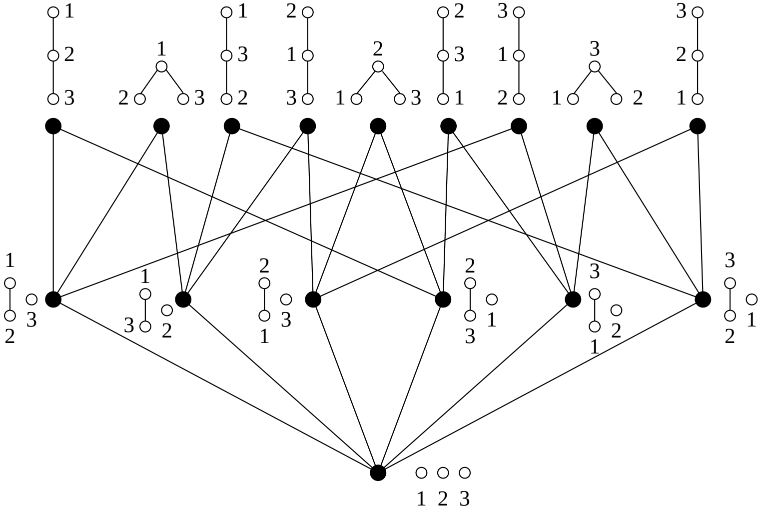
Pitman's proof

Proof #3 (**Pitman**, 1999). $\boldsymbol{P_n}$: set of all planted forests on $[n]$

$\boldsymbol{uv}$: an edge of a forest $F \in P_n$ such that u is closer than v to the root of its component.

F **covers** F': obtain F' by removing the edge uv from F, and rooting the new tree containing v at v. This defines the cover relations of a partial order on P_n.

The poset P_3



An easy exercise

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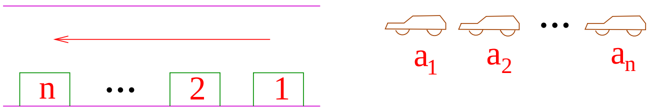
$\boldsymbol{M_n}$: number of maximal chains of P_n

Counting maximal chains from top to bottom and from bottom to top gives

$$M_n = nt(n)(n-1)! = n^{n-1}(n-1)!$$

$$\Rightarrow \boxed{t(n) = n^{n-2}}.$$

A parking scenario



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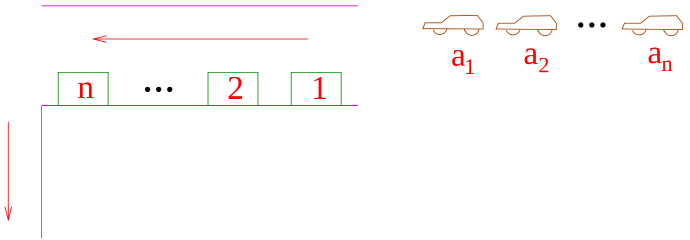

Parking functions

Car C_i prefers space a_i, drives there, and parks if possible. If a_i is occupied, then C_i takes the next available space. We call $(a_1, \ldots, a_n)$ a **parking function** (of length n) if all cars can park.

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First considered by **Ronald Pyke** (implicitly) and **Alan Konheim** and **Benjamin Weiss** (1966).

The case of the capricious wives

Konheim and Weiss:

> *Let st. be a street with p parking places. A car occupied by a man and his dozing wife enters st. at the left and moves towards the right. The wife awakens at a capricious moment and orders her husband to park immediately! He dutifully parks at his present location, if it is empty, and if not, continues to the right and parks at the next available space. If no space is available he leaves st.*

Small examples

$n = 2$: 11 12 21

$n = 3$: 111 112 121 211 113 131 311 122
212 221 123 132 213 231 312 321

Parking function characterization

Easy: Let $\alpha = (a_1, \ldots, a_n) \in \mathbb{P}^n$. Let $b_1 \leq b_2 \leq \cdots \leq b_n$ be the increasing rearrangement of α. Then α is a parking function if and only $b_i \leq i$.

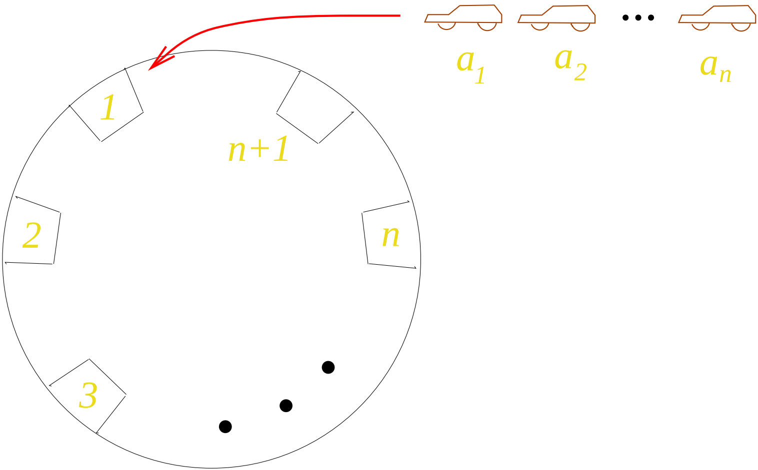
Corollary. *Every permutation of the entries of a parking function is also a parking function.*

Enumeration of parking functions

Theorem (**Pyke**, 1959; **Konheim and Weiss**, 1966). *Let $f(n)$ be the number of parking functions of length n. Then* $\boldsymbol{f(n) = (n+1)^{n-1}}$.

Proof (**Pollak**, c. 1974). Add an additional space $n+1$, and arrange the spaces in a circle. Allow $n+1$ also as a preferred space.

Pollak's proof



Conclusion of Pollak's proof

Now all cars can park, and there will be one empty space. α is a parking function $\Leftrightarrow$ if the empty space is $n+1$. If $\alpha = (a_1, \ldots, a_n)$ leads to car C_i parking at space p_i, then $(a_1 + j, \ldots, a_n + j)$ (modulo $n+1$) will lead to car C_i parking at space $p_i + j$. Hence exactly one of the vectors

$$(a_1 + i, a_2 + i, \ldots, a_n + i) \text{ (modulo } n+1)$$

is a parking function, so

$$f(n) = \frac{(n+1)^n}{n+1} = (n+1)^{n-1}.$$

Prime parking functions

Definition (**I. Gessel**). A parking function is **prime** if it remains a parking function when we delete a 1 from it.

Factorization of increasing PF's

1	2	3	4	5	6	7	8	9	10	11
1	1	3	3	4	4	7	8	8	9	10

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***p*(*n*)**: number of prime parking functions of length n

$$\sum_{n\geq 0}(n+1)^{n-1}\frac{x^n}{n!} = \frac{1}{1-\sum_{n\geq 1} p(n)\frac{x^n}{n!}}$$

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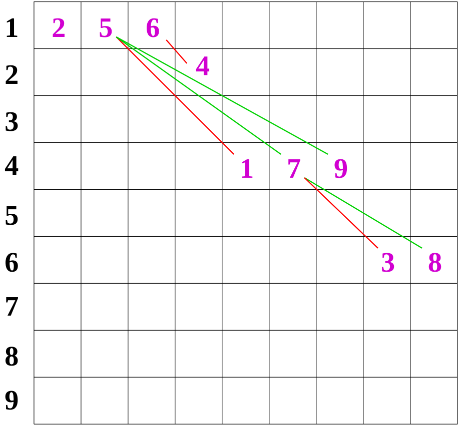
Exercise. Find a "parking" proof.

Bijection: parking functions → planted forests

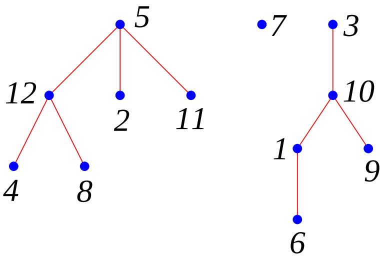


1	2	3	4	5	6	7	8	9
4	1	6	2	1	1	4	6	4

Inversions

An **inversion** in F is a pair (i, j) so that $i > j$ and i lies on the path from j to the root.

$$\mathrm{inv}(F) = \#(\text{inversions of } F)$$



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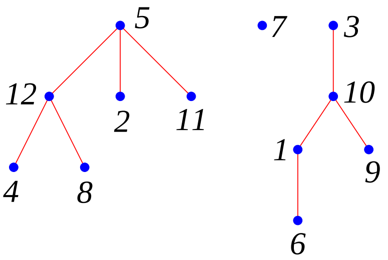

Inversions:

$(5,4),\ (5,2),\ (12,4),\ (12,8),\ (3,1),\ (10,1),\ (10,6),\ (10,9)$

$$\mathrm{inv}(F) = 8$$

The inversion enumerator

Let

$$I_n(q) = \sum_F q^{\mathrm{inv}(F)},$$

summed over all forests F with vertex set $\{1, \ldots, n\}$. E.g.,

$$\begin{aligned} I_1(q) &= 1 \\ I_2(q) &= 2 + q \\ I_3(q) &= 6 + 6q + 3q^2 + q^3 \end{aligned}$$

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Theorem (**Mallows-Riordan** 1968, **Gessel-Wang** 1979) *We have*

$$I_n(1+q) = \sum_G q^{e(G)-n},$$

where G ranges over all connected graphs (without loops or multiple edges) on $n+1$ labelled vertices, and where $e(G)$ denotes the number of edges of G.

Generating function

Corollary.

$$\sum_{n\geq 0} I_n(q)(q-1)^n \frac{x^n}{n!} = \frac{\sum_{n\geq 0} q^{\binom{n+1}{2}} \frac{x^n}{n!}}{\sum_{n\geq 0} q^{\binom{n}{2}} \frac{x^n}{n!}}$$

Connection with parking functions

Theorem (**Kreweras**, 1980) *We have*

$$q^{\binom{n}{2}} I_n(1/q) = \sum_{(a_1,\ldots,a_n)} q^{a_1+\cdots+a_n},$$

where $(a_1, \ldots, a_n)$ *ranges over all parking functions of length* n.

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Note. The earlier bijection between forests and parking functions does not send the number of inversions to the sum of the terms. Such a bijection is more complicated.

The Shi arrangement: background

Braid arrangement $\mathcal{B}_n$: the set of hyperplanes

$$x_i - x_j = 0, \quad 1 \leq i < j \leq n,$$

in $\mathbb{R}^n$.

$$\begin{aligned} \mathcal{R} &= \text{set of regions of } \mathcal{B}_n \\ \#\mathcal{R} &= \text{??} \end{aligned}$$

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To specify a region, we must specify for each $i < j$ whether $x_i < x_j$ or $x_i > x_j$. Hence the number of regions is the number of ways to linearly order $x_1, \dots, x_n$.

Labeling the regions

Let R_0 be the **base region**

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Label R_0 with

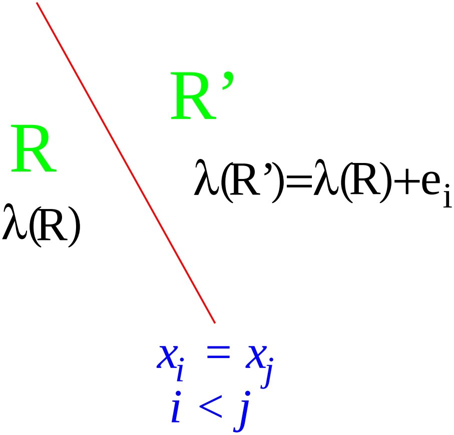
$$\boldsymbol{\lambda(R_0)} = (1, 1, \ldots, 1) \in \mathbb{Z}^n.$$

If R is labelled, R' is separated from R only by $x_i - x_j = 0$ $(i < j)$, and R' is unlabelled, then set

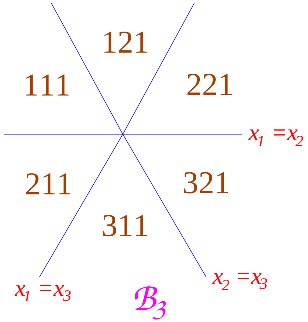
$$\boldsymbol{\lambda(R')} = \lambda(R) + e_i,$$

where $\boldsymbol{e_i}$ = ith unit coordinate vector.

The labeling rule



Description of labels



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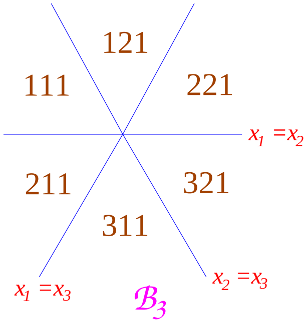

Theorem (easy). *The labels of $\mathcal{B}_n$ are the sequences $(b_1, \ldots, b_n) \in \mathbb{Z}^n$ such that $1 \le b_i \le n - i + 1$.*

The Shi arrangement

Shi Jianyi (时俭益)

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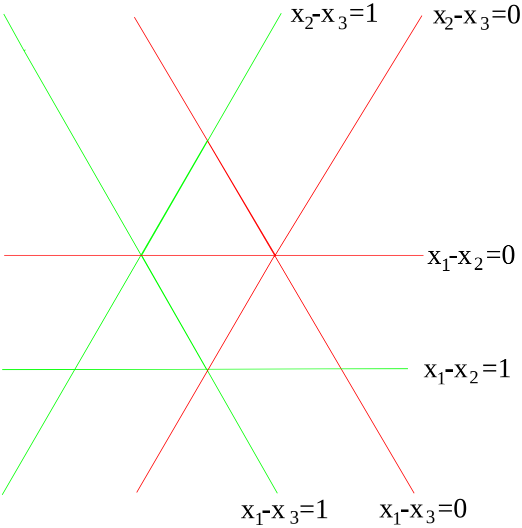
Shi arrangement $\mathcal{S}_n$: the set of hyperplanes

$$x_i - x_j = 0, 1,$$

$1 \leq i < j \leq n$, in $\mathbb{R}^n$.

The case $n = 3$



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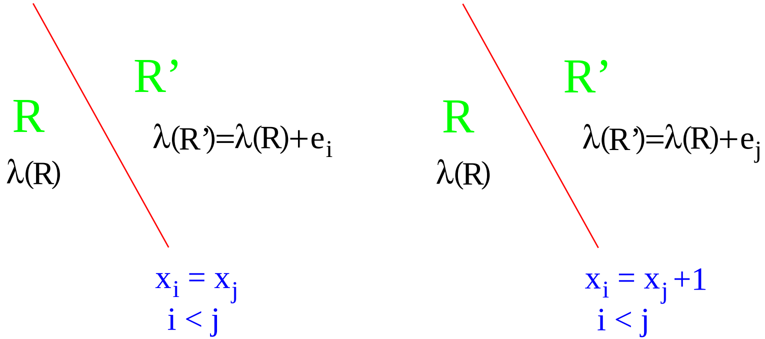
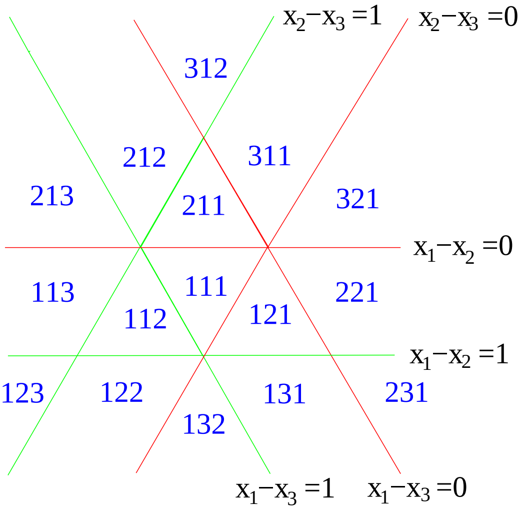
The labeling rule illustrated



The labeling for $n = 3$



Description of the labels

Theorem (**Pak, S.**). *The labels of $\mathcal{S}_n$ are the parking functions of length n (each occurring once).*

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Corollary (**Shi**, 1986).

$$r(\mathcal{S}_n) = (n+1)^{n-1}$$

The parking function polytope

Given $x_1, \ldots, x_n \in \mathbb{R}_{\geq 0}$, define $\boldsymbol{P_n = P(x_1, \ldots, x_n)} \subset \mathbb{R}^n$ by:

$(y_1, \ldots, y_n) \in P_n$ if

$$0 \leq y_i, \quad y_1 + \cdots + y_i \leq x_1 + \cdots + x_i$$

for $1 \leq i \leq n$.

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(also called **Pitman-Stanley polytope**)

Volume of P

Theorem. *Let* $x_1, \ldots, x_n \in \mathbb{R}_{\geq 0}$. *Then*

$$n!\, V(P_n) = \sum_{\substack{\text{parking functions} \\ (i_1,\ldots,i_n)}} x_{i_1} \cdots x_{i_n}.$$

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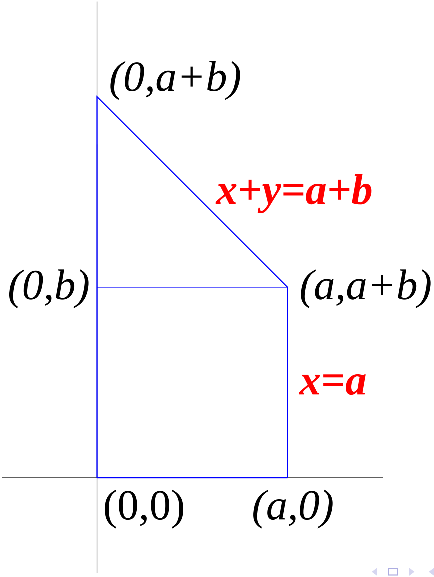
Note. If each $x_i > 0$, then P_n has the combinatorial type of an n-cube.

The case $n = 2$



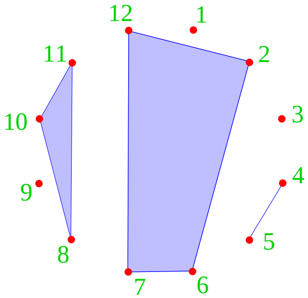
Noncrossing partitions

A **noncrossing partition** of $\{1, 2, \ldots, n\}$ is a partition $\{B_1, \ldots, B_k\}$ of $\{1, \ldots, n\}$ such that

$$a < b < c < d,\ a, c \in B_i,\ b, d \in B_j \Rightarrow i = j.$$

$(B_i \neq \emptyset,\ B_i \cap B_j = \emptyset \text{ if } i \neq j, \bigcup B_i = \{1, \ldots, n\})$

Number of noncrossing partitions



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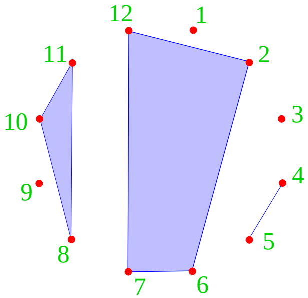

Theorem (**H. W. Becker**, 1948–49). *The number of noncrossing partitions of* $\{1, \ldots, n\}$ *is the* **Catalan number**

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Maximal chains of noncrossing partitions

A **maximal chain** $\mathfrak{m}$ of noncrossing partitions of $\{1, \ldots, n+1\}$ is a sequence

$$\pi_0, \pi_1, \pi_2, \ldots, \pi_n$$

of noncrossing partitions of $\{1, \ldots, n+1\}$ such that π_i is obtained from π_{i-1} by merging two blocks into one. (Hence π_i has exactly $n+1-i$ blocks.)

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1–2–3–4–5   1–25–3–4   1–25–34

125–34   12345

A maximal chain labeling

Define:

$$\min \mathbf{B} = \text{least element of } B$$

$$\mathbf{j} < \mathbf{B} \; : \; j < k \;\; \forall k \in B.$$

Suppose π_i is obtained from π_{i-1} by merging together blocks B and B', with $\min B < \min B'$. Define

$$\begin{aligned} \boldsymbol{\Lambda}_{\mathbf{i}}(\mathfrak{m}) &= \max\{j \in B \; : \; j < B'\} \\ \boldsymbol{\Lambda}(\mathfrak{m}) &= (\Lambda_1(\mathfrak{m}), \ldots, \Lambda_n(\mathfrak{m})). \end{aligned}$$

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For above example:

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we have

$$\Lambda(\mathfrak{m}) = (2, 3, 1, 2).$$

Labelings and parking functions

Theorem. *Λ is a bijection between the maximal chains of noncrossing partitions of $\{1, \ldots, n+1\}$ and parking functions of length n.*

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Corollary (**Kreweras**, 1972) *The number of maximal chains of noncrossing partitions of $\{1, \ldots, n+1\}$ is*

$$(n+1)^{n-1}.$$

The parking function $\mathfrak{S}_n$-action

The symmetric group $\mathfrak{S}_n$ acts on the set $\mathcal{P}_n$ of all parking functions of length n by permuting coordinates.

Sample properties

- Multiplicity of trivial representation (number of orbits) $= C_n = \frac{1}{n+1}\binom{2n}{n}$

$$n = 3 : \quad \mathbf{111} \quad \mathbf{211} \quad \mathbf{221} \quad \mathbf{311} \quad \mathbf{321}$$

Clear since orbit representatives are sequences $b_1 \leq b_2 \leq \cdots \leq b_n$, $1 \leq b_i \leq i$.

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- Multiplicity of the irreducible representation indexed by $\lambda \vdash n$: $\frac{1}{n+1} s_\lambda(1^{n+1})$

The parking function symmetric function

Let $\mathbf{PF}_n = \mathrm{PF}_n(x_1, x_2, \ldots)$ denote the Frobenius characteristic symmetric function of the action of $\mathfrak{S}_n$ on parking functions of length n. More concretely,

$$\mathrm{PF}_n = \sum_{\alpha} h_{m_1(\alpha)} h_{m_2(\alpha)} \cdots ,$$

where α ranges over all **increasing** parking functions of length n, and $m_i(\alpha)$ is the number of i's in α.

Example. $n = 3$

$$\begin{array}{ll} 111 & h_3 \\ 112 & h_2 h_1 \\ 113 & h_2 h_1 \\ 122 & h_2 h_1 \\ 123 & h_1^3 \end{array} ,$$

so $\mathrm{PF}_3 = h_3 + 3h_2 h_1 + h_1^3$.

Connection with power series inversion

- Define

$$
\begin{aligned}
F(t) &= \sum_{n\geq 1} \mathrm{PF}_n\, t^n \\
G(t) &= \sum_{n\geq 1} (-1)^{n-1} e_{n-1}\, t^n \\
&= t(1-x_1 t)(1-x_2 t)\cdots .
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Connections with Lagrange inversion, etc.

$(1 + F(t))^{-1}$

Let PPF_n be the Frobenius characteristic symmetric function of the action of $\mathfrak{S}_n$ on **prime** parking functions α of length n, i.e., α remain a parking function when we delete a 1. Let $\boldsymbol{p(n)}$ be the number of prime parking functions of length n

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Recall

$$\begin{aligned}\sum_{n\geq 0}(n+1)^{n-1}\frac{t^n}{n!} &= \frac{1}{1-\sum_{n\geq 1} p(n)\frac{t^n}{n!}} \\ \Rightarrow p(n) &= (n-1)^{n-1}\end{aligned}$$

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Theorem. $1+F(t) := 1+\sum_{n\geq 1}\mathrm{PF}_n t^n = \dfrac{1}{1-\sum_{n\geq 1}\mathrm{PPF}_n t^n}$

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Yinghui Wang (王颖慧) and **RS**: interpretation of $(1+F(t))^k$ for all $k\in\mathbb{Z}$ (rather subtle for $k<0$)

Basis expansions

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$$\begin{aligned}
\mathrm{PF}_n &= \sum_{\lambda\vdash n}(n+1)^{\ell(\lambda)-1}z_\lambda^{-1}p_\lambda \\
&= \frac{1}{n+1}\sum_{\lambda\vdash n}s_\lambda(1^{n+1})s_\lambda \\
&= \frac{1}{n+1}\sum_{\lambda\vdash n}\left[\prod_i\binom{\lambda_i+n}{\lambda_i}\right]m_\lambda \\
&= \sum_{\lambda\vdash n}\frac{n(n-1)\cdots(n-\ell(\lambda)+2)}{d_1(\lambda)!\cdots d_n(\lambda)!}h_\lambda
\end{aligned}$$

More expansions

$$\mathrm{PF}_n = \sum_{\lambda \vdash n} \varepsilon_\lambda \frac{(n+2)(n+3)\cdots(n+\ell(\lambda))}{d_1(\lambda)!\cdots d_n(\lambda)!} e_\lambda$$

$$\omega\mathrm{PF}_n = \frac{1}{n+1}\left[\prod_i \binom{n+1}{\lambda_i}\right] m_\lambda,$$

Background: invariants of $\mathfrak{S}_n$

The group $\mathfrak{S}_n$ acts on $R = \mathbb{C}[x_1, \ldots, x_n]$ by permuting variables, i.e., $w \cdot x_i = x_{w(i)}$. Let

$$R^{\mathfrak{S}_n} = \{f \in R \,:\, w \cdot f = f \text{ for all } w \in \mathfrak{S}_n\}.$$

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Well-known:

$$R^{\mathfrak{S}_n} = \mathbb{C}[e_1, \ldots, e_n],$$

where

$$\boldsymbol{e_k} = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} x_{i_1} x_{i_2} \cdots x_{i_k}.$$

The coinvariant algebra

$\boldsymbol{R_+^{\mathfrak{S}_n}}$: symmetric functions with 0 constant term

(**irrelevant ideal** of $R^{\mathfrak{S}_n}$)

$$\boldsymbol{D_n} := R/\left(R_+^{\mathfrak{S}_n}\right) = R/(e_1, \ldots, e_n).$$

Then $\dim D_n = n!$, and $\mathfrak{S}_n$ acts on D_n according to the **regular representation**.

Diagonal action of $\mathfrak{S}_n$

Now let $\mathfrak{S}_n$ act **diagonally** on

$$R = \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n],$$

i.e,

$$w \cdot x_i = x_{w(i)}, \quad w \cdot y_i = y_{w(i)}.$$

As before, let

$$\begin{aligned} \boldsymbol{R^{\mathfrak{S}_n}} &= \{f \in R \,:\, w \cdot f = f \text{ for all } w \in \mathfrak{S}_n\} \\ \boldsymbol{D_{2,n}} &= R / \left(R_+^{\mathfrak{S}_n} \right). \end{aligned}$$

Haiman's theorem

Theorem (**Haiman**, 1994, 2001). dim $D_{2,n} = (n+1)^{n-1}$, *and the action of* $\mathfrak{S}_n$ *on* $D_{2,n}$ *is isomorphic to the action on* $\mathcal{P}_n$, *tensored with the sign representation. In other words,*

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Connections with Macdonald polynomials, Hilbert scheme of points in the plane, etc.

Probabilistic aspects

Diaconis-Hicks, 2016: what does a random parking function $(a_1, \ldots, a_n)$ look like?

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Theorem. *As $n \to \infty$ and fixed j,*

$$\begin{aligned} \mathrm{Prob}(a_1 = j) &\sim \frac{1 + Q(j)}{n} \\ \mathrm{Prob}(a_1 = n - j) &\sim \frac{1 - Q(j+2)}{n}, \end{aligned}$$

where

$$Q(j) = \sum_{k \geq j} \frac{e^{-k} k^{k-1}}{k!}$$

(tail of Borel distribution on $j = 1, 2, \ldots$). Moreover,

$$\mathbb{E}(a_1) = \frac{n}{2} - \frac{\sqrt{2\pi}}{4} n^{1/2} + o(n^{1/2}).$$

Extremes

$$\begin{aligned} \mathrm{Prob}(a_1 = 1) &\sim \frac{2}{n} \\ \mathrm{Prob}(a_1 = n) &\sim \frac{1}{en}. \end{aligned}$$

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Error term?

A last sample result

Let α be a parking function. In the original parking scenario with n cars, let $L(\alpha)$ be the number of cars (**lucky** cars) which park in their preferred space. Then

$$\mathrm{Prob}\left(\frac{L(\alpha)-\frac{n}{2}}{\sqrt{n/6}}\right) \sim \int_{-\infty}^{x} \frac{e^{-t^2/2}}{\sqrt{2\pi}}\,dt.$$

What's next?

Next topic:

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