

Stern's Triangle

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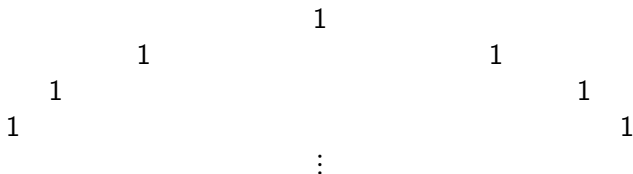
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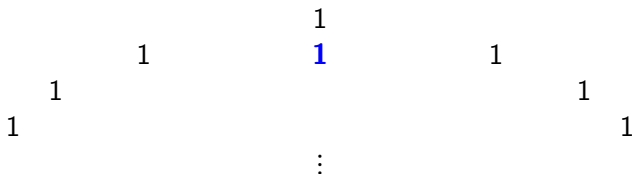
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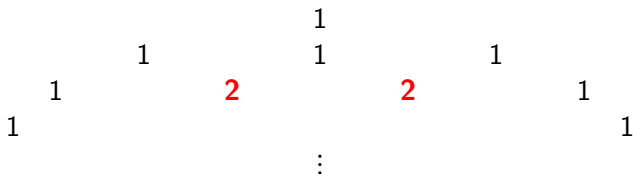
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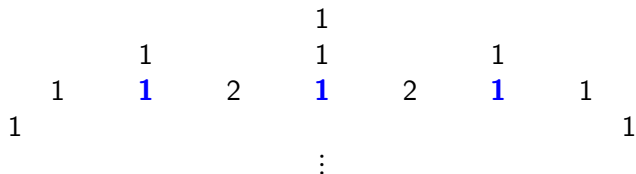
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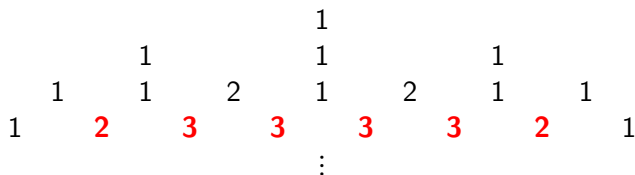
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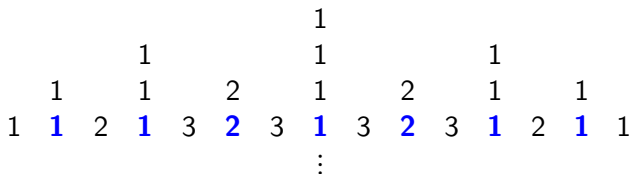
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							1							
							1					1		
			1				1		2			1		1
	1		1	2			1	3	2	3	1	2	1	1
1	1	2	1	3	2	3	1	3	2	3	1	2	1	1
							⋮							

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- Number of entries in row n (beginning with row 0): $2^{n+1} - 1$
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- Sum of entries in row n : 3^n
- Largest entry in row n : F_{n+1} (Fibonacci number)
- Let $\binom{n}{k}$ be the k th entry (beginning with $k = 0$) in row n . Write

$$P_n(x) = \sum_{k \geq 0} \binom{n}{k} x^k.$$

Then $P_{n+1}(x) = (1 + x + x^2)P_n(x^2)$, since $x P_n(x^2)$ corresponds to bringing down the previous row, and $(1 + x^2)P_n(x^2)$ to summing two consecutive entries.

Stern's diatomic sequence

- **Corollary.** $P_n(x) = \prod_{i=0}^{n-1} (1 + x^{2^i} + x^{2 \cdot 2^i})$

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- The sequence b_1, b_2, b_3, \dots is **Stern's diatomic sequence**:

1, 1, 2, 1, 3, 2, 3, 1, 4, 3, 5, 2, 5, 3, 4, 1, ...

(often prefixed with 0)

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1, 1, 2, 1, 3, 2, 3, 1, 4, 3, 5, 2, 5, 3, 4, 1, ...

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- $b_1 = 1, b_{2n} = b_n, b_{2n+1} = b_n + b_{n+1}$

Partition interpretation

$$\sum_{n \geq 0} b_n x^n = \prod_{i \geq 0} (1 + x^{2^i} + x^{2 \cdot 2^i})$$

$\Rightarrow b_n$ is the number of partitions of n into powers of 2, where each power of 2 can appear at most twice.

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Note. If each power of 2 can appear at most once, then we obtain the (unique) binary expansion of n :

$$\frac{1}{1-x} = \prod_{i \geq 0} (1 + x^{2^i}).$$

Historical note

An essentially equivalent array is due to **Moritz Abraham Stern** around 1858 and is known as **Stern's diatomic array**:

1																		1
1								2										1
1				3				2				3						1
1		4		3		5		2		5		3		4				1
1	5	4	7	3	8	5	7	2	7	5	8	3	7	4	5			1
								⋮										

Amazing property

Theorem (Stern, 1858). *Let b_0, b_1, \dots be Stern's diatomic sequence. Then every positive rational number occurs exactly once among the ratios b_i/b_{i+1} , and moreover this expression is in lowest terms.*

Sums of squares

							1												
							1							1					
							1		2				1						1
	1		1		2		1		2		1		1						
1	1	2	1	3	2	3	1	3	2	3	1	2	1	1					
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$$u_2(n) := \sum_k \binom{n}{k}^2 = 1, 3, 13, 59, 269, 1227, \dots$$

$$u_2(n+1) = 5u_2(n) - 2u_2(n-1), \quad n \geq 1$$

$$\sum_{n \geq 0} u_2(n)x^n = \frac{1-2x}{1-5x+2x^2}$$

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Equivalently, if $\prod_{i=0}^{n-1} (1 + x^{2^i} + x^{2 \cdot 2^i}) = \sum a_j x^j$, then

$$\sum a_j^3 = 3 \cdot 7^{n-1}.$$

Proof for $u_2(n)$

$$\begin{aligned}u_2(n+1) &= \dots + \binom{n}{k}^2 + \left(\binom{n}{k} + \binom{n}{k+1} \right)^2 + \binom{n}{k+1}^2 + \dots \\ &= 3u_2(n) + 2 \sum_k \binom{n}{k} \binom{n}{k+1}.\end{aligned}$$

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Thus define $u_{1,1}(n) := \sum_k \binom{n}{k} \binom{n}{k+1}$, so

$$u_2(n+1) = 3u_2(n) + 2u_{1,1}(n).$$

What about $u_{1,1}(n)$?

$$\begin{aligned}u_{1,1}(n+1) &= \cdots + \left(\binom{n}{k-1} + \binom{n}{k} \right) \binom{n}{k} + \binom{n}{k} \left(\binom{n}{k} + \binom{n}{k+1} \right) \\ &\quad + \left(\binom{n}{k} + \binom{n}{k+1} \right) \binom{n}{k+1} + \cdots \\ &= 2u_2(n) + 2u_{1,1}(n)\end{aligned}$$

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Recall also $u_2(n+1) = 3u_2(n) + 2u_{1,1}(n)$.

Two recurrences in two unknowns

Let $\mathbf{A} := \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}$. Then

$$\mathbf{A} \begin{bmatrix} u_2(n) \\ u_{1,1}(n) \end{bmatrix} = \begin{bmatrix} u_2(n+1) \\ u_{1,1}(n+1) \end{bmatrix}.$$

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Also $u_{1,1}(n+1) = 5u_{1,1}(n) - 2u_{1,1}(n-1)$.

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Now we need

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We get

$$\begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} u_3(n) \\ u_{2,1}(n) \end{bmatrix} = \begin{bmatrix} u_3(n+1) \\ u_{2,1}(n+1) \end{bmatrix}.$$

Unexpected eigenvalue

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In fact,

$$\begin{aligned}u_3(n) &= 3 \cdot 7^{n-1} \\u_{2,1}(n) &= 2 \cdot 7^{n-1}.\end{aligned}$$

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Proved by **David Speyer** (2018).

Basic idea of Speyer's proof

Theorem. *The matrix A_r is realized by the operator $\phi: V_r \rightarrow V_r$ defined by*

$$\phi(f)(x, y) = f(x + y, y) + f(x, x + y),$$

where V_r is the space of homogeneous polynomials (over \mathbb{Z}) of degree r in the variables x, y , modulo the subspace generated by all $f(x, y) - f(y, x)$.

General α

$$\alpha = (\alpha_0, \dots, \alpha_{m-1})$$

$$u_\alpha(n) := \sum_k \binom{n}{k}^{\alpha_0} \binom{n}{k+1}^{\alpha_1} \cdots \binom{n}{k+m-1}^{\alpha_{m-1}}$$

A closer look at $\alpha = (1, 1, 1, 1)$

$$u_{1,1,1,1}(n) = \sum_k \binom{n}{k} \binom{n}{k+1} \binom{n}{k+2} \binom{n}{k+3}$$

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$$A_{(1,1,1,1)} = \begin{bmatrix} 3 & 8 & 6 & 0 & 0 & 0 \\ 2 & 5 & 3 & 0 & 0 & 0 \\ 2 & 4 & 2 & 0 & 0 & 0 \\ 1 & 4 & 2 & 1 & 0 & 0 \\ 1 & 3 & 1 & 2 & 1 & 0 \\ 0 & 2 & 2 & 2 & 2 & 0 \end{bmatrix} \begin{matrix} 4 \\ 3,1 \\ 2,2 \\ 1,2,1 \\ 2,1,1 \\ 1,1,1,1 \end{matrix}$$

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$$A_{(1,1,1,1)} = \left[\begin{array}{ccc|cc|c} 3 & 8 & 6 & 0 & 0 & 0 \\ 2 & 5 & 3 & 0 & 0 & 0 \\ 2 & 4 & 2 & 0 & 0 & 0 \\ \hline 1 & 4 & 2 & 1 & 0 & 0 \\ 1 & 3 & 1 & 2 & 1 & 0 \\ \hline 0 & 2 & 2 & 2 & 2 & 0 \end{array} \right] \begin{array}{l} 4 \\ 3,1 \\ 2,2 \\ 1,2,1 \\ 2,1,1 \\ 1,1,1,1 \end{array}$$

Reduction to $\alpha = (r)$

min. polynomial for $\alpha = (4)$: $(x + 1)(2x^2 - 11x + 1)$

min. polynomial for $\alpha = (1, 1, 1, 1)$: $(x - 1)^2(x + 1)(2x^2 - 11x + 1)$

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mp(α): minimum polynomial of A_α

Theorem. Let $\alpha \in \mathbb{N}^m$ and $\sum \alpha_j = r$. Then $\text{mp}(\alpha)$ has the form $x^{w_\alpha}(x - 1)^{z_\alpha} \text{mp}(r)$ for some $w_\alpha, z_\alpha \in \mathbb{N}$.

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No conjecture for value of w_α, z_α .

A generalization

Let $p(x), q(x) \in \mathbb{C}[x]$, $\alpha = (\alpha_0, \dots, \alpha_{m-1}) \in \mathbb{N}^r$, and $b \geq 2$. Set

$$q(x) \prod_{i=0}^{n-1} p(x^{b^i}) = \sum_k \langle n \rangle_{p,q,\alpha,b}^k x^k = \sum_k \langle n \rangle^k x^k$$

and

$$u_{p,q,\alpha,b}(n) = \sum_k \langle n \rangle^{\alpha_0} \langle n \rangle_{k+1}^{\alpha_1} \cdots \langle n \rangle_{k+m-1}^{\alpha_{m-1}}.$$

Main theorem

Theorem. For fixed p, q, α, b , the function $u_{p,q,\alpha,b}(n)$ satisfies a linear recurrence with constant coefficients ($n \gg 0$). Equivalently, $\sum_n u_{p,q,\alpha,b}(n)x^n$ is a rational function of x .

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Note. \exists multivariate generalization.

Some data

$$q(x) = 1, p(x) = 1 + x + x^2, b = 2, \alpha = (r)$$

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$$\frac{p(x)}{1 + x + x^2} \parallel \begin{array}{|l} r = 2 \\ x^2 - 5x + 2 \end{array} \mid \begin{array}{|l} r = 3 \\ x - 7 \end{array} \mid \begin{array}{|l} r = 4 \\ (x + 1)(x^2 - 11x + 2) \end{array}$$

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$1 + 2x + x^2$	$(x - 2)(x - 8)$	$(x - 4)(x - 16)$	$(x - 2)(x - 8)(x - 32)$
$1 + 3x + x^2$	$x^2 - 17x + 54$	$x^2 - 47x + 450$	$x^3 - \dots - 30618$
$1 + 4x + x^2$	$x^2 - 26x + 128$	$x^2 - 94x + 1728$	$x^3 - \dots - 458752$

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Aside. $30618 = 2 \cdot 3^7 \cdot 7$, $458752 = 2^{16} \cdot 7$

An example

Example. Let $p(x) = (1+x)^2$, $q(x) = 1$. Then

$$u_{p,(2),2}(n) = \frac{1}{3} (2 \cdot 2^{3n} + 2^n)$$

$$u_{p,(3),2}(n) = \frac{1}{2} (2^{4n} + 2^{2n})$$

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What's going on?

$$\begin{aligned} p(x)p(x^2)p(x^4)\cdots p(x^{2^{n-1}}) &= \left((1+x)(1+x^2)(1+x^4)\cdots(1+x^{2^{n-1}}) \right)^2 \\ &= \left(1+x+x^2+x^3+\cdots+x^{2^n-1} \right)^2. \end{aligned}$$

The rest of the story

Example. Let

$$(1 + x + x^2 + x^3 + \cdots + x^{2^n-1})^3 = \sum_j a_j x^j.$$

What is $\sum_j a_j^r$?

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$$\begin{aligned}(1 + x + \dots + x^{m-1})^3 &= \left(\frac{1-x^m}{1-x}\right)^3 \\ &= \frac{1 - 3x^m + 3x^{2m} - x^{3m}}{(1-x)^3} \\ &= \sum_{k=0}^{m-1} \binom{k+2}{2} x^k + \sum_{k=m}^{2m-1} \left[\binom{k+2}{2} - 3 \binom{k-m+2}{2} \right] x^k \\ &+ \sum_{k=2m}^{3m-1} \left[\binom{k+2}{2} - 3 \binom{k-m+2}{2} + 3 \binom{k-2m+2}{2} \right] x^k.\end{aligned}$$

The rest of the story (cont.)

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for some polynomial $P(m) \in \mathbb{Q}[m]$.

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So $P(2^n)$ is a \mathbb{Q} -linear combination of terms 2^{jn} , as desired.

Evenness and oddness

Fact. $P(m)$ is either even ($P(m) = P(-m)$) or odd ($P(m) = -P(-m)$) (depending on degree).

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Proof surprisingly tricky. Is there a simple proof?

Modular properties

Sample result for Pascal's triangle:

$$\#\{k : \binom{n}{k} \equiv 1 \pmod{2}\} = 2^{b(n)},$$

where $b(n)$ is the number of 1's in the binary expansion of n (**Lucas**).

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Behavior for Stern's triangle is entirely different!

Rationality

Let $0 \leq a < m$.

$$g_{m,a}(n) = \# \left\{ k : 0 \leq k \leq 2^{n+1} - 2, \binom{n}{k} \equiv a \pmod{m} \right\}.$$

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Theorem (Reznick). $G_{m,a}(x)$ is a rational function.

The cases $m = 2$ and $m = 3$

$$G_{2,0}(x) = \frac{2x^2}{(1-x)(1+x)(1-2x)}$$

$$G_{2,1}(x) = \frac{1+2x}{(1+x)(1-2x)}$$

$$G_{3,0}(x) = \frac{4x^3}{(1-x)(1-2x)(1+x+2x^2)}$$

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... and more ($m = 4$)

$$G_{4,0}(x) = \frac{4x^4}{(1-x)(1+x)(1-2x)(1-x+2x^2)}$$

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$$G_{4,3}(x) = \frac{4x^3}{(1-x)(1+x)(1-2x)}$$

... and even more ($m = 5$)

$$G_{5,0}(x) = \frac{4x^4}{(1-x)(1+x)(1-2x)(1-x+2x^2)}$$

$$G_{5,1}(x) = \frac{1-x^2-x^4-8x^5+5x^6-4x^7-16x^8+8x^9-32x^{10}-32x^{11}}{(1-x)(1+x)(1-2x)(1+x^2)(1-x+2x^2)(1-x^2+4x^4)}$$

$$G_{5,2}(x) = \frac{2x^2+8x^5+2x^6-4x^7+12x^8-16x^{10}}{(1+x)(1-2x)(1+x^2)(1-x+2x^2)(1-x^2+4x^4)}$$

$$G_{5,3}(x) = \frac{4x^3+4x^5+4x^6+12x^7-4x^8+16x^{10}}{(1+x)(1-2x)(1+x^2)(1-x+2x^2)(1-x^2+4x^4)}$$

$$G_{5,4}(x) = \frac{4x^4-4x^5+8x^6+8x^7+8x^8+16x^{10}+32x^{11}}{(1+x)(1-2x)(1+x^2)(1-x+2x^2)(1-x^2+4x^4)}$$

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E.g., $(m, a) =$

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 $(6, 0), (6, 3), (7, 0), (8, 4), (8, 6), (9, 0), (10, 5), (11, 0)$ (but **not** $(8, 0)$).
- Why are so many numerator coefficients a power of 2? (In all computed cases, coefficients are $\pm 2^k$ or $\pm 3 \cdot 2^k$.)

$g_{m,a,b}(n)$

For the proof of rationality, need to define

$$g_{m,a,b}(n) = \# \left\{ k : \binom{n}{k} \equiv a \pmod{m} \binom{n}{k+1} \equiv b \pmod{m} \right\},$$

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with $\binom{n}{-1} = 0$ and $\binom{n}{2^n-1} = 0$.

$$\gcd \left(\binom{n}{k}, \binom{n}{k+1} \right) = 1$$

$$\Rightarrow g_{m,a,b} = 0 \text{ unless } \mathbb{Z}/m\mathbb{Z} = \langle a, b \rangle$$

A recurrence

Let $\langle a, b \rangle = \mathbb{Z}/m\mathbb{Z}$. How to get

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Answer. Either

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Hence

$$g_{m,a,b}(n+1) = g_{m,a,b-a}(n) + g_{m,a-b,b}(n),$$

where we take $b - a$ and $a - b$ modulo m .

Example: $n = 3$

Write $\mathbf{g}_{ab} = \mathbf{g}_{3,a,b}$.

$$\begin{bmatrix} g_{01}(n+1) \\ g_{02}(n+1) \\ g_{11}(n+1) \\ g_{12}(n+1) \\ g_{22}(n+1) \\ g_{10}(n+1) \\ g_{20}(n+1) \\ g_{21}(n+1) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} g_{01}(n) \\ g_{02}(n) \\ g_{11}(n) \\ g_{12}(n) \\ g_{22}(n) \\ g_{10}(n) \\ g_{20}(n) \\ g_{21}(n) \end{bmatrix}$$
$$= \mathbf{A}_3 \mathbf{v}$$

Characteristic polynomial

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Explains why $G_{3,a}$ has denominator $(1 - x)(1 - 2x)(1 + x + 2x^2)$.

Size of A_m

Size (number of rows and number of columns) of A_m is the number $\nu(m)$ of pairs $(a, b) \in \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$ that generate $\mathbb{Z}/m\mathbb{Z}$.

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Theorem (nice exercise).

$$\begin{aligned}\nu(m) &= m^2 \prod_{p|m} \frac{(p-1)(p+1)}{p^2} \\ &= \phi(m)\psi(m),\end{aligned}$$

where ϕ is the Euler phi function and ψ is the **Dedekind psi function**.

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$$\phi(m) = m \prod_{p|m} \frac{p-1}{p}$$

$$\psi(m) = m \prod_{p|m} \frac{p+1}{p}$$

The final slide

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THE END