$(2n-1)!!$

April 14, 2020

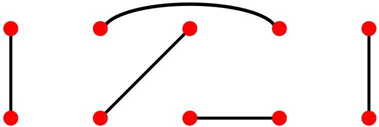
Semifactorials

$$\mathbf{(2n-1)!!} = 1 \cdot 3 \cdot 5 \cdots (2n-1) = \frac{(2n)!}{2^n n!},$$

called $2n-1$ **double factorial** (bad?) or **semifactorial**

Matchings

(complete) **matching** on $2n$-element set:



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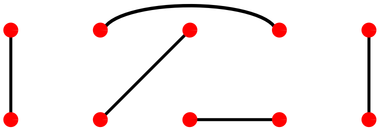

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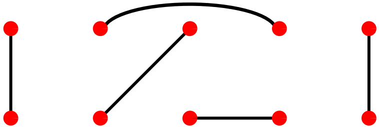

Theorem. *The number of matchings on* $[2n]$ *is* $(2n-1)!!$.

Proof. Pick $i \in [2n]$ and match it in $2n-1$ ways. Then pick some unmatched element j and match it in $(2n-3)$ ways, etc. $\square$

Schröder's third problem

Ernst Schröder, *Vier kombinatorische Probleme*, 1870

Problem 3 (**complete binary partitions**). How many ways to partition an n-set ($n > 1$) into two nonempty blocks, then partition each nonsingleton block into two nonempty blocks, etc., until only singletons remain?

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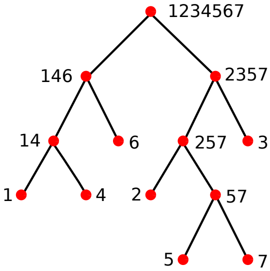

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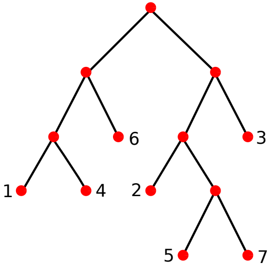
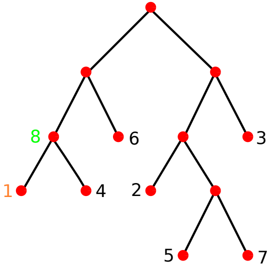


leaf labelled (unordered) binary tree

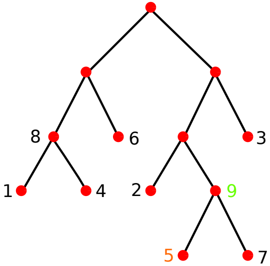
Bijection with matchings

Label by $n+1$ the unlabelled vertex with two labelled children, with the least possible label of a child.



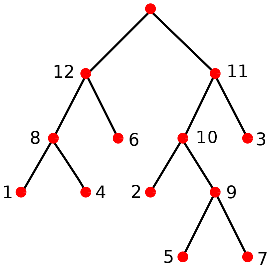
Bijection with matchings

Label by $n+2$ the unlabelled vertex with two labelled children, with the least possible label of a child.



Bijection with matchings

Continue until all nonroot vertices are labelled $1, 2, \ldots, 2n - 2$.



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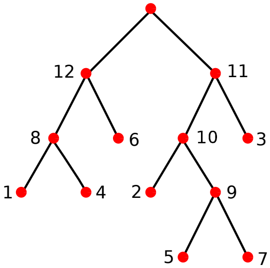

Now match the two children of any nonleaf vertex: 5,7 – 2,9 – 3,10 – 1,4 – 6,8 – 11,12.

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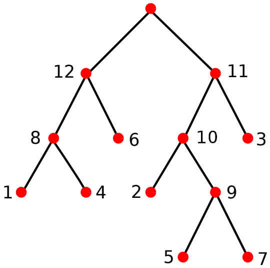

Now match the two children of any nonleaf vertex: 5,7 – 2,9 – 3,10 – 1,4 – 6,8 – 11,12.

Theorem. *The number of leaf-labelled binary trees with n leaves is $(2n-3)!!$.*

Inkling of probability theory

Theorem.

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^{2n} e^{-\frac{1}{2}x^2} dx = (2n-1)!!$$

the $(2n)$th moment of the standard normal distribution.

An $\mathfrak{S}_{2n}$ action

$\mathcal{M}_n$: set of all matchings on $[2n]$, so $\#\mathcal{M}_n = (2n-1)!!$

$\mathfrak{S}_{2n}$ acts of $\mathcal{M}_n$ by permuting vertices. What is this action? I.e., what is the multiplicity of each irreducible character χ^λ, $\lambda \vdash 2n$?

The subgroup $\mathfrak{S}_2^n$

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$N(\mathfrak{S}_2^n)$: the **normalizer** of $\mathfrak{S}_2^n$, i.e., all $w \in \mathfrak{S}_{2n}$ such that

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$N(\mathfrak{S}_2^n)$ consists of all $w \in \mathfrak{S}_{2n}$ that permute the elements in each row and permute the rows among themselves of the array ($n = 5$)

1	2
3	4
5	6
7	8
9	10

Action on cosets

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The action on $\mathfrak{S}_{2n}$ on the left cosets of $N(\mathfrak{S}_2^n)$ is isomorphic to the action of $\mathfrak{S}_{2n}$ on $\mathcal{M}_n$. Thus, as $\mathfrak{S}_{2n}$-modules,

$$\mathcal{M}_n \cong \uparrow_{N(\mathfrak{S}_2^n)}^{\mathfrak{S}_{2n}} 1.$$

Plethysm

Let **ch** denote the Frobenius characteristic symmetric function of an $\mathfrak{S}_m$ action. By the theory of plethysm,

$$\mathrm{ch}\,\mathcal{M}_n = (\mathrm{ch}\,1_{\mathfrak{S}_n})[\mathrm{ch}\,1_{\mathfrak{S}_2}] = h_n[h_2].$$

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Theorem. *Let $\lambda \vdash 2n$. The multiplicity of χ^λ in the action of $\mathfrak{S}_{2n}$ on $\mathcal{M}_n$ is 1 if $\lambda = 2\mu$, and 0 otherwise.*

Zonal polynomials

$\boldsymbol{H_n} = N(\mathfrak{S}_2^n)$ (**hyperoctahedral group**)

Because $\mathcal{M}_n$ is **multiplicity-free** as an $\mathfrak{S}_{2n}$-module, the pair $(\mathfrak{S}_{2n}, H_n)$ is a **Gelfand pair**.

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Let $\lambda \vdash n$ and $\chi^{2\lambda}$ be the irreducible character of $\mathfrak{S}_{2n}$ indexed by 2λ. Let $\boldsymbol{s} \in \mathfrak{S}_{2n}$ of cycle type $\rho \vdash 2n$.

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Define the **zonal polynomial**

$$Z_\lambda = 2^n n! \sum_{\rho \vdash n} z_{2\rho}^{-1} \omega_\rho^\lambda p_\rho,$$

a homogeneous symmetric function of degree n.

Some properties of zonal polynomials

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$$\prod_{u\in\lambda}(2a(u)+l(u)+1).$$
- $Z_\lambda = J_\lambda^{(2)}$, where J_λ^α $(\alpha\in\mathbb{R})$ is a **Jack symmetric function** (a limiting case of **Macdonald polynomials**)

The Brauer algebra

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Let $\dim_{\mathbb{C}} V = k$. The general linear group $\mathrm{GL}(V)$ acts diagonally on $V^{\otimes n}$. The linear transformations $V^{\otimes n} \to V^{\otimes n}$ commuting with this action are generated by the $n!$ permutations of tensor coordinates. For $k \geq n$ these linear transformations form the algebra $\mathbb{C}[\mathfrak{S}_n]$ (the group algebra of $\mathfrak{S}_n$).

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Let $\dim_{\mathbb{C}} V = k$. The orthogonal group $O(V)$ (i.e., $A(A^*)^t = I$) acts diagonally on $V^{\otimes n}$. For $k \geq n$, the linear transformations $V^{\otimes n} \to V^{\otimes n}$ commuting with this action form an algebra $\mathfrak{B}_n$ of dimension $(2n-1)!!$ (the **Brauer algebra**).

Brauer algebra multiplication

Let z be a parameter. Take $\mathcal{M}_n$ as a basis for an algebra $\mathfrak{B}_n(z)$, where $\mathfrak{B}_n(1) = \mathfrak{B}_n$ (not semisimple). For "generic" z (e.g., $z \notin \mathbb{Z}$), $\mathfrak{B}_n(z)$ is semisimple.

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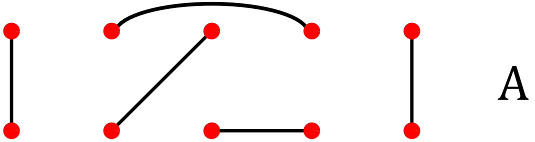

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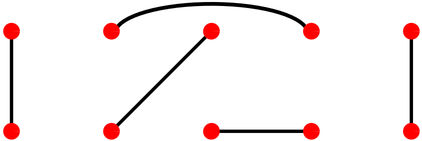

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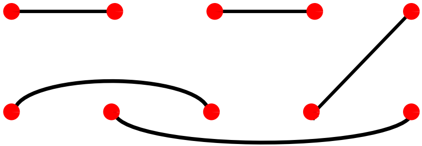

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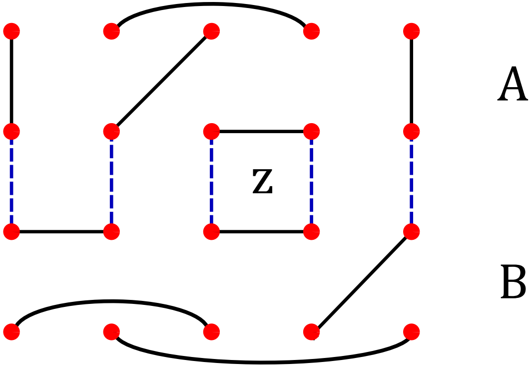

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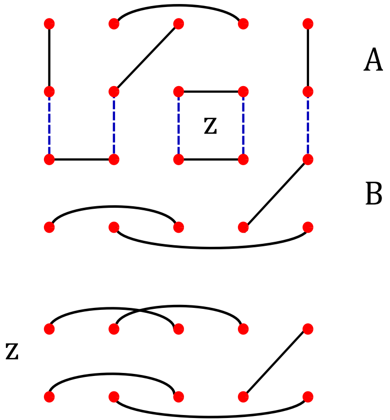

Oscillating tableaux

An **oscillating tableau** T of shape λ and length n is a sequence

$$\emptyset = \lambda^0, \lambda^1, \dots, \lambda^m = \lambda$$

of partitions (identified with their Young diagrams) such that λ^i is obtained from λ^{i-1} by adding a box or removing a box.

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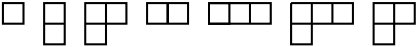
Example. Shape $\lambda = (2,1)$, length $n = 7$:

$\emptyset$ 

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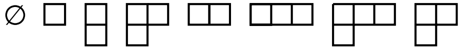

$o^{\lambda,n}$: number of oscillating tableau of shape λ and length n

Dimension of $\mathfrak{B}_n$ irreps

Theorem. *Fix $n \geq 1$. Irreps of $\mathfrak{B}_n(z)$ (z generic) are indexed by partitions $\lambda \vdash m$, where $m \leq n$, $n \equiv m \pmod 2$. The dimension of the irrep indexed by such λ is $o^{\lambda,n}$.*

Corollary. $\sum_\lambda (o^{\lambda,n})^2 = (2n-1)!!$

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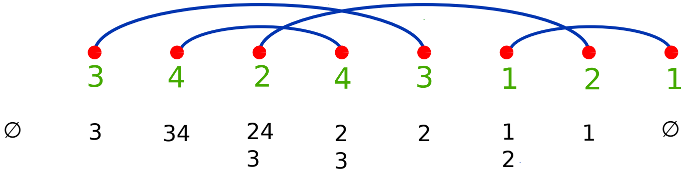
First combinatorial proof (bijection with $\mathcal{M}_n$) by **RS** and **S. Sundaram**.

Sundaram's bijection



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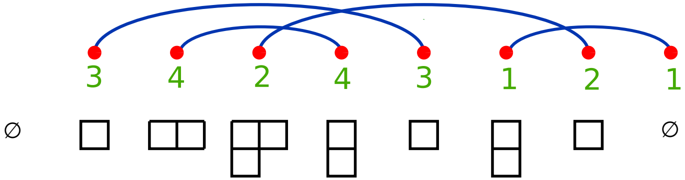
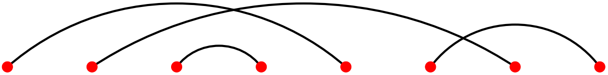


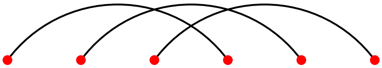
Crossings and nestings



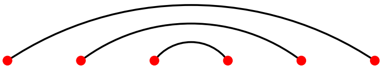
crossing:

nesting:

k-crossings and k-nestings



3–crossing



3–nesting

$$\begin{aligned} M &= \text{matching} \\ \mathbf{cr}(M) &= \max\{k : \exists\ k\text{-crossing}\} \\ \mathbf{ne}(M) &= \max\{k : \exists\ k\text{-nesting}\}. \end{aligned}$$

Some consequences

Theorem (**Bill Yongchuan Chen** (陈永川), **Eva Yuping Deng** (邓玉平), **Rosena Ruoxia Du** (杜若霞), **Catherine Huafei Yan** (颜华菲), **RS**) *Let* $M \mapsto (\emptyset = T_0, T_1, \ldots, T_{2n} = \emptyset)$ *in the bijection from matchings to oscillating tableau of shape* $\emptyset$. *Then* $\mathrm{cr}(M)$ *is equal to the most number of rows of any* T_i, *and* $\mathrm{ne}(M)$ *is equal to the most number of columns of any* T_i.

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Theorem (**Bill Yongchuan Chen** (陈永川), **Eva Yuping Deng** (邓玉平), **Rosena Ruoxia Du** (杜若霞), **Catherine Huafei Yan** (颜华菲), **RS**) *Let* $M \mapsto (\emptyset = T_0, T_1, \dots, T_{2n} = \emptyset)$ *in the bijection from matchings to oscillating tableau of shape* $\emptyset$. *Then* $\mathrm{cr}(M)$ *is equal to the most number of rows of any* T_i, *and* $\mathrm{ne}(M)$ *is equal to the most number of columns of any* T_i.

Corollary. *Let* $f_n(i,j) = \#$ *matchings* M *on* $[2n]$ *with* $\mathrm{cr}(M) = i$ *and* $\mathrm{ne}(M) = j$. *Then* $\boldsymbol{f_n(i,j) = f_n(j,i)}$.

Corollary. $\#$ *matchings* M *on* $[2n]$ *with* $\mathrm{cr}(M) = k$ *equals* $\#$ *matchings* M *on* $[2n]$ *with* $\mathrm{ne}(M) = k$.

An enumerative consequence

Theorem (**Grabiner-Magyar**, essentially) *Let* $f_k(n)$ *be the number of matchings* $M \in \mathcal{M}_n$ *satisfying* $\mathrm{cr}(M) \leq k$. *Define*

$$F_k(x) = \sum_n f_k(n) \frac{x^{2n}}{(2n)!}.$$

Then

$$F_k(x) = \det \left[I_{|i-j|}(2x) - I_{i+j}(2x) \right]_{i,j=1}^{k}$$

where

$$I_m(2x) = \sum_{j \geq 0} \frac{x^{m+2j}}{j!(m+j)!}$$

*(***hyperbolic Bessel function*** of the first kind of order m).*

A probabilistic consequence

Note. $\mathrm{cr}(M)$ is the matching analogue of the length of the longest increasing subsequence of $w \in \mathfrak{S}_n$, and $\mathrm{ne}(M)$ is the analogue of the length of the longest decreasing subsequence.

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$$u''(x) = 2u(x)^3 + xu(x).$$

Tracy-Widom distribution:

$$\boldsymbol{F(t)} = \exp\left(-\int_t^\infty (x-t)u(x)^2\,dx\right)$$

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Theorem.

$$\lim_{n\to\infty} \mathrm{Prob}\left(\frac{\mathrm{cr}_n(M)-\sqrt{2n}}{(2n)^{1/6}} \leq \frac{t}{2}\right) = F(t)^{1/2}\exp\left(\frac{1}{2}\int_t^\infty u(s)ds\right)$$

The final slide

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Hope you enjoyed the lectures!

Thanks for listening!