



Alternating Permutations

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M.I.T.

Basic definitions

A sequence a_1, a_2, \dots, a_k of distinct integers is **alternating** if

$$a_1 > a_2 < a_3 > a_4 < \dots ,$$

and **reverse alternating** if

$$a_1 < a_2 > a_3 < a_4 > \dots .$$

Euler numbers

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$$\begin{aligned} E_n &= \#\{w \in \mathfrak{S}_n : w \text{ is alternating}\} \\ &= \#\{w \in \mathfrak{S}_n : w \text{ is reverse alternating}\} \end{aligned}$$

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E.g., $E_4 = 5$: 2143, 3142, 3241, 4132, 4231

André's theorem

Theorem (Désiré André, 1879)

$$y := \sum_{n \geq 0} E_n \frac{x^n}{n!} = \sec x + \tan x$$

$$\begin{aligned} &= 1 + 1x + 1 \frac{x^2}{2!} + 2 \frac{x^3}{3!} + 5 \frac{x^4}{4!} + 16 \frac{x^5}{5!} \\ &\quad + 61 \frac{x^6}{6!} + 272 \frac{x^7}{7!} + \dots \end{aligned}$$

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E_{2n} is a **secant number**.

E_{2n+1} is a **tangent number**.

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Choose a reverse alternating permutation $v = b_1 b_2 \cdots b_{n-k}$ of $[n] - S$.

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Let $w = a_k \cdots a_2 a_1, n + 1, b_1 b_2 \cdots b_{n-k}$.

Proof (continued)

$$w = a_k \cdots a_2 a_1, n + 1, b_1 b_2 \cdots b_{n-k}$$

Given k , there are:

- $\binom{n}{k}$ choices for $\{a_1, a_2, \dots, a_k\}$
- E_k choices for $a_1 a_2 \cdots a_k$
- E_{n-k} choices for $b_1 b_2 \cdots b_{n-k}$.

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We obtain each alternating and reverse alternating $w \in \mathfrak{S}_{n+1}$ once each.

Completion of proof

$$\Rightarrow 2E_{n+1} = \sum_{k=0}^n \binom{n}{k} E_k E_{n-k}, \quad n \geq 1$$

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$$2y' = 1 + y^2, \quad y(0) = 1.$$

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Define

$$\tan x = \sum_{n \geq 0} E_{2n+1} \frac{x^{2n+1}}{(2n+1)!}$$

$$\sec x = \sum_{n \geq 0} E_{2n} \frac{x^{2n}}{(2n)!}$$

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⇒ **combinatorial trigonometry**

Example of combinatorial trig.

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Equate coefficients of $x^{2n}/(2n)!$:

$$\sum_{k=0}^n \binom{2n}{2k} E_{2k} E_{2(n-k)} = \sum_{k=0}^{n-1} \binom{2n}{2k+1} E_{2k+1} E_{2(n-k)-1}.$$

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$$\mathcal{A}_n \text{ (resp. } \mathcal{B}_n) = \{(u, v) \in \mathfrak{S}_{2n}^2 :$$

alt. perms. on $S, [2n] - S, \#S \text{ even (resp. odd)}\}$

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Exercise: bijection $\varphi: \mathcal{A}_n \rightarrow \mathcal{B}_n$

Another exercise

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - (\tan x)(\tan y)}$$

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From Greek *boustrophēdon* (*βουστροφηδόν*), turning like an ox while plowing: *bous*, ox + *strophē*, a turning (from *strephein*, to turn)

The boustrophedon array

1
0 → 1
1 ← 1 ← 0
0 → 1 → 2 → 2
5 ← 5 ← 4 ← 2 ← 0
0 → 5 → 10 → 14 → 16 → 16
61 ← 61 ← 56 ← 46 ← 32 ← 16 ← 0.
...

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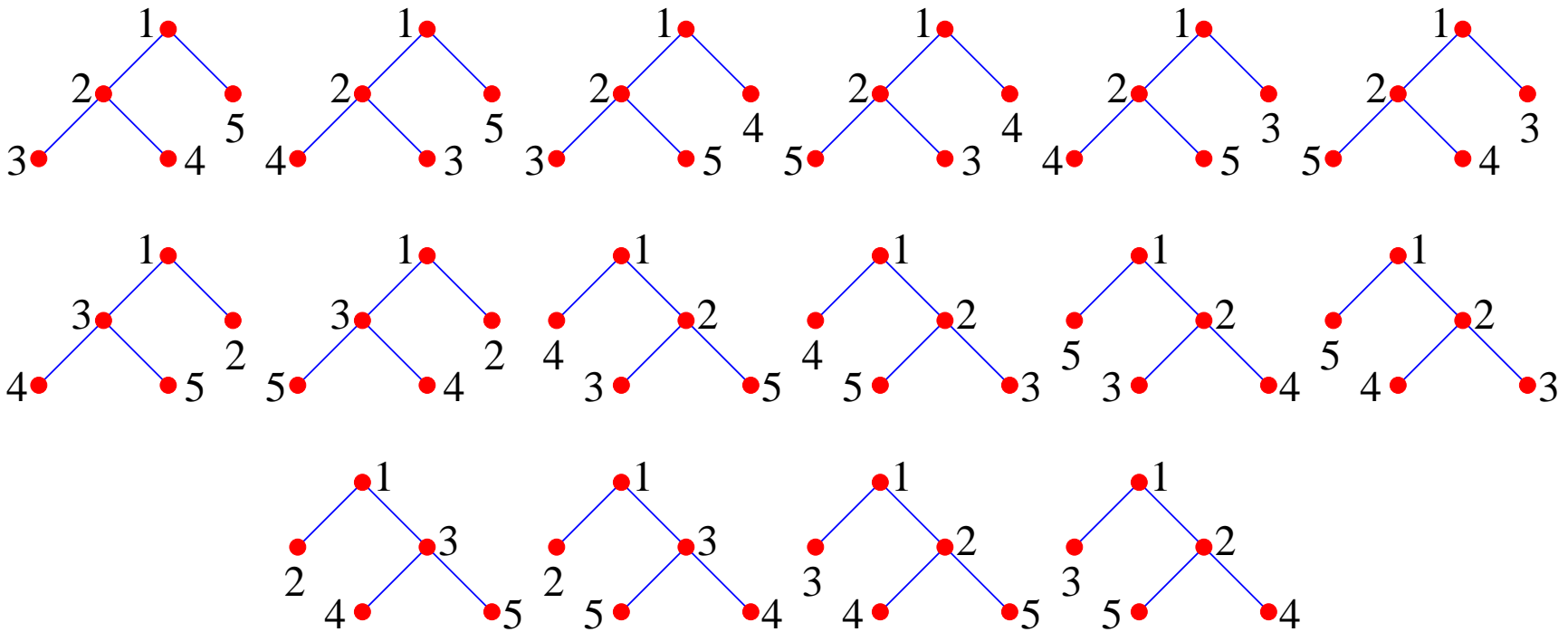
Boustrophedon entries

- last term in row n : E_{n-1}
- sum of terms in row n : E_n
- k th term in row n : number of alternating permutations in \mathfrak{S}_n with first term k

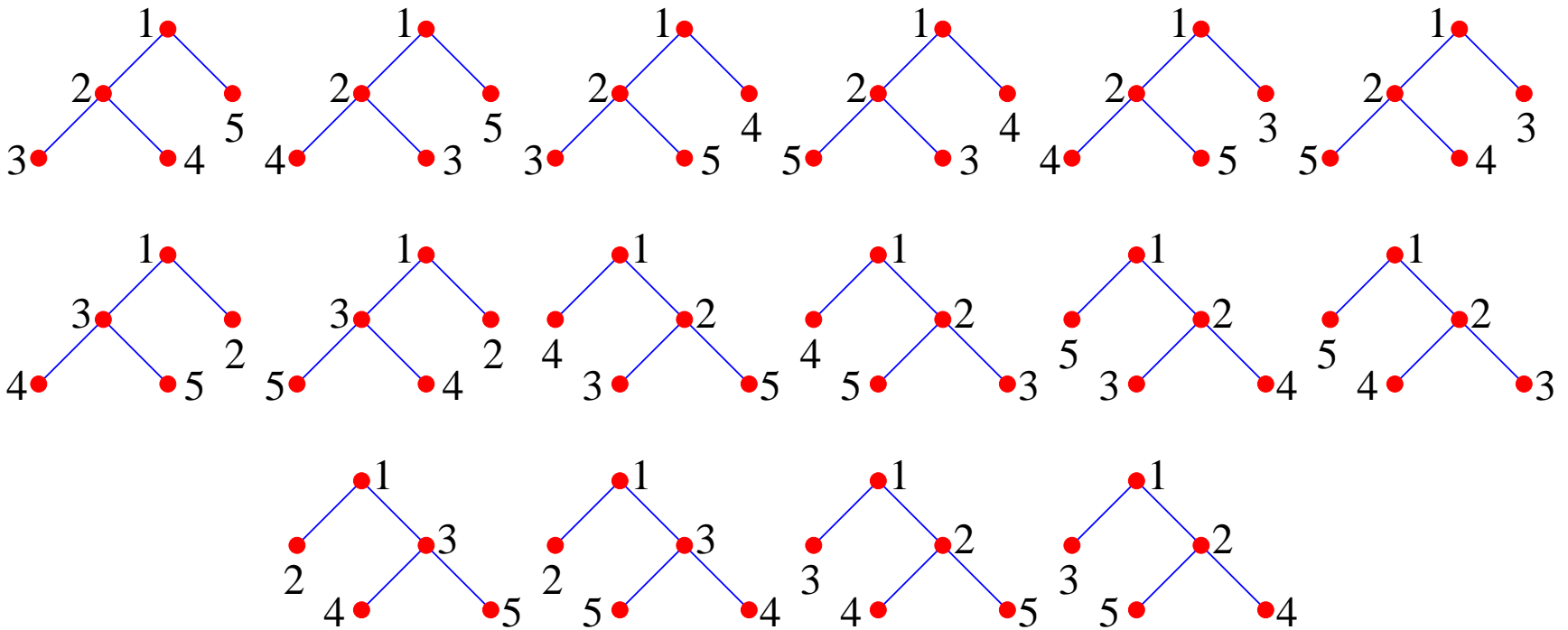
Some occurrences of Euler numbers

(1) E_{2n-1} is the number of complete increasing binary trees on the vertex set $[2n + 1] = \{1, 2, \dots, 2n + 1\}$.

Five vertices



Five vertices



Slightly more complicated for E_{2n}

Proof for $2n + 1$

$b_1 b_2 \cdots b_m$: sequence of distinct integers

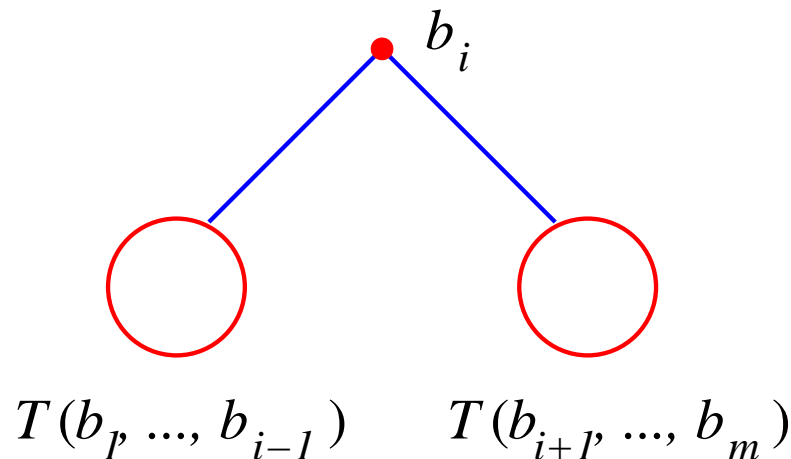
$$b_i = \min\{b_1, \dots, b_m\}$$

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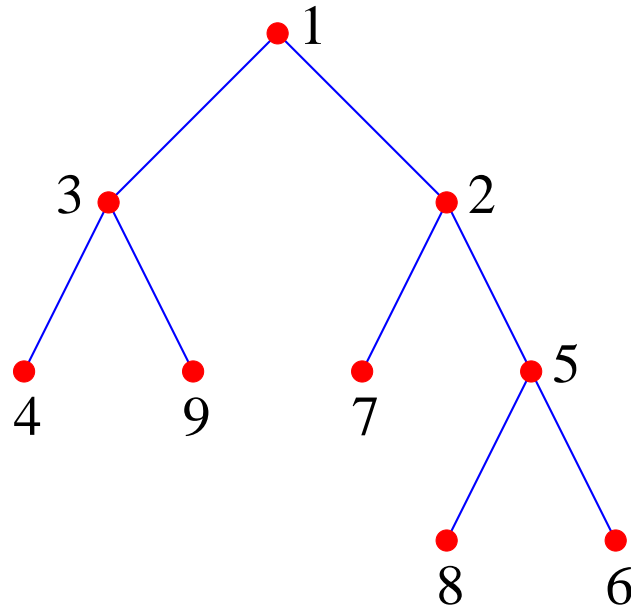
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Define recursively a binary tree $T(b_1, \dots, b_m)$ by



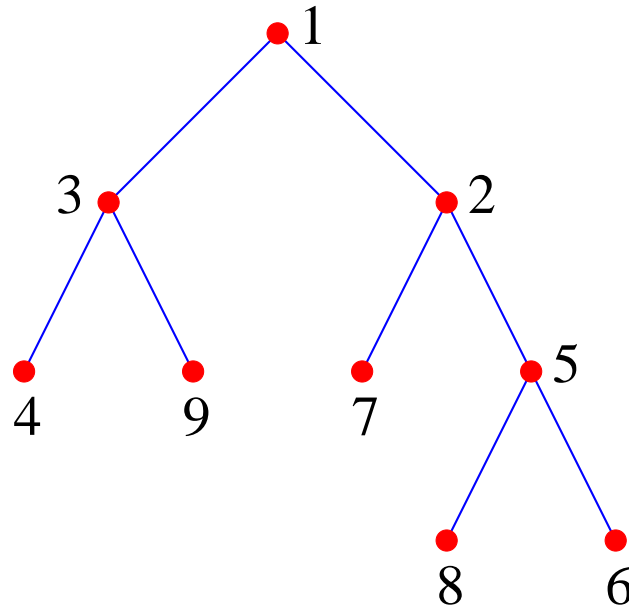
Completion of proof

Example. 439172856



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Let $w \in \mathfrak{S}_{2n+1}$. Then $T(w)$ is complete if and only if w is alternating, and the map $w \mapsto T(w)$ gives the desired bijection.

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$1-2-3-4-5-6, 12-3-4-5-6, 12-34-5-6$
 $125-34-6, 125-346, 123456$

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Proof. Exercise.

Orbit representatives for $n = 5$

12-3-4-5

123-4-5

1234-5

12-3-4-5

123-4-5

123-45

12-3-4-5

12-34-5

125-34

12-3-4-5

12-34-5

12-345

12-3-4-5

12-34-5

1234-5

Volume of a polytope

(3) Let \mathcal{E}_n be the convex polytope in \mathbb{R}^n defined by

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Theorem. *The volume of \mathcal{E}_n is $E_n/n!$.*

Naive proof

$$\text{vol}(\mathcal{E}_n) = \int_{x_1=0}^1 \int_{x_2=0}^{1-x_1} \int_{x_3=0}^{1-x_2} \cdots \int_{x_n=0}^{1-x_{n-1}} dx_1 dx_2 \cdots dx_n$$

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$$f_n(t) := \int_{x_1=0}^t \int_{x_2=0}^{1-x_1} \int_{x_3=0}^{1-x_2} \cdots \int_{x_n=0}^{1-x_{n-1}} dx_1 dx_2 \cdots dx_n$$

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$$f'_n(t) = \int_{x_2=0}^{1-t} \int_{x_3=0}^{1-x_2} \cdots \int_{x_n=0}^{1-x_{n-1}} dx_1 dx_2 \cdots dx_n$$

$$= f_{n-1}(1-t).$$

$F(y)$

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$$\mathbf{F(y)} = \sum_{n \geq 0} f_n(t) y^n$$

$$\Rightarrow \frac{\partial^2}{\partial t^2} F(y) = -y^2 F(y),$$

etc.

Conclusion of proof

$$F(y) = (\sec y)(\cos(t - 1)y + \sin ty)$$

$$\Rightarrow F(y)|_{t=1} = \sec y + \tan y.$$

Tridiagonal matrices

An $n \times n$ matrix $M = (m_{ij})$ is **tridiagonal** if $m_{ij} = 0$ whenever $|i - j| \geq 2$.

doubly-stochastic: $m_{ij} \geq 0$, row and column sums equal 1

\mathcal{T}_n : set of $n \times n$ tridiagonal doubly stochastic matrices

Polytope structure of \mathcal{T}_n

Easy fact: the map

$$\begin{aligned}\mathcal{T}_n &\rightarrow \mathbb{R}^{n-1} \\ M &\mapsto (m_{12}, m_{23}, \dots, m_{n-1,n})\end{aligned}$$

is a (linear) bijection from \mathcal{T} to \mathcal{E}_n .

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Application (**Diaconis** et al.): random doubly stochastic tridiagonal matrices and random walks on \mathcal{T}_n

A modification

Let \mathcal{F}_n be the convex polytope in \mathbb{R}^n defined by

$$\begin{aligned}x_i &\geq 0, & 1 \leq i \leq n \\x_i + x_{i+1} + x_{i+2} &\leq 1, & 1 \leq i \leq n - 2.\end{aligned}$$

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$$V_n = \text{vol}(\mathcal{F}_n)$$

n	1-3	4	5	6	7	8	9	10
$n!V_n$	1	2	5	14	47	182	786	3774

A “naive” recurrence

$$V_n = \mathbf{f}_n(1, 1),$$

where

$$f_0(a, b) = 1, \quad f_n(0, b) = 0 \text{ for } n > 0$$

$$\frac{\partial}{\partial a} f_n(a, b) = f_{n-1}(b - a, 1 - a).$$

$f_n(a, b)$ for $n \leq 3$

$$f_1(a, b) = a$$

$$f_2(a, b) = \frac{1}{2}(2ab - a^2)$$

$$f_3(a, b) = \frac{1}{6}(a^3 - 3a^2 - 3ab^2 + 6ab)$$

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Is there a “nice” generating function for $f_n(a, b)$ or $V_n = f_n(1, 1)$?

Distribution of $\text{is}(w)$

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Vershik-Kerov, Logan-Shepp:

$$\begin{aligned} E(n) &:= \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \text{is}(w) \\ &\sim 2\sqrt{n} \end{aligned}$$

Limiting distribution of $\text{is}(w)$

Baik-Deift-Johansson:

For fixed $t \in \mathbb{R}$,

$$\lim_{n \rightarrow \infty} \text{Prob} \left(\frac{\text{is}_n(w) - 2\sqrt{n}}{n^{1/6}} \leq t \right) = F(t),$$

the **Tracy-Widom distribution**.

Alternating analogues

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The first is much easier!

Longest alternating subsequences

$as(w)$ = length longest alt. subseq. of w

$$D(n) = \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} as(w) \sim ?$$

$$w = \mathbf{56218347} \Rightarrow as(w) = 5$$

Definitions of $a_k(n)$ and $b_k(n)$

$$a_k(n) = \#\{w \in \mathfrak{S}_n : \text{as}(w) = k\}$$

$$b_k(n) = a_1(n) + a_2(n) + \cdots + a_k(n)$$

$$= \#\{w \in \mathfrak{S}_n : \text{as}(w) \leq k\}$$

The case $n = 3$

w	$as(w)$
123	1
132	2
213	3
231	2
312	3
321	2

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$$a_1(3) = 1, a_2(3) = 3, a_3(3) = 2$$

$$b_1(3) = 1, b_2(3) = 4, b_3(3) = 6$$

The main lemma

Lemma. $\forall w \in \mathfrak{S}_n \exists$ *alternating subsequence of maximal length that contains n .*

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Corollary.

$$\Rightarrow a_k(n) = \sum_{j=1}^n \binom{n-1}{j-1}$$

$$\sum_{2r+s=k-1} (a_{2r}(j-1) + a_{2r+1}(j-1)) a_s(n-j)$$

The main generating function

$$B(x, t) = \sum_{k, n \geq 0} b_k(n) t^k \frac{x^n}{n!}$$

Theorem.

$$B(x, t) = \frac{2/\rho}{1 - \frac{1-\rho}{t} e^{\rho x}} - \frac{1}{\rho},$$

where $\rho = \sqrt{1 - t^2}$.

Formulas for $b_k(n)$

Corollary.

$$\Rightarrow b_1(n) = 1$$

$$b_2(n) = n$$

$$b_3(n) = \frac{1}{4}(3^n - 2n + 3)$$

$$b_4(n) = \frac{1}{8}(4^n - (2n - 4)2^n)$$

⋮

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⋮

no such formulas for longest **increasing** subsequences

Mean (expectation) of $\text{as}(w)$

$$D(n) = \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \text{as}(w) = \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} k \cdot a_k(n),$$

the **expectation** of $\text{as}(n)$ for $w \in \mathfrak{S}_n$

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the **expectation** of $as(n)$ for $w \in \mathfrak{S}_n$

Let

$$\begin{aligned} A(x, t) &= \sum_{k, n \geq 0} a_k(n) t^k \frac{x^n}{n!} = (1-t)B(x, t) \\ &= (1-t) \left(\frac{2/\rho}{1 - \frac{1-\rho}{t} e^{\rho x}} - \frac{1}{\rho} \right). \end{aligned}$$

Formula for $D(n)$

$$\begin{aligned}\sum_{n \geq 0} D(n)x^n &= \frac{\partial}{\partial t} A(x, 1) \\ &= \frac{6x - 3x^2 + x^3}{6(1-x)^2} \\ &= x + \sum_{n \geq 2} \frac{4n+1}{6} x^n.\end{aligned}$$

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$$\Rightarrow D(n) = \frac{4n+1}{6}, \quad n \geq 2$$

Formula for $D(n)$

$$\begin{aligned}\sum_{n \geq 0} D(n)x^n &= \frac{\partial}{\partial t} A(x, 1) \\ &= \frac{6x - 3x^2 + x^3}{6(1-x)^2} \\ &= x + \sum_{n \geq 2} \frac{4n+1}{6} x^n.\end{aligned}$$

$$\Rightarrow D(n) = \frac{4n+1}{6}, \quad n \geq 2$$

Compare $E(n) \sim 2\sqrt{n}$.

Variance of $as(w)$

$$V(n) = \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \left(as(w) - \frac{4n+1}{6} \right)^2, \quad n \geq 2$$

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similar results for higher moments

A new distribution?

$$P(t) = \lim_{n \rightarrow \infty} \text{Prob}_{w \in \mathfrak{S}_n} \left(\frac{\text{as}(w) - 2n/3}{\sqrt{n}} \leq t \right)$$

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Stanley distribution?

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Theorem (Pemantle, Widom, (Wilf)).

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Umbral enumeration

Umbral formula: involves E^k , where E is an indeterminate (the **umbra**). Replace E^k with the Euler number E_k . (Technique from 19th century, modernized by **Rota** et al.)

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Example.

$$\begin{aligned}(1 + E^2)^3 &= 1 + 3E^2 + 3E^4 + E^6 \\ &= 1 + 3E_2 + 3E_4 + E_6 \\ &= 1 + 3 \cdot 1 + 3 \cdot 5 + 61 \\ &= 80\end{aligned}$$

Another example

$$\begin{aligned}(1+t)^E &= 1 + Et + \binom{E}{2}t^2 + \binom{E}{3}t^3 + \dots \\ &= 1 + Et + \frac{1}{2}E(E-1)t^2 + \dots \\ &= 1 + E_1t + \frac{1}{2}(E_2 - E_1)t^2 + \dots \\ &= 1 + t + \frac{1}{2}(1-1)t^2 + \dots \\ &= 1 + t + O(t^3).\end{aligned}$$

Alt. fixed-point free involutions

fixed point free involution $w \in \mathfrak{S}_{2n}$: all cycles of length two

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$$n = 3 : \quad 214365 = (1, 2)(3, 4)(5, 6)$$

$$645231 = (1, 6)(2, 4)(3, 5)$$

$$f(3) = 2$$

An umbral theorem

Theorem.

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An umbral theorem

Theorem.

$$\begin{aligned} F(x) &= \sum_{n \geq 0} f(n)x^n \\ &= \left(\frac{1+x}{1-x} \right)^{(E^2+1)/4} \end{aligned}$$

Proof idea

Proof. Uses representation theory of the symmetric group \mathfrak{S}_n .

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$$\chi(w) = 0 \text{ or } \pm E_k.$$

Now use known results on combinatorial properties of characters of \mathfrak{S}_n .

Ramanujan's Second Notebook

Theorem (Ramanujan, Berndt, implicitly) As $x \rightarrow 0+$,

$$2 \sum_{n \geq 0} \left(\frac{1-x}{1+x} \right)^{n(n+1)} \sim \sum_{k \geq 0} f(k) x^k = F(x),$$

an **analytic** (non-formal) identity.

A formal identity

Corollary (via Ramanujan, Andrews).

$$F(x) = 2 \sum_{n \geq 0} q^n \frac{\prod_{j=1}^n (1 - q^{2j-1})}{\prod_{j=1}^{2n+1} (1 + q^j)},$$

where $q = \left(\frac{1-x}{1+x}\right)^{2/3}$, a **formal identity**.

Simple result, hard proof

Recall: number of n -cycles in \mathfrak{S}_n is $(n - 1)!$.

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Theorem. Let $b(n)$ be the number of **alternating** n -cycles in \mathfrak{S}_n . Then if n is odd,

$$b(n) = \frac{1}{n} \sum_{d|n} \mu(d) (-1)^{(d-1)/2} E_{n/d}.$$

Special case

Corollary. *Let p be an odd prime. Then*

$$b(p) = \frac{1}{p} \left(E_p - (-1)^{(p-1)/2} \right).$$

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Combinatorial proof?

Inc. subsequences of alt. perms.

Recall: $\text{is}(w)$ = length of longest increasing subsequence of $w \in \mathfrak{S}_n$. Define

$$C(n) = \frac{1}{E_n} \sum_w \text{is}(w),$$

where w ranges over all E_n alternating permutations in \mathfrak{S}_n .

β

Little is known, e.g., what is

$$\beta = \lim_{n \rightarrow \infty} \frac{\log C(n)}{\log n} ?$$

I.e., $C(n) = n^{\beta + o(1)}$.

Compare $\lim_{n \rightarrow \infty} \frac{\log E(n)}{\log n} = 1/2$.

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Easy: $\beta \geq \frac{1}{2}$.

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.....

Darn!

That's
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